On Generalized Lee Weights for Codes over $\mathbb{Z}_4$

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1 Introduction

For a linear code over a finite field, Helleseth, Klove and Mykkeltveit [9] introduced the generalized Hamming weights while studying the weight distribution of irreducible cyclic codes and later Wei ([18]) rediscovered the idea of generalized Hamming weights. After that a lot of papers dealing with the weights have been published (cf. [17] etc.). Recently, the generalized Hamming weights for codes over $\mathbb{Z}_4$ have been defined and studied, see [1], [19], [20], [3] and [10] for example.

In this note, we shall define a type of generalized Lee weights for codes over $\mathbb{Z}_4$ and give some fundamental results.

A linear code of length $n$ over $\mathbb{Z}_4$ is a $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. For a linear code $C$ of length $n$ over $\mathbb{Z}_4$, we define the rank of $C$, denoted by $\text{rank}(C)$, by the minimum number of generators of $C$. It is known that a linear code $C$ of length $n$ over $\mathbb{Z}_4$ is permutation-equivalent to a linear code with generator matrix of the form

$$
\begin{pmatrix}
I_{k_1} & X & Y \\
0 & 2I_{k_2} & 2Z
\end{pmatrix},
$$

where $X$ and $Z$ are binary matrices and $Y$ is a $\mathbb{Z}_4$-matrix. In this case, it finds that $|C| = 4^{k_1}2^{k_2}$ and $\text{rank}(C) = k_1 + k_2$. We shall define a code with a generator matrix of the form in 1 as being of type $\{k_1, k_2\}$.

For a vector $\mathbf{x} \in \mathbb{Z}_4^n$, we denote the Hamming weight and Lee weight by $\text{wt}(\mathbf{x})$ and $\text{L-wt}(\mathbf{x})$, respectively.

For a linear code $C$ of length $n$ over $\mathbb{Z}_4$, let $A(C)$ be the $|C| \times n$ array of all codewords in $C$. It is well-known that each column of $A(C)$ corresponds to the following three cases: (i)

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the column contains only 0 (ii) the column contains 0 and 2 equally often (iii) the column contains all elements of \( \mathbb{Z}_4 \) equally often (cf. [20]). For the three columns (i), (ii) and (iii), we define the Lee weights of these columns by 0, 2 and 1 respectively. Thus we define the Lee weight \( \text{wt}_L(C) \) of \( C \) by the sum of the Lee weights of all columns of \( A(C) \). For example, if
\[
C = \{(0,0,0), (1,0,1), (2,0,2), (3,0,3), (0,2,2), (1,2,3), (2,2,0), (3,2,1)\},
\]
then \( \text{wt}_L(C) = 1 + 2 + 1 = 4 \). We remark that if \( C \) is generated by only one vector \( x \), then the Lee weight \( \text{wt}_L(C) \) corresponds to the original Lee weight \( \text{L-wt}(x) \) of \( x \). Then we have the following theorem.

**Theorem 1.1** Let \( C \) be a linear code \( C \) of length \( n \) over \( \mathbb{Z}_4 \) with type \( 4^{k_1}2^{k_2} \). Then we have
\[
\text{wt}_L(C) = \frac{1}{4^{k_1-1}2^{k_2}} \sum_{x \in C} (\text{L-wt}(x) - \text{wt}(x))
\]
\[
= \frac{1}{4^{k_1-1}2^{k_2}} \sum_{x \in C} |\{i : x_i = 2\}|.
\]

Now, for \( 1 \leq r \leq \text{rank}(C) \), we define the \( r \)-th generalized Lee weight with respect to rank (GLWR) \( d_f^L(C) \) of \( C \) as follows:
\[
d_f^L(C) := \min \{\text{wt}_L(D) : D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } \text{rank}(D) = r\}.
\]
We note that \( d_f^L(C) \) corresponds to the minimum Lee weight of \( C \).

## 2 Bounds for GLWR

In this section, we give some bounds for GLWR of linear codes over \( \mathbb{Z}_4 \).

**Lemma 2.1** If \( C \) is a linear code of length \( n \) over \( \mathbb{Z}_4 \) with \( \text{rank}(C) = 2 \), then there exists a codeword \( 0 \neq v \in C \) such that \( \text{L-wt}(v) \leq \text{wt}_L(C) \).

Using the above lemma, we have the following result.

**Theorem 2.2** Let \( C \) be a linear code of length \( n \) over \( \mathbb{Z}_4 \) with \( \text{rank}(C) \geq 2 \). Then we have
\[
1 \leq d_f^L(C) \leq d_f^H(C).
\]

In [11], the \( r \)-th generalized Hamming weight with respect to rank (GHWR) of a linear code \( C \) is defined by
\[
d_f^H(C) := \min |\text{Supp}(D)| : D \text{ is a } \mathbb{Z}_4\text{-submodule of } C \text{ with } \text{rank}(D) = r,
\]
where \( \text{Supp}(D) := \cup_{x \in D} \text{supp}(x) \). We remark that
\[
d_f^L(C) \leq 2d_f^H(C).
\]

The following lemma is called the generalized Singleton bound for linear codes over \( \mathbb{Z}_4 \) (see
Lemma 2.3 Let $C$ be a linear code of length $n$ over $\mathbb{Z}_4$. Then, for any $r$, $1 \leq r \leq \text{rank}(C)$,

$$d^H_r(C) \leq n - \text{rank}(C) + r.$$ 

Now, we give a similar type bound for GLWR.

Theorem 2.4 For a linear code $C$ of length $n$ over $\mathbb{Z}_4$ and any $r$, $1 \leq r \leq \text{rank}(C)$,

$$\left\lfloor \frac{d^L_r(C) - 2r + 1}{2} \right\rfloor \leq n - \text{rank}(C).$$

Remark 2.5 In [7] and [15], it is shown that for a linear code $C$ of length $n$ over $\mathbb{Z}_4$ with minimum Lee weight $d_L$,

$$\left\lfloor \frac{d_L - 1}{2} \right\rfloor \leq n - \text{rank}(C).$$

Since $d_L = d^L_1(C)$, the bound in Theorem 2.4 is a generalization of the above bound.

If a linear code $C$ of length $n$ over $\mathbb{Z}_4$ meets the bound in Theorem 2.4 for $r$, that is, $\left\lfloor \frac{(d^L_r(C) - 2r + 1)/2 \right\rfloor = n - \text{rank}(C)$, then we shall call the code $C$ as $r$-th maximum Lee distance separable with respect to rank ($r$-th MLDR) code. Similarly if a linear code $C$ of length $n$ over $\mathbb{Z}_4$ meets the bound in Lemma 2.3 for $r$, that is, $d^H_r(C) = n - \text{rank}(C) + r$, then the code $C$ is called $r$-th maximum Hamming distance separable with respect to rank ($r$-th MHDR) code. Now we shall give a connection between $r$-th MLDR codes and $r$-th MHDR codes.

Lemma 2.6 If $C$ is an $r$-th MLDR code, then $d^L_r(C) = 2d^H_r(C) - 1$ or $2d^H_r(C)$.

Theorem 2.7 Let $C$ be a linear code $C$ of length $n$ over $\mathbb{Z}_4$. If $C$ is an $r$-th MLDR code, then $C$ is an $r$-th MHDR code.

Theorem 2.8 Let $C$ be an $r$-th MHDR code of length $n$ over $\mathbb{Z}_4$. $C$ is an $r$-th MLDR code if and only if $d^L_r(C) = 2d^H_r(C) - 1$ or $2d^H_r(C)$.

It is known that if $C$ is a linear code of length $n$ over $\mathbb{Z}_4$ with minimum Hamming weight $d_H$ and minimum Lee weight $d_L$, then

$$(3) \quad d_H \geq \left\lfloor \frac{d_L}{2} \right\rfloor$$

(cf. [14]). In [16], they have proved the following Griesmer type bound for linear codes over finite quasi-Frobenius rings.
Lemma 2.9 Let $C$ be a linear code of length $n$ over $\mathbb{Z}_4$ with $\text{rank}(C) = k$ and minimum Hamming weight $d_H$. Then

$$n \geq \sum_{i=0}^{k-1} \left\lfloor \frac{d_H}{2^i} \right\rfloor.$$

Using (3) and Lemma 2.9, we have the following Griesmer type bound for minimum Lee weights of linear codes over $\mathbb{Z}_4$.

Proposition 2.10 Let $C$ be a linear code of length $n$ over $\mathbb{Z}_4$ with $\text{rank}(C) = k$ and minimum Lee weight $d_L$. Then

$$n \geq \sum_{i=0}^{k-1} \left\lfloor \frac{\lceil d_L/2 \rceil}{2^i} \right\rfloor.$$

Now we have a generalized Griesmer type bound for GLWR.

Theorem 2.11 For a linear code $C$ of length $n$ over $\mathbb{Z}_4$ and any $r$, $1 \leq r \leq \text{rank}(C)$, we have

$$d_r^L(C) \geq \sum_{i=0}^{r-1} \left\lfloor \frac{d_1^L(C)/2}{2^i} \right\rfloor.$$

Let $C$ be a linear code $C$ of length $n$ over $\mathbb{Z}_4$. From the definitions of GLWR and GHWR, we have

(4) $$d_r^H \geq \left\lceil \frac{d_r^L}{2} \right\rceil$$

for any $r$. We define the socle of $C$ as follows:

$$\text{Soc}(C) := \{x \in C \mid 2x = 0\}.$$ It is known that if $C$ is a linear code $C$ of length $n$ over $\mathbb{Z}_4$ with $\text{rank}(C) = k$ and minimum hamming weight $d_H$, then $\text{Soc}(C)$ is isomorphic to a binary $[n, k, d]$ code (cf. [11]).

Lemma 2.12 ([11]) For any $r$, $1 \leq r \leq \text{rank}(C)$, we have

$$d_r^H(C) = d_r^H(\text{Soc}(C)).$$

Using the above lemma and Theorem 3.19 (p. 35 in [5]), the lemma follows:

Lemma 2.13 Let $C$ be a linear code $C$ of length $n$ over $\mathbb{Z}_4$ with $\text{rank}(C) = k$. Then

$$n \geq d_r^H(C) + \sum_{i=1}^{k-r} \left\lfloor \frac{d_r^H(C)}{2^i(2^i - 1)} \right\rfloor,$$

for any $r$, $1 \leq r \leq k$. 

Now we have a generalized Griesmer type bound for GLWR.

**Theorem 2.14** Let $C$ be a linear code $C$ of length $n$ over $\mathbb{Z}_4$ with $\text{rank}(C) = k$. Then

$$n \geq \left\lceil \frac{d_f^L(C)}{2} \right\rceil + \sum_{i=1}^{k-r} \left\lceil \frac{\left\lceil \frac{d_f^L(C)/2}{2^i} \right\rceil}{2^{i}(2^i - 1)} \right\rceil,$$

for any $r$, $1 \leq r \leq k$.

**References**


