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Sperner Matroid and Sperner Map

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Abstract. A reconstruction of a Sperner map from a Sperner matroid is illustrated. As an application of this reconstruction, we give a new proof of a completion theorem of Lovász's matroid version of Sperner's lemma.

The purpose of this note is to give a simple procedure which allows us to retrieve a Sperner map from a Sperner matroid.

1. Reconstruction of Sperner map from Sperner matroid

Let K be a triangulation of a d-simplex $a_0a_1\dots a_d$ in a Euclidean space, and $V(K)$ the vertex set of K. A map $\varphi : V(K) \to \{0,1,\dots,d\}$ is said to be a *Sperner map* if for each $i_0, i_1, \dots, i_k$ with $0 \leq i_0 < i_1 < \dots < i_k \leq d$ and for each $v \in V(K) \cap a_{i_0}a_{i_1}\dots a_{i_k}$, $\varphi(v) \in \{i_0, i_1, \dots, i_k\}$. A matroid M on $V(K)$ is called a *Sperner matroid* over K if for each $S \subset \{a_0, a_1, \dots, a_d\}$ and for each $v \in V(K) \cap conv(S)$, $v \in cl_M(S)$, where $conv(S)$ stands for the convex hull of S and $cl_M(S)$ denotes the closure of S in M. Let M be a Sperner matroid over K such that $\{a_0, a_1, \dots, a_d\}$ forms a basis of M. Put

$$F_j \equiv cl_M(\{a_0, a_1, \dots, a_j\}) \quad (j = 0, 1, \dots, d).$$

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Let us define $\varphi : V(K) \to \{0, 1, \ldots, d\}$ by setting

$$\varphi(v) = 0 \quad \text{if} \quad v \in F_0,$$

$$\varphi(v) = j \quad \text{if} \quad v \in F_j \setminus F_{j-1} \quad (j = 1, 2, \ldots, d).$$

The problem that we consider in this paper is the following : *Under what condition is φ a Sperner map ?* It is clearly necessary that the $(d-1)$-face $a_1 a_2 \ldots a_d$ contains no loops. To see this, if $a_1 a_2 \ldots a_d$ contains a loop v of M, then $\varphi(v) = 0$ and $v \in V(K) \cap a_1 a_2 \ldots a_d$. As φ is a Sperner map, we have $\varphi(v) \in \{1, 2, \ldots, d\}$, in contradiction. What is perhaps surprising is that this condition is also sufficient. Indeed, we have

Theorem 1. *Let M be a Sperner matroid over a triangulation K of a d-simplex $a_0 a_1 \ldots a_d$ such that $\{a_0, a_1, \ldots, a_d\}$ forms a basis. Put $F_j \equiv cl_M(\{a_0, a_1, \ldots, a_j\})$ $(j = 0, 1, \ldots, d)$, and let $\varphi : V(K) \to \{0, 1, \ldots, d\}$ be defined by*

$$\varphi(v) = 0 \textit{ if } v \in F_0, \ \varphi(v) = j \textit{ if } v \in F_j \setminus F_{j-1} \quad (j = 1, 2, \ldots, d).$$

Then φ is a Sperner map if and only if the $(d-1)$-face $a_1 a_2 \ldots a_d$ contains no loops of M.

The proof of Theorem 1 is based on the following :

Lemma. *Let B be a basis of a matroid M. Suppose*

(a) $S \subset T \subset B$,

(b) $X \subset B, \ X \cap T = \emptyset$,

(c) $y \in cl_M(T) \setminus cl_M(S)$.

Then $y \in cl_M(T \cup X) \setminus cl_M(S \cup X)$.

2. A sign function

Let K be a triangulation of a d-simplex $a_0 a_1 \ldots a_d$ in a Euclidean space, M a Sperner matroid over K, $B \equiv (a_0, a_1, \ldots, a_d)$ an ordered basis of M. Let $\Lambda_B : K \to \{-1, 0, 1\}$. We define $\Lambda_B(v_0 v_1 \ldots v_d) = 1$ (resp.-1) if $v_0 \in F_0$ and $v_j \in F_j \setminus F_{j-1}$ $(j = 1, 2, \ldots, d)$, where $F_j \equiv cl_M(\{v_0, v_1, \ldots, v_j\})$ $(j = 0, 1, \ldots, d)$, and $\det(\alpha_{ij}) > 0$ (resp. $\det(\alpha_{ij}) < 0$), where

$$\begin{pmatrix} v_0 \\ \vdots \\ v_d \end{pmatrix} = \begin{pmatrix} \alpha_{00} & \cdots & \alpha_{0d} \\ \vdots & & \vdots \\ \alpha_{d0} & \cdots & \alpha_{dd} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_d \end{pmatrix}, \quad \sum_{j=0}^{d} \alpha_{ij} = 1 \quad (0 \le i \le d). \tag{$*$}$$

We define $\Lambda_B(v_0 v_1 \ldots v_d) = 0$ otherwise. Now, let $\varphi : V(K) \to \{0, 1, \ldots, d\}$. We call a d-simplex $v_0 v_1 \ldots v_d \in K$ *positively* (*resp. negatively*) *completely labelled* if $\varphi(v_j) = j$ $(j = 0, 1, \ldots, d)$, and $\det(\alpha_{ij}) > 0$ (resp.

$\det(\alpha_{ij}) < 0$), where the matrix (α_{ij}) is given in $(*)$. A d-simplex of K is *completely labelled* if it is positively or negatively completely labelled. It is obvious that $v_0 v_1 \dots v_d \in K$ is completely labelled if and only if $\{\varphi(v_0), \varphi(v_1), \dots, \varphi(v_d)\} = \{0, 1, \dots, d\}$. The celebrated Sperner lemma[7] asserts thet if φ is a Sperner map then $\sharp\{\sigma \in K\ ;\ \sigma \text{ is completely labelled }\} \equiv 1 \pmod 2$. The oriented Sperner lemma[1] states that

$$\sharp\{\sigma \in K\ ;\ \sigma \text{ is positively completely labelled }\} - \sharp\{\sigma \in K\ ;\ \sigma \text{ is negatively completely labelled }\} = 1.$$

By Thoerem 1 and the oriented Sperner lemma, we have

Theorem 2. *Let K be a triangulation of a d-simplex $a_0 a_1 \dots a_d$ in a Euclidean space, and M a Sperner matroid over K such that the $(d-1)$-face $a_1 a_2 \dots a_d$ contains no loops of M. If $B \equiv (a_0, a_1, \dots, a_d)$ is an ordered basis of M, then*

$$\sum_{\sigma \in K} \Lambda_B(\sigma) = 1.$$

Theorem 2 was recently obtained by the authors in [4] with a completely different proof. An example given in [4] shows that the condition " the $(d-1)$-face $a_1 a_2 \dots a_d$ contains no loops " cannot be dispensed with.

By disregarding orientation in Theorem 2, we have

Theorem 3. *Under the assumptions of Theorem 2,*

$$\sum_{\sigma \in K} |\Lambda_B(\sigma)| \equiv 1 \ (mod\ 2).$$

Theorem 3 is an extension of Lovász's theorem. It complements Lovász's theorem in two aspects : one concerns an arbitrary matroid while the other is the assertion of oddness. Indeed, Lovász[2] proved the following :

Theorem 4 (L. Lovász). *Let K be a triangulation of a d-simplex $a_0 a_1 \dots a_d$ in a Euclidean space, and M a Sperner matroid over K such that M contains no loops. If $\{a_0, a_1, \dots, a_d\}$ is a basis of M , then K has a d-simplex $v_0 v_1 \dots v_d$ such that $\{v_0, v_1, \dots, v_d\}$ is also a basis of M.*

Finally, let us mention that it is perhaps worth developing a general matroid version which contains multiple balanced Sperner lemma[6], combinatorial Lefschetz fixed-point formula[5], and multiple combinatorial Stokes' theorem[3].

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