数理解析研究所講究録 1298 巻 2002 年 65-69

On optimal 2-uniform convexity inequalities

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This is a résumé of some recent results of the authors on optimal 2-uniform convexity inequalities.

A Banach space X is called *q-uniformly convex* $(2 \leq q < \infty)$ if there is $C > 0$ such that

$$\delta_X(\varepsilon) \geq C\varepsilon^q \quad \text{for all } \varepsilon > 0, \tag{1}$$

where $\delta_X(\varepsilon)$ is the modulus of convexity,

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \ \|x\| = \|y\| = 1,\ \|x-y\| = \epsilon\right\}. \tag{2}$$

The q-unform convexity of X is characterized by the following "q-uniform convexity inequality"

$$\frac{\|x+y\|^q + \|x-y\|^q}{2} \geq \|x\|^q + \|Cy\|^q, \tag{3}$$

where $0 < C \leq 1$, independent on $x, y \in X$ (cf. [1,2,4]).

Clarkson's inequalities imply that L_q $(2 \leq q < \infty)$ is q-uniformly convex and L_p $(1 < p \leq 2)$ is p'-uniformly convex, where $1/p + 1/p' = 1$, whereas, as is well known, L_p $(1 < p \leq 2)$ is in fact 2-uniformly convex; Ball-Carlen-Lieb [1] gave a proof which uses Hanner's and Gross' inequality. The *optimal 2-uniform convexity inequality* for L_p $(1 < p \leq 2)$ is the following:

$$\frac{\|f+g\|_p^2 + \|f-g\|_p^2}{2} \geq \|f\|_p^2 + (p-1)\|g\|_p^2, \tag{4}$$

where the constant $p-1$ is optimal. This is equivalent to the following more sharp inequality

$$\left(\frac{\|f+g\|_p^p + \|f-g\|_p^p}{2}\right)^{1/p} \geq \left(\|f\|_p^2 + (p-1)\|g\|_p^2\right)^{1/2}, \tag{5}$$

where $p-1$ is optimal ([1]). (For $2 \leq p < \infty$ these inequalities are reversed; see Ball-Carlen-Lieb [1].) The inequality (5) yields the following best estimate in (1) for L_p $(1 < p \leq 2)$:

$$\delta_{L_p}(\varepsilon) \geq \{(p-1)/8\}\varepsilon^q \quad \text{for all } \varepsilon > 0.$$

In the recent paper [5] Takahashi-Hashimoto-Kato presented some generalizations of the q-uniform convexity inequality (3), and showed that these inequalities are inherited to the Lebesgue-Bochner space $L_r(X)$. In this note, by using their results, we shall present some generalizations of the optimal 2-uniform convexity inequalities (4) and (5).

First we state the following inequalities which are fundamental in our discussion:

Lemma 1 ([4, p.76]). Let $1 < p \leq q < \infty$ and $\gamma = \sqrt{(p-1)/(q-1)}$. Then:
(i) For any $x, y \in X$

$$\left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} \leq \left(\frac{\|x+y\|^q + \|x-y\|^q}{2}\right)^{1/q} \tag{6}$$

(ii) For any $x, y \in X$

$$\left(\frac{\|x+\gamma y\|^q + \|x-\gamma y\|^q}{2}\right)^{1/q} \leq \left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} \tag{7}$$

Theorem 1 (Takahashi-Hashimoto-Kato [5]). Let $2 \leq q < \infty$ and $1 < t \leq \infty$. The following are equivalent.
(i) X is q-uniformly convex.
(ii) For any $1 < t \leq \infty$ there exists $0 < C \leq 1$ such that

$$\left(\frac{\|x+y\|^t + \|x-y\|^t}{2}\right)^{1/t} \geq (\|x\|^q + \|Cy\|^q)^{1/q} \quad \forall x,\, y \in X. \tag{8}$$

(iii) For some $1 < t \leq \infty$ there exists $0 < C \leq 1$ such that the inequality (8) holds.

In particular, we have

Theorem 2 (2-uniform convexity inequalities). The following are equivalent.
(i) X is 2-uniformly convex.
(ii) For any $1 < t \leq \infty$ there exists $0 < C \leq 1$ such that

$$\left(\frac{\|x+y\|^t + \|x-y\|^t}{2}\right)^{1/t} \geq \left(\|x\|^2 + \|Cy\|^2\right)^{1/2} \quad \forall x,\, y \in X. \tag{9}$$

(iii) For some $1 < t \leq \infty$ there exists $0 < C \leq 1$ such that (9) holds.

Remark 1. In Theorem 2 (ii) and (iii) we have $0 < C \leq \min\{1,\, t-1\}$, where equality holds if X is a Hilbert space.

Proposition 1. Assume that the following 2-uniform convexity inequality

$$\max\{\|x+y\|,\ \|x-y\|\} \geq \left(\|x\|^2 + C\|y\|^2\right)^{1/2} \tag{10}$$

holds in X. Then,

$$\delta_X(\epsilon) \geq \frac{C}{8}\epsilon^2 \qquad \text{for all } \ 0 < \epsilon < 2. \tag{11}$$

One should note that for $1 < t < \infty$

$$\max\{\|x+y\|,\ \|x-y\|\} \geq \left(\frac{\|x+y\|^t + \|x-y\|^t}{2}\right)^{1/t}.$$

Now, 2-uniform convexity inequality is inherited to $L_r(X)$ as follows.

Theorem 3. Let $1 < p,\ r \leq 2$. Assume that the inequality

$$\left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} \geq \left(\|x\|^2 + C\|y\|^2\right)^{1/2} \tag{12}$$

holds in X. Then

$$\left(\frac{\|f+g\|_r^p + \|f-g\|_r^p}{2}\right)^{1/p} \geq \left(\|f\|_r^2 + C'\|g\|_r^2\right)^{1/2} \tag{13}$$

holds in $L_r(X)$, where

$$C' = \begin{cases} C & \textit{if } p \leq r \leq 2, \\ \{(r-1)/(p-1)\}C & \textit{if } 1 < r < p. \end{cases}$$

Remark 2. The constant C' is optimal in general.

Since X is isometrically embedded into $L_r(X)$, it is trivial that any inequality valid in $L_r(X)$ holds in X. The next result asserts that from a 2-uniform convexity inequality in $L_r(X)$ we have a stonger one in X.

Theorem 4. Let $1 < r \leq 2$ and $r < p$. Assume that

$$\left(\frac{\|f+g\|_r^p + \|f-g\|_r^p}{2}\right)^{1/p} \geq (\|f\|_r^2 + C\|g\|_r^2)^{1/2} \tag{14}$$

holds in $L_r(X)$. Then

$$\left(\frac{\|x+y\|^r + \|x-y\|^r}{2}\right)^{1/r} \geq (\|x\|^2 + C\|y\|^2)^{1/2} \tag{15}$$

holds in X.

Indeed take any non-zero x, $y \in X$ and put $f = (x, x)$, $g = (y, -y) \in \ell_r^2(X) \subset L_r(X)$ in (14).

By Theorems 3 and 4 we have the following optimal 2-uniform convexity inequality for L_r (use the parallelogram law for scalars).

Theorem 5 (Optimal 2-uniform convexity inequality for L_r, $1 < r \leq 2$). Let $1 \leq r \leq 2$ and $1 < p \leq \infty$. Then

$$\left(\frac{\|f+g\|_r^p + \|f-g\|_r^p}{2}\right)^{1/p} \geq \left(\|f\|_r^2 + C\|g\|_r^2\right)^{1/2} \tag{16}$$

holds in L_r, where $C = \min\{p-1,\, r-1\}$.

Remark 3. (i) The constant C in (16) is best possible.
(ii) The inequality (16) for L_p, $1 < p \leq 2$ with $C = p - 1$, that is,

$$\left(\frac{\|f+g\|_p^p + \|f-g\|_p^p}{2}\right)^{1/p} \geq \left(\|f\|_p^2 + (p-1)\|g\|_p^2\right)^{1/2} \tag{5}$$

was proved in Ball-Carlen-Lieb [1]. Their proof used Hanner's inequality and Gross' inequality, whereas we derived (5) from Theorems 3 and 4 and the parallelogram law for scalars.

Theorem 3 yields the following

Theorem 6 (Optimal 2-uniform convexity inequality for $L_r(L_s)$, $1 < r, s \leq 2$). Let $1 \leq r$, $s \leq 2$ and $1 < p \leq \infty$. Then

$$\left(\frac{\|f+g\|_r^p + \|f-g\|_r^p}{2}\right)^{1/p} \geq \left(\|f\|_r^2 + C\|g\|_r^2\right)^{1/2} \tag{17}$$

holds in $L_r(L_s)$, where $C = \min\{p-1,\, r-1,\, s-1\}$. In particular, if $1 < p \leq \min\{r,\, s\}$, then $C = p - 1$.

Remark 4. The constant C in (17) is best possible.

References

[1] K. Ball, E. A. Carlen and E. H. Lieb, Sharp uniform convexity and smoothness inequalities for trace norms, Invent. Math. **115** (1994), 463–482.

[2] B. Beauzamy, Introduction to Banach Spaces and Their Geometry 2nd Ed., 1985.

[3] M. Kato and Y. Takahashi, Type, cotype constants and Clarkson's inequalities for Banach spaces, Math. Nachr. **186** (1997), 187–196.

[4] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces II, 1979.

[5] Y. Takahashi, K. Hashimoto and M. Kato, On sharp uniform convexity, smoothness, and strong type, cotype inequalities, J. Nonlinear Convex Anal. **3** (2002), 267-281.

[6] Y. Takahashi and M. Kato, Clarkson and Random Clarkson inequalities for $L_r(X)$, Math. Nachr. **188** (1997), 341–348.

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