数理解析研究所講究録 1300 巻 2003 年 88-101

A summary of the works "A class of C^{*}-algebras generalizing both graph algebras and homeomorphism C^{*}-algebras"

Takeshi Katsura (勝良 健史)
Department of Mathematical Sciences
University of Tokyo, Komaba, Tokyo, 153-8914, JAPAN
e-mail: katsu@ms.u-tokyo.ac.jp

0 Introduction

In this note, we summarize the definitions and the results in [Ka1, Ka2, Ka3] where structures of C^{*}-algebras associated with topological graphs are examined. Topological graphs generalize (discrete) graphs as well as topological dynamical systems, and our construction of C^{*}-algebras from topological graphs is a common generalization of those of graph algebras [KPRR, KPR, FLR] and homeomorphism C^{*}-algebras.

Sections 1 and 2 contain definitions of topological graphs and C^{*}-algebras associated with them. In Sections 3, we give the 6-term exact sequences of KK-groups and K-groups. Sections 4 is devoted to a summary of examples of C^{*}-algebras associated with topological graphs. For the detail, see [Ka2]. The rest of the sections contain results of [Ka3] where we generalize many notion such as minimality from dynamical systems to topological graphs.

Some of the results in this note are generalized to the non-commutative setting in [Ka5, Ka6]. We changed some notations from [Ka4].

1 Topological correspondences

In this section, we introduce a notion of topological correspondences between locally compact spaces, and construct C^{*}-correspondences from them.

Let A, B be C^{*}-algebras. A (right) Hilbert B-module X is a Banach space with a B-valued inner product $\langle\cdot,\cdot\rangle$ and a right action of B satisfying certain conditions (for the detail, see [L]). For a Hilbert B-module X, we denote by $\mathcal{L}(X)$ the set of adjointable operators on X, and by $\mathcal{K}(X)$ the ideal of $\mathcal{L}(X)$ spanned by the elements of the form $\theta_{\xi,\eta}$ for $\xi,\eta\in X$ where $\theta_{\xi,\eta}\in\mathcal{L}(X)$ is defined by $\theta_{\xi,\eta}(\zeta)=\xi\langle\eta,\zeta\rangle$ for $\zeta\in X$. By a left action of the C^{*}-algebra A on the Hilbert B-module X, we mean a $*$-homomorphism $\pi: A\to\mathcal{L}(X)$. A Hilbert B-module X together with a left action of A on X is called a *C^{*}-correspondence* from A to B. From a $*$-homomorphism

$\varphi : A \to B$, we can define a C^*-correspondence from A to B by taking $X = B$. Thus we consider C^*-correspondences as a generalization of $*$-homomorphisms.

Definition 1.1 Let E^0 and F^0 be locally compact (Hausdorff) spaces. A *topological correspondence* (E^1, d, r) from E^0 to F^0 consists of a locally compact space E^1, a local homeomorphism $d : E^1 \to E^0$, and a continuous map $r : E^1 \to F^0$.

A continuous map $\sigma : E^0 \to F^0$ gives a topological correspondence $(E^0, \mathrm{id}, \sigma)$. More generally, a set of continuous maps $\sigma_i : O_i \to F^0$ defined only on open subsets O_i of E^0 gives a topological correspondence (E^1, d, r) by setting $E^1 = \coprod_i O_i$ and defining d by the embedding and r by σ_i's. We can say that the subset $r(d^{-1}(v)) \subset F^0$ is the "image" of a point $v \in E^0$ under a topological correspondence (E^1, d, r), which can be empty or infinite. Thus topological correspondences are "multi-valued" generalizations of continuous maps.

Let us take a topological correspondence (E^1, d, r) from E^0 to F^0. Denote by $C_d(E^1)$ the set of continuous functions ξ of E^1 such that $\langle \xi, \xi \rangle (v) = \sum_{e \in d^{-1}(v)} |\xi(e)|^2 < \infty$ for any $v \in E^0$ and $\langle \xi, \xi \rangle \in C_0(E^0)$. For $\xi, \eta \in C_d(E^1)$ and $f \in C_0(E^0)$, we define $\xi f \in C_d(E^1)$ and $\langle \xi, \eta \rangle \in C_0(E^0)$ by

$$(\xi f)(e) = \xi(e) f(d(e)) \quad \text{for } e \in E^1,$$

$$\langle \xi, \eta \rangle (v) = \sum_{e \in d^{-1}(v)} \overline{\xi(e)} \eta(e) \quad \text{for } v \in E^0.$$

With these operations, $C_d(E^1)$ is a Hilbert $C_0(E^0)$-module ([Kal, Proposition 1.10]). We define a left action π_r of $C_0(F^0)$ on $C_d(E^1)$ by $(\pi_r(f)\xi)(e) = f(r(e))\xi(e)$ for $e \in E^1$, $\xi \in C_d(E^1)$ and $f \in C_0(F^0)$. Thus we get a C^*-correspondence $C_d(E^1)$ from $C_0(F^0)$ to $C_0(E^0)$. A composition of two topological correspondences can be defined naturally, and this relates to the internal tensor product of C^*-correspondences.

2 C^*-algebras arising from topological graphs

A topological dynamical system $\Sigma = (X, \sigma)$ consists of a locally compact space X and a homeomorphism σ on X. Since topological correspondences generalize continuous maps, a pair of a locally compact space E^0 and a topological correspondence (E^1, d, r) from E^0 to itself generalizes a topological dynamical system. Such pair is called a topological graph.

Definition 2.1 A *topological graph* $E = (E^0, E^1, d, r)$ consists of two locally compact spaces E^0, E^1, a local homeomorphism $d : E^1 \to E^0$, and a continuous map $r : E^1 \to E^0$.

We think that E^0 is a set of vertices and E^1 is a set of edges and that an edge $e \in E^1$ is directed from its domain $d(e) \in E^0$ to its range $r(e) \in E^0$. We denote by $E_\Sigma = (X, X, \mathrm{id}, \sigma)$ the topological graph defined by a topological dynamical system $\Sigma = (X, \sigma)$. As we saw in Section 1, a topological graph $E = (E^0, E^1, d, r)$ gives us a C^*-correspondence $C_d(E^1)$ from $C_0(E^0)$ to itself.

Definition 2.2 Let $E = (E^0, E^1, d, r)$ be a topological graph. A *Toeplitz E-pair* on a C^*-algebra A is a pair of maps $T = (T^0, T^1)$ where $T^0 : C_0(E^0) \to A$ is a $*$-homomorphism and $T^1 : C_d(E^1) \to A$ is a linear map satisfying that

(i) $T^1(\xi)^* T^1(\eta) = T^0(\langle \xi, \eta \rangle)$ for $\xi, \eta \in C_d(E^1)$,
(ii) $T^0(f) T^1(\xi) = T^1(\pi_r(f)\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$.

For $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$, the equation $T^1(\xi)T^0(f) = T^1(\xi f)$ holds automatically from the condition (i). For a Toeplitz E-pair $T = (T^0, T^1)$, we write $C^*(T)$ for denoting the C^*-algebra generated by the images of the maps T^0 and T^1. We can define a $*$-homomorphism $\Phi^1 : \mathcal{K}(C_d(E^1)) \to C^*(T)$ by $\Phi^1(\theta_{\xi,\eta}) = T^1(\xi)T^1(\eta)^*$ for $\xi, \eta \in C_d(E^1)$.

Definition 2.3 Let $E = (E^0, E^1, d, r)$ be a topological graph. We define an open subset E^0_{rg} of E^0 by

$$E^0_{\mathrm{rg}} = \{v \in E^0 \mid \text{there exists a neighborhood } V \text{ of } v \text{ such that } r^{-1}(V) \subset E^1 \text{ is compact, and } r(r^{-1}(V)) = V\},$$

and set $E^0_{\mathrm{sg}} = E^0 \setminus E^0_{\mathrm{rg}}$.

A vertex in E^0_{rg} is called *regular*, and a vertex in E^0_{sg} is called *singular*. We can show that the restriction of π_r to $C_0(E^0_{\mathrm{rg}})$ is an injection into $\mathcal{K}(C_d(E^1))$ [Ka1].

Definition 2.4 Let $E = (E^0, E^1, d, r)$ be a topological graph. A Toeplitz E-pair $T = (T^0, T^1)$ is called a *Cuntz-Krieger E-pair* if $T^0(f) = \Phi^1(\pi_r(f))$ for all $f \in C_0(E^0_{\mathrm{rg}})$. We denote by $\mathcal{O}(E)$ the universal C^*-algebra generated by a Cuntz-Krieger E-pair $t = (t^0, t^1)$.

When E is a discrete graph, $\mathcal{O}(E)$ is isomorphic to the graph algebra of the opposite graph of E. For a topological graph E_Σ defined by a topological dynamical system $\Sigma = (X, \sigma)$, the C^*-algebra $\mathcal{O}(E_\Sigma)$ is isomorphic to the homeomorphism C^*-algebra $A(\Sigma) = C_0(X) \rtimes_\sigma \mathbb{Z}$. The universal Cuntz-Krieger pair can be characterized by the following two conditions.

Definition 2.5 A Toeplitz pair $T = (T^0, T^1)$ is called *injective* when T^0 is injective, and said to *admit a gauge action* when for each complex number z with $|z| = 1$ there exists an automorphism β'_z on $C^*(T)$ such that $\beta'_z(T^0(f)) = T^0(f)$ and $\beta'_z(T^1(\xi)) = zT^1(\xi)$.

For an injective Toeplitz pair $T = (T^0, T^1)$, the linear map T^1 is isometric.

Theorem 2.6 ([Ka1, Theorem 4.5]) *A Cuntz-Krieger pair T is universal if and only if it is injective and admits a gauge action.*

We can define a C^*-algebra $\mathcal{O}(E)$ without using the open set E^0_{rg} nor a notion of Cuntz-Krieger pairs.

Proposition 2.7 ([Ka2]) *Let E be a topological graph. For an injective Toeplitz E-pair T admitting a gauge action, there exists a (unique) surjection $\rho : C^*(T) \to \mathcal{O}(E)$ such that $t^i = \rho \circ T^i$ for $i = 0, 1$.*

Thus $\mathcal{O}(E)$ can be defined as the smallest C^*-algebra generated by an injective Toeplitz E-pair admitting a gauge action. Note that the existence of such smallest C^*-algebra is a non-trivial fact. Now Theorem 2.6 can be rephrased as follows.

Proposition 2.8 *Let E be a topological graph. For an injective Toeplitz E-pair T admitting a gauge action, the surjection $\rho : C^*(T) \to \mathcal{O}(E)$ in Proposition 2.7 is an isomorphism if and only if T is a Cuntz-Krieger E-pair.*

We can construct the C^*-algebra $\mathcal{O}(E)$ concretely using the Fock space. This construction gives us an isomorphism between the C^*-algebra $\mathcal{O}(E)$ and the relative Cuntz-Pimsner algebra of $C_d(E^1)$ with respect to the ideal $C_0(E^0_{\mathrm{rg}})$ of $C_0(E^0)$ defined in [MS].

3 Nuclearity and KK-groups

For all topological graph E, the C^*-algebra $\mathcal{O}(E)$ is nuclear ([Ka1, Proposition 6.1]), and it satisfies the Universal Coefficient Theorem (UCT) of [RoSc] when it is separable ([Ka1, Proposition 6.6]). The C^*-algebra $\mathcal{O}(E)$ is separable if and only if E is second countable, which means both E^0 and E^1 are second countable. We get 6-term exact sequences of KK-groups and K-groups which help to compute those of the C^*-algebra $\mathcal{O}(E)$.

Let us denote by $\iota_* \in KK(C_0(E^0_{\mathrm{rg}}), C_0(E^0))$ the element defined by the inclusion $\iota : C_0(E^0_{\mathrm{rg}}) \hookrightarrow C_0(E^0)$, and by $[\pi_r] \in KK(C_0(E^0_{\mathrm{rg}}), C_0(E^0))$ the element defined by the triple $(C_d(E^1), \pi_r, 0)$. Note that the element $[\pi_r]$ is obtained from the map $\pi_r : C_0(E^0_{\mathrm{rg}}) \to \mathcal{K}(C_d(E^1))$, the strong Morita equivalence between $\mathcal{K}(C_d(E^1))$ and $C_0(d(E^1))$ defined by the Hilbert module $C_d(E^1)$, and the inclusion $C_0(d(E^1)) \subset C_0(E^0)$.

Proposition 3.1 **([Ka1, Proposition 6.9])** *Let E be a second countable topological graph. For any separable C^*-algebra B we have the following two exact sequences:*

$$\begin{tikzcd}
KK_0(B, C_0(E^0_{\mathrm{rg}})) \arrow[r, "\iota_* - [\pi_r]"'] & KK_0(B, C_0(E^0)) \arrow[r, "t^0_*"'] & KK_0(B, \mathcal{O}(E)) \arrow[d] \\
KK_1(B, \mathcal{O}(E)) \arrow[u] & KK_1(B, C_0(E^0)) \arrow[l, "t^0_*"'] & KK_1(B, C_0(E^0_{\mathrm{rg}})) \arrow[l, "\iota_* - [\pi_r]"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
KK_0(C_0(E^0_{\mathrm{rg}}), B) \arrow[d] & KK_0(C_0(E^0), B) \arrow[l, "\iota_* - [\pi_r]"] & KK_0(\mathcal{O}(E), B) \arrow[l, "t^0_*"] \\
KK_1(\mathcal{O}(E), B) \arrow[r, "t^0_*"] & KK_1(C_0(E^0), B) \arrow[r, "\iota_* - [\pi_r]"] & KK_1(C_0(E^0_{\mathrm{rg}}), B). \arrow[u]
\end{tikzcd}$$

Corollary 3.2 ([Ka1, Corollary 6.10]) *For a second countable topological graph E, the following sequence of K-groups is exact:*

$$\begin{array}{ccccc} K_0(C_0(E^0_{\mathrm{rg}})) & \xrightarrow[\iota_*-[\pi_r]]{} & K_0(C_0(E^0)) & \xrightarrow[t^0_*]{} & K_0(\mathcal{O}(E)) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}(E)) & \xleftarrow{t^0_*} & K_1(C_0(E^0)) & \xleftarrow{\iota_*-[\pi_r]} & K_1(C_0(E^0_{\mathrm{rg}})). \end{array}$$

4 Examples

Topological graphs are generalizations of not only (discrete) graphs and topological dynamical systems but also other notions such as partial homeomorphisms [E], singly generated dynamical systems [Re] and so on. Moreover the construction of C^*-algebras from topological graphs generalizes those of

- homeomorphism C^*-algebras,
- graph algebras [KPRR, KPR, FLR],
- crossed products by partial homeomorphisms [E],
- C^*-algebras associated with branched coverings [DM],
- C^*-algebras associated with singly generated dynamical systems [Re],
- C^*-algebras associated with infinite matrices [EL],
- C^*-algebras associated with subshifts [M].

The class of C^*-algebras arising from topological graphs contains many examples of nuclear C^*-algebras such as;

- all AF-algebras
- all simple AT-algebras with real rank zero,
- many AH-algebras including all Goodearl algebras (see [RøSt, Example 3.1.7]) and purely infinite AH-algebras constructed in [Rø],
- all simple separable nuclear purely infinite C^*-algebras satisfying UCT,
- many simple stably projectionless C^*-algebras.

The class of C^*-algebras arising from topological graphs is closed under taking

- direct sums,
- unitizations,
- tensor products with $\mathbb{M}_n$ or $\mathbb{K}$,

- tensor products with commutative C^*-algebras,
- inductive limits by certain connecting maps,
- ideals which is invariant under the gauge action,
- quotients by ideals which is invariant under the gauge action.

5 Orbits and invariant sets

In this section, we introduce a notion of orbits and invariant sets of topological graphs. These are closely related ideal structures of the C^*-algebra $\mathcal{O}(E)$ of a topological graph E. The difference of definitions of positive orbit spaces and negative orbit spaces comes from the irreversible feature of topological correspondences.

Let us fix a topological graph $E = (E^0, E^1, d, r)$. We set $d^0 = r^0 = \mathrm{id}_{E^0}$ and $d^1 = d, r^1 = r$. For $n = 2, 3, \ldots$, we define a space E^n of *paths* with length n by

$$E^n = \{(e_1, e_2, \ldots, e_n) \in E^1 \times \cdots \times E^1 \times E^1 \mid d(e_k) = r(e_{k+1}) \text{ for } k = 1, 2, \ldots, n-1\}.$$

We define domain and range maps $d^n, r^n : E^n \to E^0$ by $d^n(e) = d(e_n)$ and $r^n(e) = r(e_1)$ for $e = (e_1, e_2, \ldots, e_n) \in E^n$. Note that (E^n, d^n, r^n) is the n-times composition of the topological correspondence (E^1, d, r) on E^0. An *infinite path* $e = (e_1, e_2, \ldots, e_n, \ldots)$ means that $e_k \in E^1$ and $d(e_k) = r(e_{k+1})$ for each $k = 1, 2, \ldots$. The set of all infinite paths is denoted by E^∞. The range $r^\infty(e) \in E^0$ of an infinite path $e = (e_1, e_2, \ldots, e_n, \ldots) \in E^\infty$ is defined by $r(e_1)$.

Definition 5.1 We define the *positive orbit space* $\mathrm{Orb}^+(v)$ of $v \in E^0$ by

$$\mathrm{Orb}^+(v) = \{r^n(e) \in E^0 \mid e \in (d^n)^{-1}(v) \subset E^n, n \in \mathbb{N}\}.$$

Definition 5.2 For $n \in \mathbb{N} \cup \{\infty\}$, a path $e \in E^n$ is called a *negative orbit* of $v \in E^0$ if $r^n(e) = v$ and $d^n(e) \in E^0_{\mathrm{sg}}$ when $n < \infty$.

Note that each $v \in E^0$ has at least one negative orbit, but may have many negative orbits in general.

Definition 5.3 For a negative orbit $e = (e_1, e_2, \ldots, e_n) \in E^n$ of $v \in E^0$ with $n \in \mathbb{N}$, the *negative orbit space* $\mathrm{Orb}^-(v, e)$ is defined by

$$\mathrm{Orb}^-(v, e) = \{v, d(e_1), d(e_2), \ldots, d(e_n)\} \subset E^0.$$

Similarly, for a negative orbit $e = (e_1, e_2, \ldots, e_k, \ldots) \in E^\infty$ of $v \in E^0$, the *negative orbit space* $\mathrm{Orb}^-(v, e)$ is defined by

$$\mathrm{Orb}^-(v, e) = \{v, d(e_1), d(e_2), \ldots, d(e_k), \ldots\} \subset E^0.$$

Definition 5.4 We define the *orbit space* $\mathrm{Orb}(v,e)$ of $v \in E^0$ with respect to a negative orbit e of v by

$$\mathrm{Orb}(v,e) = \bigcup_{v' \in \mathrm{Orb}^-(v,e)} \mathrm{Orb}^+(v').$$

Remark 5.5 A negative orbit e of $v \in E^0$ determines "the past" of v, and the negative orbit space $\mathrm{Orb}^-(v,e)$ consists of the points in "the past", while the positive orbit space $\mathrm{Orb}^+(v)$ is the set of all points in "the future" of v. The orbit space $\mathrm{Orb}(v,e)$ of v with respect to the negative orbit e consists of the all points which are reached from some point in "the past". "The past" $e \in E^n$ may have an "origin" $d^n(e)$ which should be a singular point (when $n < \infty$), or may come from long, long time ago (when $n = \infty$). When "the past" e has an "origin" $v' \in E^0_{\mathrm{sg}}$, the orbit space $\mathrm{Orb}(v,e)$ coincides with the positive orbit space $\mathrm{Orb}^+(v')$ of the "origin" v'.

Definition 5.6 A subset X of E^0 is said to be *positively invariant* if $d(e) \in X$ implies $r(e) \in X$ for each $e \in E^1$, and *negatively invariant* if for $v \in X \cap E^0_{\mathrm{rg}}$, there exists $e \in E^1$ with $r(e) = v$ and $d(e) \in X$. A subset X of E^0 is said to be *invariant* if X is both positively and negatively invariant.

It is easy to see the following two lemmas.

Lemma 5.7 *For each $v \in E^0$, the positive orbit space $\mathrm{Orb}^+(v)$ is positively invariant. A subset X of E^0 is positively invariant if and only if $\mathrm{Orb}^+(v) \subset X$ for all $v \in X$.*

Lemma 5.8 *For each $v \in E^0$ and each negative orbit e of v, the negative orbit space $\mathrm{Orb}^-(v,e)$ is negatively invariant. A subset X of E^0 is negatively invariant if and only if for each $v \in X$, there exists a negative orbit e of v such that $\mathrm{Orb}^-(v,e) \subset X$.*

From these lemmas, we get the following.

Proposition 5.9 *For each $v \in E^0$ and each negative orbit e of v, the orbit space $\mathrm{Orb}(v,e)$ is invariant. A subset X of E^0 is invariant if and only if for each $v \in X$, there exists a negative orbit e of v such that $\mathrm{Orb}(v,e) \subset X$.*

We are interested in closed invariant subsets.

Lemma 5.10 *If a subset X of E^0 is positively invariant or negatively invariant, then so is the closure $\overline{X}$. Hence $\overline{X}$ is invariant for an invariant set $X \subset E^0$.*

By this lemma, $\overline{\mathrm{Orb}(v,e)}$ is a closed invariant set for a negative orbit e of $v \in E^0$. Let X^0 be a closed subset of E^0, and define $X^1 = d^{-1}(X^0) \subset E^1$. If X^0 is positively invariant, then we have $r(X^1) \subset X^0$ and so $X = (X^0, X^1, d, r)$ is a topological graph. We can state a condition for a closed positively invariant set X^0 to be negatively invariant (hence invariant) using the topological graph X.

Proposition 5.11 *A closed positively invariant subset X^0 of E^0 is invariant if and only if $X^0_{\mathrm{sg}} \subset E^0_{\mathrm{sg}}$.*

By this proposition, for a closed invariant subset X^0 of E^0, we have the inclusion $X^0_{\mathrm{sg}} \subset E^0_{\mathrm{sg}} \cap X^0$. In general, this inclusion is not equal. We will see this difference in the study of ideals of $\mathcal{O}(E)$ (see Definition 7.1). We finish this section by studying complements of invariant subsets.

Definition 5.12 A subset V of E^0 is said to be *hereditary* if V satisfies $d(r^{-1}(V)) \subset V$, and said to be *saturated* if we have $v \in V$ for $v \in E^0_{\mathrm{rg}}$ satisfying $d(r^{-1}(v)) \subset V$.

Proposition 5.13 *A set X is positively invariant if and only if the complement V of X is hereditary, and negatively invariant if and only if V is saturated.*

Definition 5.14 For a subset V of E^0, we define $H(V), S(V) \subset E^0$ by

$$H(V) = \bigcup_{n=0}^{\infty} d^n\big((r^n)^{-1}(V)\big).$$

and by $S(V) = \bigcup_{n=0}^{\infty} V_n$ where $V_0 = V$ and for $n = 1, 2, \ldots$, V_n is defined inductively by

$$V_n = V_{n-1} \cup \{v \in E^0_{\mathrm{rg}} \mid d(r^{-1}(v)) \subset V_{n-1}\}.$$

Proposition 5.15 *For a subset V of E^0, $H(V)$ is the smallest hereditary subset containing V and $S(V)$ is the smallest saturated subset containing V.*

It is not difficult to see that if a subset V is hereditary then so is $S(V)$. Hence we have the following.

Proposition 5.16 *For a subset V of E^0, $S(H(V))$ is the smallest hereditary and saturated subset containing V.*

By noting that if V is open then so is both $H(V)$ and $S(V)$, we get the following.

Proposition 5.17 *For an open subset V of E^0, the open set $S(H(V))$ is the smallest open set which contains V and whose complement is a closed invariant subset.*

6 The space of negative orbits, and the one-sided Markov shift

We denote by E^0_∞ the set of all negative orbits, and by E^1_∞ the subset of E^0_∞ consisting of the negative orbits whose length is grater than or equal to 1. We define topologies on E^0_∞ and E^1_∞ as follows.

Let $\widetilde{E}^1 = E^1 \cup \{\infty\}$ be the one-point compactification of E^1. We consider a negative orbit $e \in E^n$ with $n \geq 1$ as an element of the infinite direct product $\widetilde{E}^1 \times \widetilde{E}^1 \times \cdots$ of the compact space $\widetilde{E}^1$ by

$$E^n \ni (e_1, \ldots, e_n) \mapsto (e_1, \ldots, e_n, \infty, \infty, \ldots) \in \widetilde{E}^1 \times \widetilde{E}^1 \times \cdots \qquad \text{when } n < \infty,$$
$$E^\infty \ni (e_1, \ldots, e_k, \ldots) \mapsto (e_1, \ldots, e_k, \ldots) \in \widetilde{E}^1 \times \widetilde{E}^1 \times \cdots \qquad \text{when } n = \infty.$$

Thus we can consider E^1_∞ as a subset of the compact set $\widetilde{E}^1 \times \widetilde{E}^1 \times \cdots$, and we define the relative topology on E^1_∞.

The set E^0_∞ is a disjoint union of E^0_{sg} and E^1_∞. We consider E^0_∞ as a subset of $E^0 \times \widetilde{E}^1 \times \widetilde{E}^1 \times \cdots$ by the embeddings

$$E^0_{\text{sg}} \ni v \mapsto (v, \infty, \infty, \ldots) \in E^0 \times \widetilde{E}^1 \times \widetilde{E}^1 \times \cdots,$$
$$E^1_\infty \ni (e_1, e_2, \ldots) \mapsto (r(e_1), e_1, e_2, \ldots) \in E^0 \times \widetilde{E}^1 \times \widetilde{E}^1 \times \cdots,$$

and define the relative topology on E^0_∞. We denote by r_∞ the embedding $E^1_\infty \to E^0_\infty$. Then we have the following.

Proposition 6.1 *The topological spaces E^0_∞ and E^1_∞ are locally compact, and the map $r_\infty : E^1_\infty \to E^0_\infty$ is a homeomorphism onto an open subset of E^0_∞.*

We define a map $d_\infty : E^1_\infty \to E^0_\infty$ by $d_\infty(e_1, e_2, \ldots) = (d(e_1), e_2, e_3, \ldots)$. Then d_∞ is a local homeomorphism, and so we get a topological graph $E_\infty = (E^0_\infty, E^1_\infty, d_\infty, r_\infty)$. In the case that a topological graph E has finitely many vertices and edges, and has no sinks or sources (which means that d and r are surjective), the topological graph E_∞ is nothing but the one-sided Markov shift considered in [CK].

We define two maps $m^0 : E^0_\infty \to E^0$ and $m^1 : E^1_\infty \to E^1$ by

$$m^0(v, e_1, e_2, \ldots) = v, \quad m^1(e_1, e_2, \ldots) = e_1.$$

Then both m^0 and m^1 are surjective proper continuous maps and we have $m^0 \circ d_\infty = d \circ m^1$ and $m^0 \circ r_\infty = r \circ m^1$. The pair $m = (m^0, m^1)$ satisfying these conditions (and one more condition) is called a *factor map* from E_∞ to E [Ka2]. Let us define a $*$-homomorphism $\mu^0 : C_0(E^0) \ni f \mapsto f \circ m^0 \in C_0(E^0_\infty)$ and a linear map $\mu^1 : C_d(E^1) \ni \xi \mapsto \xi \circ m^1 \in C_{d_\infty}(E^1_\infty)$. Since the factor map $m = (m^0, m^1)$ satisfies the condition called regularity, we get a $*$-homomorphism $\mu : \mathcal{O}(E) \to \mathcal{O}(E_\infty)$ such that $\mu \circ t^i = t^i_\infty \circ \mu^i$ for $i = 0, 1$ where $t = (t^0, t^1)$ is the universal Cuntz-Krieger E-pair on $\mathcal{O}(E)$ and $t_\infty = (t^0_\infty, t^1_\infty)$ is the universal Cuntz-Krieger E_∞-pair on $\mathcal{O}(E_\infty)$. The following is one of the main theorems of [Ka7].

Theorem 6.2 *The $*$-homomorphism $\mu : \mathcal{O}(E) \to \mathcal{O}(E_\infty)$ is an isomorphism.*

By this theorem, the C^*-algebra $\mathcal{O}(E)$ is shown to be related to the dynamical system $E_\infty = (E^0_\infty, E^1_\infty, d_\infty, r_\infty)$ which can be considered as a generalization of one-sided Markov shifts. Recall that this observation was important in the work of [CK]. We also see from Theorem 6.2 that the C^*-algebra $\mathcal{O}(E)$ is obtained from a topological groupoid whose unit space is E^0_∞.

7 Gauge invariant ideals

The set of all gauge invariant ideals is parameterized by pairs of two closed subsets of E^0 called admissible pairs.

Definition 7.1 A pair $\rho = (X^0, Z)$ of closed subsets of E^0 satisfying the following two conditions is called an *admissible pair*;

(i) X^0 is invariant,

(ii) $X^0_{\mathrm{sg}} \subset Z \subset E^0_{\mathrm{sg}} \cap X^0$.

Define a C^*-subalgebra $\mathcal{F}^1 \subset \mathcal{O}(E)$ and a $*$-homomorphism $\pi_0^1 : \mathcal{F}^1 \to C_0(E^0_{\mathrm{sg}})$ by

$$\mathcal{F}^1 = \{t^0(f) + \varphi^1(x) \mid f \in C_0(E^0), x \in \mathcal{K}(C_d(E^1))\},$$

and $\pi_0^1(t^0(f) + \varphi^1(x)) = f|_{E^0_{\mathrm{sg}}}$. For an ideal I of $\mathcal{O}(E)$, we define closed subsets X^0_I and Z_I of E^0 by

$$X^0_I = \{v \in E^0 \mid f(v) = 0 \text{ for all } f \in C_0(E^0) \text{ with } t^0(f) \in I\},$$
$$Z_I = \{v \in E^0_{\mathrm{sg}} \mid f(v) = 0 \text{ for all } f \in \pi_0^1(I \cap \mathcal{F}^1)\}.$$

Proposition 7.2 *For an ideal I of $\mathcal{O}(E)$, the pair $\rho_I = (X^0_I, Z_I)$ is an admissible pair.*

Definition 7.3 For an admissible pair $\rho = (X^0, Z)$, we define a topological graph $E_\rho = (E^0_\rho, E^1_\rho, d_\rho, r_\rho)$ as follows. Set $Y_\rho = Z \setminus X^0_{\mathrm{sg}}$, $\partial Y_\rho = \overline{Y_\rho} \setminus Y_\rho$, and define

$$E^0_\rho = X^0 \amalg_{\partial Y_\rho} \overline{Y_\rho}\,, \qquad E^1_\rho = X^1 \amalg_{d^{-1}(\partial Y_\rho)} d^{-1}(\overline{Y_\rho}).$$

The domain map $d_\rho : E^1_\rho \to E^0_\rho$ is defined from $d : X^1 \to X^0$ and $d : d^{-1}(\overline{Y_\rho}) \to \overline{Y_\rho}$. The range map $r_\rho : E^1_\rho \to E^0_\rho$ is defined from $r : X^1 \to X^0$ and $r : d^{-1}(\overline{Y_\rho}) \to X^0$.

Note that for an admissible pair $\rho = (X^0, Z)$ with $Z = X^0_{\mathrm{sg}}$, we have $E_\rho = X$. By using Theorem 2.6, we can show the following.

Proposition 7.4 *For a gauge-invariant ideal I of $\mathcal{O}(E)$, there exists a natural isomorphism $\mathcal{O}(E)/I \cong \mathcal{O}(E_{\rho_I})$.*

From this proposition and some computation, we get the next theorem.

Theorem 7.5 *The map $I \mapsto \rho_I$ gives us an inclusion reversing one-to-one correspondence between the set of all gauge-invariant ideals and the set of all admissible pairs.*

This theorem is a continuous counterpart of [BHRS, Theorem 3.6].

8 Freeness and topological freeness

A path $e \in E^n$ with $n \geq 1$ is called a *loop* if $r^n(e) = d^n(e)$. The vertex $r^n(e) = d^n(e)$ is called the *base point* of the loop e. A loop $e = (e_1, \ldots, e_n)$ is said to be *simple* if $r(e_i) \neq r(e_j)$ for $i \neq j$, and *without entrances* if $r^{-1}(r(e_i)) = \{e_i\}$ for $i = 1, \ldots, n$.

Definition 8.1 A topological graph E is said to be *topologically free* if the set of base points of loops without entrances has an empty interior.

This generalizes topological freeness of ordinary dynamical systems and Condition L of graph algebras (see, for example, [T] and [KPR]).

Theorem 8.2 ([Ka1, Theorem 5.12]) *If a topological graph $E = (E^0, E^1, d, r)$ is topologically free, then the natural surjection $\mathcal{O}(E) \to C^*(T)$ is an isomorphism for all injective Cuntz-Krieger E-pair $T = (T^0, T^1)$.*

The necessity of topological freeness in Theorem 8.2 is proved in [Ka3]. By Theorem 8.2, we have the following (cf. Proposition 7.4).

Proposition 8.3 *Let I be an ideal of $\mathcal{O}(E)$. If the topological graph E_{ρ_I} is topologically free, then I is gauge-invariant.*

Definition 8.4 For a positive integer n, we denote by $\operatorname{Per}_n(E)$ the set of vertices v satisfying the following three conditions;

(i) there exists a simple loop $(e_1, \ldots, e_n) \in E^n$ whose base point is v,

(ii) for each $i = 1, 2, \ldots, n$, there exist no $e \in E^1$ satisfying $r(e) = r(e_i)$ and $d(e) \in \operatorname{Orb}^+(v)$ other than e_i,

(iii) v is isolated in $\operatorname{Orb}^+(v)$.

We set $\operatorname{Per}(E) = \bigcup_{n=1}^{\infty} \operatorname{Per}_n(E)$ and $\operatorname{Aper}(E) = E^0 \setminus \operatorname{Per}(E)$.

An element in $\operatorname{Per}(E)$ is called a *periodic point* while an element in $\operatorname{Aper}(E)$ is called an *aperiodic point*. The conditions (i) and (ii) above mean that $v \in E^0$ is a base point of exactly one simple loop, and the condition (iii) says that there exist no "approximate loops" whose "base points" are v.

Definition 8.5 A topological graph E is said to be *free* if $\operatorname{Aper}(E) = E^0$.

This is a generalization of freeness of ordinary dynamical systems and Condition K of graph algebras (see, for example, [KPRR]).

Proposition 8.6 *A topological graph E is free if and only if E_ρ is topologically free for every admissible pair ρ.*

In particular, free topological graphs are topologically free. From Theorem 7.5, Proposition 8.3 and Proposition 8.6, we have the following.

Theorem 8.7 *If a topological graph E is free, then every ideal is gauge-invariant. Hence the set of all ideals corresponds bijectively to the set of all admissible pairs by the map $I \mapsto \rho_I$.*

9 Minimality and topological transitivity

In [Ka3], we generalize minimality and topological transitivity from topological dynamical systems to topological graphs.

Definition 9.1 A topological graph E is said to be *minimal* if there exists no closed invariant sets other than $\emptyset$ or E^0.

The following characterization of minimality is naturally expected.

Proposition 9.2 *For a topological graph E, the following conditions are equivalent.*

(i) *E is minimal.*

(ii) *An orbit space* $\mathrm{Orb}(v, e)$ *is dense in E^0 for all $v \in E^0$ and all negative orbit e of v.*

(iii) *For every non-empty open set $V \subset E^0$, we have $S(H(V)) = E^0$.*

The condition (ii) in Proposition 9.2 is related to cofinality of (discrete) graphs [KPRR]. By Theorem 7.5, E is minimal if and only if $\mathcal{O}(E)$ has no non-trivial gauge invariant ideals. We can prove the following.

Theorem 9.3 *For a topological graph E, the following conditions are equivalent.*

(i) *The C^*-algebra $\mathcal{O}(E)$ is simple.*

(ii) *E is minimal and topologically free.*

(iii) *E is minimal and free.*

For topological dynamical systems $\Sigma = (X, \sigma)$, minimality implies topological freeness when X is infinite. This is not the case for topological graphs (or even discrete graphs).

Definition 9.4 A topological graph E is called *topologically transitive* if we have $H(V_1) \cap H(V_2) \neq \emptyset$ for two non-empty open sets $V_1, V_2 \subset E^0$.

Proposition 9.5 *If there exist $v \in E^0$ and a negative orbit e of v such that the orbit space* $\mathrm{Orb}(v, e)$ *is dense in E^0, then E is topologically transitive.*

The converse of Proposition 9.5 is true when E^0 is second countable, but in general it is false even for topological dynamical systems.

Proposition 9.6 *For a topological graph E, the following are equivalent.*

(i) *E is topologically transitive.*

(ii) *For two non-empty open sets $V_1, V_2 \subset E^0$, we have $S(H(V_1)) \cap S(H(V_2)) \neq \emptyset$.*

(iii) *If two closed invariant subsets X_1^0, X_2^0 satisfies $X_1^0 \cup X_2^0 = E^0$, then either $X_1^0 = E^0$ or $X_2^0 = E^0$ holds.*

Theorem 9.7 *A C^*-algebra $\mathcal{O}(E)$ is primitive if and only if E is topologically free and topologically transitive.*

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