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A PROBLEM CONCERNING MAPPINGS WITH CONSTANT DISPLACEMENT.

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ABSTRACT. We present here an open problem concerning lipschitzian self mappings of closed convex subsets of Banach spaces.

Let X be a Banach space with norm $\|\cdot\|$ and let C be a nonempty, convex closed and bounded subset of X. A lot of attention has been focused recently on the behavior of lipschitzian self mappings of such sets C. Let us recall that the mapping $T : C \to C$ is *lipschitzian* (satisfies Lipschitz condition) if there exists $k \geq 0$ such that

$$\|Tx - Ty\| \leq k \|x - y\|, \tag{1}$$

for all $x, y \in C$. The smallest k for which (1) holds is said to be the Lipschitz constant for T and is denoted by $k(T)$. If (1) holds we also say that T is k-lipschitzian or that T is of class $\mathcal{L}(k), T \in \mathcal{L}(k)$.

If C is compact then due to the Schauder Fixed Point Theorem any continuous (thus also any lipschitzian) mapping $T : C \to C$ has a point x satisfying $x = Tx$, *a fixed point of* T. If C is not compact, it is no longer true. The strongest known result due to P. K. Lin and Y. Sternfeld [6] states:

- **If C is not compact then for any $k > 1$ there exists a mapping $T : C \to C$ of class $L(k)$ such that,**

$$d(T) = \inf\{\|x - Tx\| : x \in C\} > 0. \tag{2}$$

The number $d(T)$ defined by (2) is called the minimal displacement of T and mappings T which satisfy (2) are called mappings with positive displacement.

Once we have a lipschitzian mapping T with positive displacement $d = d(T) > 0$ we can define a modified mapping $\widetilde{T} : C \to C$ by

$$\widetilde{T}x = x + d\frac{Tx - x}{\|Tx - x\|}.$$

It is easy to observe that $\widetilde{T}$ is also lipschitzian but the Lipschitz constant $k\left(\widetilde{T}\right)$ is not necessarily the same as $k(T)$.

This modified mapping has *constant positive displacement* equal d, which means that for all $x \in C$ we have

$$\left\|x - \widetilde{T}x\right\| = d = d(T) > 0.$$

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Now we can observe that for any $c \in (0,1]$ the convex combination of the mapping $\widetilde{T}$ with the identity mapping I,

$$\widetilde{T_c} = (1-c)\, I + c\widetilde{T},$$

is also of positive displacement equal cd. Moreover, we have

$$k\left(\widetilde{T_c}\right) = k\left((1-c)\, I + c\widetilde{T}\right) \leq 1 - c + ck\left(\widetilde{T}\right)$$

and consequently, $\lim_{c\to 1} k\left(\widetilde{T_c}\right) = 1$.

The above allows us to formulate an equivalent modification of the Lin Sternfeld result.

- **If C is not compact then for any $k > 1$ there exists a mapping $T : C \to C$ of class $L(k)$ with constant positive displacement.**

From now on we shall discuss only mappings with constant positive displacement. Suppose $T : C \to C$ is such a mapping with $d(T) = d > 0$. The iterated mapping $T^2 = T \circ T : C \to C$ is not necessarily of constant displacement. For any $x \in C$ we have an obvious inequality

$$0 \leq \left\|T^2x - x\right\| \leq \left\|T^2x - Tx\right\| + \|Tx - x\| = 2d.$$

If $\left\|T^2x - x\right\| = 2d$ then the line consisting of two linear segments $[x, Tx]$ and $\left[Tx, T^2x\right]$ is isometric to the segment $\left[x, T^2x\right]$ and consequently to the interval $[0, 2d]$. If $\left\|T^2x - x\right\| < 2d$ it means that the vector $T^2x - Tx$ is in some metric sense "rotated" with respect to the vector $Tx - x$. For this reason it is natural to introduce two coefficients

$$a_-(T) = \inf\left\{\frac{1}{d}\left\|T^2x - x\right\| : x \in C\right\}$$

and

$$a_+(T) = \sup\left\{\frac{1}{d}\left\|T^2x - x\right\| : x \in C\right\}.$$

Intuitively, they represent the *minimal metric rotation* and *global metric rotation.* It is understood in the sense that if $a_-(T) < 2$ then for any $\varepsilon > 0$ there are some $x \in C$ for which the vector $T^2x - Tx$ of length d is *rotated* with respect to the vector $Tx - x$ of the same length in such a way that

$$\left\|\left(T^2x - Tx\right) + (Tx - x)\right\| < (a_-(T) + \varepsilon)\, d. \tag{3}$$

If $a_+(T) < 2$, then (3) holds for all $x \in C$. Especially if $a_+(T) = 0$ then T is an *involution*, $T^2 = I$ on C.

There are several open problems and questions concerning mutual relations between constants $k(T), a_-(T)$ and $a_+(T)$. Here is the first observation.

Let $T : C \to C$ be a mapping of class $\mathcal{L}(k)$ with constant displacement $d(T) = d$. Take any point $x \in C$ and put $u = \frac{1}{2}\left(Tx + T^2x\right), v = \frac{1}{2}(x + Tx)$. Then we have

$$\begin{aligned} d = \|u - Tu\| &= \left\|\frac{1}{2}\left(Tx + T^2x\right) - Tu\right\| \leq \\ &\leq \frac{1}{2}\|Tx - Tu\| + \frac{1}{2}\left\|T^2x - Tu\right\| \leq \\ &\leq \frac{k}{2}\|x - u\| + \frac{k}{2}\|Tx - u\| \leq \\ &\leq \frac{k}{2}\|x - v\| + \frac{k}{2}\|v - u\| + \frac{k}{2}\|Tx - u\| = \\ &= \frac{k}{4}d + \frac{k}{4}\left\|x - T^2x\right\| + \frac{k}{4}d = \\ &= \frac{k}{2}d + \frac{k}{4}\left\|x - T^2x\right\|. \end{aligned}$$

The conclusion of it can be written in the form

$$\frac{\left\|x - T^2x\right\|}{d} \geq 2\left(\frac{2}{k} - 1\right)$$

and this shows that the rotation constants and Lipschitz constant of T must satisfy

$$a_+(T) \geq a_-(T) \geq 2\left(\frac{2}{k(T)} - 1\right). \tag{4}$$

In other words we have

- **If $T : C \to C$ is a lipschitzian mapping with constant positive displacement then**

$$k(T) \geq \frac{4}{a_-(T) + 2}. \tag{5}$$

The above evaluation is probably not sharp. The main open problem connected with mappings of constant displacement can be described as follows.

Problem 1. *For any $a \in [0,2]$ find the value*

$\varkappa(a) = \inf\{k :$ *there exists a mapping $T : C \to C$ with $k(T) = k$ and $a_-(T) = a$*$\}$

The evaluation (5) shows that

$$\varkappa(a) \geq \frac{4}{a + 2}. \tag{6}$$

The above has been shown without taking into account any geometrical properties of the set C. One can restrict himself to some particular situation of a given set C and define relative function $\varkappa_C(a)$. We shall stay here with the general case. However to estimate $\varkappa(a)$ from above we have to discuss a concrete construction.

Let $X = C[0,1]$ with the usual uniform norm and let the set K be defined by

$$K = \{x \in C[0,1] : 0 = x(0) \leq x(t) \leq x(1) = 1\}.$$

Let e be the identity function on $[0,1], e(t) \equiv t$. Any function $\alpha \in K$ generates a mapping $T_\alpha : K \to K$ defined for $x \in K$ by

$$(T_\alpha x)(t) = (\alpha \circ x)(t) = \alpha(x(t)).$$

If α is lipschitzian, so is T_α and we have

$$k(T_\alpha) = k(\alpha) = \sup\left\{\frac{|\alpha(t) - \alpha(s)|}{|t-s|} : t, s \in [0,1], t \neq s\right\}.$$

Moreover, since any $x \in K$ takes all the values between 0 and 1, we have

$$\|x - T_\alpha x\| = \max_{t\in[0,1]} |x(t) - \alpha(x(t))| = \max_{s\in[0,1]} |s - \alpha(s)| = \|e - \alpha\|.$$

Thus for $\alpha \neq e$, T_α has constant positive displacement $d(T_\alpha) = \|e - \alpha\| > 0$. The iterated mapping $T_\alpha^2 = T_\alpha \circ T_\alpha = T_{\alpha\circ\alpha}$ is of the same type with $k(T_\alpha^2) = k(\alpha\circ\alpha) \leq k(\alpha)^2$ and $d(T_\alpha^2) = \|e - \alpha\circ\alpha\| > 0$. In this case

$$a_+(T_\alpha) = a_-(T_\alpha) = \frac{k(T_\alpha^2)}{k(T_\alpha)} = \frac{\|e - \alpha\circ\alpha\|}{\|e-\alpha\|}. \tag{7}$$

The relation between Lipschitz and rotation constants in this case can be evaluated as follows. There exists at least one point $t \in [0,1]$ such that

$$|\alpha(\alpha(t)) - \alpha(t)| = \|e - \alpha\| = d(T_\alpha) > 0.$$

Let us assume that at this point $\alpha(\alpha(t)) > \alpha(t)$. The case with converse inequality can be treated the same way. Let t_0 be the minimal point for which the above holds. It means that

$$t_0 = \min\{t : \alpha(\alpha(t)) - \alpha(t) = \|e-\alpha\| = d(T_\alpha)\}. \tag{8}$$

Obviously

$$\alpha(\alpha(t_0)) - \alpha(t_0) = \|e - \alpha\|.$$

Observe that at t_0 we have $\alpha(t_0) \geq t_0$. Indeed,since $\alpha(0) = 0$, if $t_1 = \alpha(t_0) < t_0$ then $\alpha(t_1) = \alpha(\alpha(t_0)) > \alpha(t_0)$ implies existence of a point $t_2 < t_0$ for which $\alpha(t_2) = \alpha(t_0)$. For this point we would have $\alpha(\alpha(t_2)) - \alpha(t_2) = \|e - \alpha\|$ which contradicts (8).

Now, assume that α is lipschitzian with $k(\alpha) = k$. Then, we have

$$\alpha(\alpha(t_0)) - \alpha(t_0) \leq \|\alpha\circ\alpha - \alpha\| = \|T_\alpha \alpha - T_\alpha e\| \leq k\|\alpha - e\| = k(\alpha(t_0) - t_0).$$

Consequently

$$\begin{aligned}\|\alpha\circ\alpha - e\| &\geq \alpha(\alpha(t_0)) - t_0 = [\alpha(\alpha(t_0)) - \alpha(t_0)] + [\alpha(t_0) - t_0] \geq \\ &\geq \left(1 + \frac{1}{k}\right)[\alpha(\alpha(t_0)) - \alpha(t_0)] = \left(1+\frac{1}{k}\right)\|\alpha - e\|\end{aligned}$$

and finally, in view of (7)

$$2 \geq a_+(T_\alpha) = a_-(T_\alpha) \geq 1 + \frac{1}{k}. \tag{9}$$

Both inequalities in (9) are sharp. The case $a_+(T_\alpha) = 2$ occurs for any function α of the form

$$\alpha(t) = \begin{cases} \left(1 + \frac{\varepsilon}{b}\right)t & \text{for } 0 \leq t \leq b < 1 \\ t + \varepsilon & \text{for } b < t \leq 1 - \varepsilon \\ 1 & \text{for } 1 - \varepsilon < t \leq 1 \end{cases},$$

where $b \in (0,1)$ is arbitrary and ε is sufficiently small.

The equalities $k(\alpha) = k > 1$ and $a_-(T_\alpha) = 1 + \frac{1}{k}$ are satisfied for specially chosen family of functions

$$\alpha_k(t) = \begin{cases} kt & \text{for } 0 \leq t \leq \frac{1}{k} \\ 1 & \text{for } \frac{1}{k} < t \leq 1 \end{cases}.$$

In this setting the family of mappings $T_k = T_{\alpha_k}, k \geq 1$ fulfils the conditions

$$k(T_k) = k, \text{ with } d(T_k) = 1 - \frac{1}{k},$$

$$k\left(T_k^2\right) = k, \text{ with } d(T_k) = 1 - \frac{1}{k^2},$$

$$a_+(T_\alpha) = a_-(T_\alpha) = 1 + \frac{1}{k}. \tag{10}$$

Comparing (10) with the definition of the function $\varkappa(a)$ (see 1) and substituting $a = 1 + \frac{1}{k}$ we get

$$\varkappa(a) \leq \frac{1}{a-1} \tag{11}$$

for all $a \in (1, 2]$.

Summing up the estimates (6)and (11) we conclude with

$$\frac{4}{a+2} \leq \varkappa(a) \leq \begin{cases} +\infty & \text{if } a \in [0,1] \\ \frac{1}{a-1} & \text{if } a \in (1,2] \end{cases}. \tag{12}$$

The gap between the lower and upper bound given by (12) is large. Both inequalities are probably not sharp. The main problem of finding exact formula for $\varkappa(a)$ (see Problem 1) leads to some more specific, seemingly simpler, but still remaining without answer partial questions.

Problem 2. *Find better then (12) estimate for $\varkappa(a)$.*

Problem 3. *Is $\varkappa(a) < \infty$ on $[0,1]$? If not, then for which $a \in [0,1]$, $\varkappa(a) < \infty$?*

In other words. Can one construct an example of a bounded closed convex set C and a lipschitzian mapping with constant positive displacement $T : C \to C$ such that $a_-(T) \leq 1$? The same with $a_+(T) \leq 1$?

Problem 4. *Is $\varkappa(0) < \infty$?*

More specifically, does there exist a bounded closed and convex set C and a lipschitzian mapping $T : C \to C$ of constant minimal displacement and such that $a_-(T) = 0$? Replacing $a_-(T)$ in the last question to $a_+(T)$ we obtain the last „exotic" question.

Problem 5. *Does there exists a bounded, closed and convex set C for which there is a lipschitzian involution $\left(T : C \to C, T^2 = I\right)$ having constant positive displacement ?*

The notion of rotation constant and rotative mappings has been introduced by K. Goebel and M. Koter in [2] and [3]. More informations about these notions can be found in an expository article [4] and books [5] and [1].

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