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STATE SPACE DYNAMICS AND ENTROPY

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ABSTRACT. We give a summary of our recent work on the topological entropy of state space homeomorphisms induced from C^*-dynamical systems. Our main result asserts that, for an automorphism of a separable unital exact C^*-algebra, zero Voiculescu-Brown entropy implies zero topological entropy on the state space.

One of the basic problems in dynamics is to identify systems with positive entropy, i.e., systems which exhibit "chaotic" behaviour. Glasner and Weiss showed in [6] that if a homeomorphism of a compact metric space has zero topological entropy, then so does the homeomorphism induced on the space of probability measures. By developing a matrix version of the key geometric lemma from [6], we showed in [8] that, for an automorphism of a separable unital exact C^*-algebra, if the Voiculescu-Brown entropy is zero then the induced homeomorphism on the state space has zero topological entropy. In this article we give a description of the ideas and techniques involved in the proof of this theorem, along with a summary of examples and related results from [8] involving the topological entropy of induced state space homeomorphisms. For general references on topological entropy and C^*-dynamical entropy we refer the reader to [4, 7] and [15], respectively.

1. TOPOLOGICAL ENTROPY OF INDUCED STATE SPACE HOMEOMORPHISMS

We begin by recalling that the topological entropy of a homeomorphism $T : X \to X$ of a compact metric space is defined by

$$h_{\text{top}}(T) = \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n} \log N(\mathcal{U} \vee T\mathcal{U} \vee \cdots \vee T^{n-1}\mathcal{U})$$

where the supremum is taken over all open covers $\mathcal{U}$ and $N(\cdot)$ denotes the smallest cardinality of a subcover. The entropy may also be expressed in terms of separated and spanning sets:

$$h_{\text{top}}(T) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log \text{sep}_n(T,\varepsilon) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log \text{spn}_n(T,\varepsilon)$$

where $\text{sep}_n(T,\varepsilon)$ denotes the largest cardinality of an (n,ε)-separated set and $\text{spn}_n(T,\varepsilon)$ the smallest cardinality of an (n,ε)-spanning set (see [4, 7]).

Let A be a unital C^*-algebra. We will denote by $S(A)$ its state space, which is compact under the weak* topology. Given an automorphism α of A we will denote by T_α the homeomorphism of $S(A)$ defined by $T_\alpha(\sigma) = \sigma \circ \alpha$ for all $\sigma \in S(A)$.

In [14] Sigmund showed that, given a homeomorphism of a compact metric space, the topological entropy of the induced homeomorphism on the space of probability measures is either zero or infinity. The argument given there also applies to automorphisms of unital C^*-algebras, and so we have the following.

Proposition 1.1. Let A be a separable unital C^*-algebra. For any automorphism α of A we have either $h_{\text{top}}(T_\alpha) = 0$ or $h_{\text{top}}(T_\alpha) = \infty$.

As an example, consider the full group algebra $C^*(\mathbb{F}_\infty)$ of the free group on countable many generators, and define the shift automorphism α of $C^*(\mathbb{F}_\infty)$ by setting $\alpha(u_k) = u_{k+1}$ for all $k \in \mathbb{Z}$, where $\{u_k\}_{k\in\mathbb{Z}}$ is the set of canonical unitary generators. In this case we have $h_{\text{top}}(T_\alpha) = \infty$. This is a consequence of the fact that α has as a C^*-dynamical factor the automorphism of $C(\{-1,1\}^{\mathbb{Z}})$ arising from the topological 2-shift (which has entropy $\log 2$), since topological entropy is nonincreasing under passing to subsystems. Another more geometrically explicit way of showing that $h_{\text{top}}(T_\alpha) = \infty$ is to notice that for each $n \in \mathbb{N}$ the set $\{u_1, \dots, u_n\}$ forms a standard basis for a copy of ℓ_1^n, in which case we can construct, for each $f \in \{-1,1\}^{\{1,\dots,n\}}$, a norm-one linear functional σ_f on $C^*(\mathbb{F}_\infty)$ satisfying

$$\sigma_f(u_k) = f(k)$$

for every $k = 1, \dots, n$. Then any two distinct linear functionals of the form σ_f are separated by a distance of 2 upon evaluation at at least one of the unitaries $u_1, \dots u_n$, so that the collection of such functionals for a given $n \in \mathbb{N}$ is an (n, ε)-separated set with respect to a fixed metric on the unit ball of the dual A^* for some ε not depending on n. This means that the topological entropy of the induced homeomorphism of the unit ball of A^* is at least $\log 2$, whence by a decomposition argument the topological entropy of T_α is infinite (see Section 2 of [8]).

2. Voiculescu-Brown entropy and induced state space dynamics

We begin by recalling the definition of Voiculescu-Brown entropy [18, 2], which is based on completely positive approximation. We work here with the unital definition, which can be shown to coincide with the general definition in the unital case. Let A be an exact (equivalently, nuclearly embeddable [10]) C^*-algebra and $\pi : A \to \mathcal{B}(\mathcal{H})$ a faithful representation. Nuclear embeddability guarantees, for each finite $\Omega \subset A$ and $\delta > 0$, the non-emptiness of the collection $\mathrm{CPA}(\pi, \Omega, \delta)$ of triples (ϕ, ψ, B) where B is a finite-dimensional C^*-algebra and $\phi : A \to B$ and $\psi : B \to \mathcal{B}(\mathcal{H})$ are unital completely positive maps. We define $\mathrm{rcp}(\pi, \Omega, \delta)$ to be the infimum of $\operatorname{rank} B$ over all $(\phi, \psi, B) \in \mathrm{CPA}(\pi, \Omega, \delta)$, with rank referring to the dimension of a maximal Abelian C^*-subalgebra. For an automorphism α of A we set

$$\begin{aligned} ht(\pi, \alpha, \Omega, \delta) &= \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{rcp}(\pi, \Omega \cup \alpha\Omega \cup \cdots \cup \alpha^{n-1}\Omega, \delta), \\ ht(\pi, \alpha, \Omega) &= \sup_{\delta>0} ht(\pi, \alpha, \Omega, \delta), \\ ht(\pi, \alpha) &= \sup_{\Omega} ht(\pi, \alpha, \Omega) \end{aligned}$$

with the last supremum taken over all finite sets $\Omega \subset A$. The value $ht(\pi, \alpha)$ does not depend on the particular faithful representation π, and we define the Voiculescu-Brown entropy $ht(\alpha)$ to be this common value over all such π.

Our main result [8, Thm. 4.3] is the following.

Theorem 2.1. Let A be a separable unital exact C^*-algebra and α an automorphism of A. Then $ht(\alpha) = 0$ implies $h_{\text{top}}(T_\alpha) = 0$.

In fact Theorem 4.3 in [8] also addresses the non-unital case, asserting essentially the same conclusion as above with the state space replaced by the quasi-state space. For simplicity however we have restricted our attention here to the unital case.

The proof of Theorem 2.1 as given in [8] relies on the following two lemmas. The first lemma provides the key geometric fact. I am grateful to Nicole Tomczak-Jaegermann for having communicated to me this result and its proof.

Lemma 2.2. Let $K \geq 1$, and let X be a k-dimensional subspace of the Schatten $p = \infty$ class C_∞^n such that the Banach-Mazur distance

$$d(X, \ell_1^k) = \inf\{\|\Gamma\|\|\Gamma^{-1}\| : \Gamma : X \to \ell_1^k \text{ is an isomorphism}\}$$

is no greater than K. Then

$$k \leq aK^2 \log n$$

for some universal constant $a > 0$.

To establish Lemma 2.2 we use the fact that the (Rademacher) type 2 constant of ℓ_1^k is at least $\sqrt{k}$ (see §4 in [16]), while the type 2 constant of C_∞^n is at most $C\sqrt{\log n}$ for some $C > 0$ not depending on n. The latter follows from upper bounds on the type 2 constant for the Schatten p-classes for $2 \leq p < \infty$ which can be obtained from [17], along with the equality $d(C_\infty^n, C_p^n) = n^{1/p}$ for $1 \leq p < \infty$ [16, Thm. 45.2].

The second lemma is a matrix analogue of Proposition 2.1 of [6]. We can adapt the proof from [6], but we must substitute Lemma 2.2 for the part of the argument in [6] involving almost Hilbertian sections of unit balls, which doesn't work in our case.

Here C_1^n denotes the Schatten 1-class, i.e., the space of trace class matrices.

Lemma 2.3. Given $\varepsilon > 0$ and $\lambda > 0$ there exist $n_0 \in \mathbb{N}$ and $\mu > 0$ such that, for all $n \geq n_0$, if $\phi : C_1^{r_n} \to \ell_\infty^n$ is a *-linear map of norm at most 1 such that the image of the unit ball of $C_1^{r_n}$ under ϕ contains an ε-separated set of self-adjoint elements of cardinality at least $e^{\lambda n}$, then $r_n \geq e^{\mu n}$.

A key ingredient in the proof of this lemma, as adapted from Glasner and Weiss's proof of [6, Prop. 2.1], is the combinatorial Sauer-Perles-Shelah lemma [12, 13], which gives precise information about how large a subset $A \subset \{-1, 1\}^{\{1,\ldots,n\}}$ must be in general so that its restriction to some subindex set $I_n \subset \{1, \ldots, n\}$ of a prescribed cardinality is equal to $\{-1, 1\}^{I_n}$. In our analytic situation the Sauer-Perles-Shelah lemma has the consequence that there exist a $d > 0$ and a $\delta > 0$ such that, for sufficiently large $n \in \mathbb{N}$, there is a subset $I_n \subset \{1, \ldots, n\}$ of cardinality at least dn such that the dual map $(\pi \circ \phi)^*$ from $(\ell_\infty^{I_n})^* \cong \ell_1^{I_n}$ to $(C_1^{r_n})^* \cong C_\infty^{r_n}$ is an embedding of norm at most 1 whose inverse has norm at most $2/\delta$, where $\pi : \ell_\infty^n \to \ell_\infty^{I_n}$ is the canonical projection (see the proof of Lemma 2.3 in [6]). We can then apply Lemma 2.2 to obtain Lemma 2.3.

Finally, to prove the theorem we fix a metric ρ on $S(A)$ and suppose that $h_{\text{top}}(T_\alpha) > 0$. Then there exist an $\varepsilon > 0$, a $\lambda > 0$, and an infinite set $J \subset \mathbb{N}$ such that for all $n \in J$ there is an $(n, 4\varepsilon)$-separated set $E_n \subset S(A)$ of cardinality $\geq e^{\lambda n}$. By compactness we can find a finite set $\Omega \subset K$ such that, for all $\sigma, \omega \in S(A)$,

$$\rho(\sigma, \omega) \leq \sup_{x \in \Omega} |\sigma(x) - \omega(x)| + \varepsilon.$$

To complete the proof we show that $ht(\pi, \Omega, \varepsilon) > 0$ for a given faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$. This is done by taking, for each $n \in J$, a triple $(\phi_n, \psi_n, B_n) \in \mathrm{CPA}(\pi, A, \Omega, \varepsilon)$ with the rank of B_n as small as possible and defining a map Γ_n from the Schatten 1-class $C_1^{r_n}$ to $(\ell_\infty^n)^m \cong \ell_\infty^{nm}$ by

$$\Gamma_n(h) = ((\mathrm{Tr}(h\phi_n(\alpha^k(x_i))))_{k=1}^{n-1})_{i=1}^m$$

for all $h \in C_1^{r_n}$, where $r_n = \operatorname{rank} B_n$, $x_1, \dots, x_m$ are the elements of Ω, Tr is the trace on $M_{r_n}(\mathbb{C})$ which takes value 1 on minimal projections, and $B_n \subset M_{r_n}(\mathbb{C})$ under some fixed embedding. Extending $\sigma \circ \pi^{-1}$ on $\pi(A)$ to a state on $\mathcal{B}(\mathcal{H})$ for each $\sigma \in E_n$, it is then readily checked that $\Gamma(\{\sigma' \circ \psi_n : \sigma \in E_n\})$ is an ε-separated set of self-adjoint elements with cardinality $\geq e^{\lambda n}$, so that we can apply Lemma 2.3 to finish the proof of the theorem.

Since topological entropy does not increase under taking factors or restrictions to closed invariant subsets, we obtain the following corollary to Theorem 2.1.

Corollary 2.4. Let A and B be separable exact C^*-algebras and $\alpha : A \to A$ and $\beta : B \to B$ automorphisms with $h_{\mathrm{top}}(T_\alpha) > 0$. Suppose that α can be obtained from β via a finite sequence of taking unital subsystems and C^*-dynamical factors (i.e., quotients intertwining the actions). Then $ht(\beta) > 0$.

Corollary 2.4 gives us in particular some information concerning the behaviour of Voiculescu-Brown entropy under taking extensions, about which little seems to be known in general (it is even unknown, for example, whether or not a positive entropy system can have an extension with zero entropy).

We also note that, for the shift α on reduced crossed product $C_r^*(\mathbb{F}_\infty)$ of the free group on countably many generators, we have $ht(\alpha) = 0$ by [5], so that by Theorem 2.1 the topological entropy of T_α is zero, in contrast to the case of the shift on the full group C^*-algebra $C^*(\mathbb{F}_\infty)$.

Question 2.5. Does the converse of Theorem 2.1 hold?

All we have been able to come up with concerning Question 2.5 are some examples for which we can show the Voiculescu-Brown entropy is positive without having been able to determine the topological entropy on the state space [8, Example 4.6]. The examples in question involve the collection of automorphisms α_θ of the rotation C^*-algebras A_θ associated to a fixed matrix $S \in SL(2, \mathbb{Z})$ with eigenvalues off the unit circle (see [19, 1]). In [9] it is established that the Voiculescu-Brown entropy of α_θ is positive for a residual set of rotation parameters θ. On the other hand, we have only been able to show that $h_{\mathrm{top}}(T_{\alpha_\theta}) = \infty$ for the set of rotation parameters θ for which the Connes-Narnhofer-Thirring entropy with respect to the canonical tracial state is positive, and this is a meager set, as implicitly demonstrated in [11].

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