数理解析研究所講究録 1303 巻 2003 年 12-19

Products of k-spaces, and questions

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As is well-known, every product of a locally compact space with a k-space is a k-space, but not every product of a metric space with a k-space is a k-space. We consider characterizations or conditions for (finite) products of k-spaces to be k-spaces, and pose related questions. For other topics on the products of k-spaces, see [T3], [T4], for example.

We assume that spaces are regular T_1, and maps are continous and onto.

1 Definitions and Preliminaries

Let X be a space, and let $\mathcal{P}$ be a (not necessarily open or closed) cover of X. Then X *is determined by* a cover $\mathcal{P}$, [1] if $U \subset X$ is open in X if and only if $U \cap P$ is relatively open in P for every $P \in \mathcal{P}$. Here, we can replace "open" by "closed". Every space is determined by its open (or hereditarily closure-preserving closed) cover.

Let us recall that a space is a *k-space* (resp. *sequential space*) it it is determined by a cover of compact (resp. compact metric) subsets. Sequential space are k-spaces, and the converse hols if points are G_δ-sets. A space X is called a *k_ω-space* [M3] (resp. *s_ω-space*) if X is determined by a countable cover of compact (resp. compact metric) subsets.

A space X is called a *bi-k-space* (resp. *bi-quasi-k-space*) [M3] if, whenever a filter base $\mathcal{F}$ accumulates at $x \in X$, then there exists a k-sequence (resp. q-sequence) $\{A_n : n \in N\}$ such that $x \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathcal{F}$. When the filter base $\mathcal{F}$ is a decreasing sequence, then such a space X is a *countably bi-k-space* (resp. *countably bi-quasi-k-space*) [M3]. Here, a *k-sequence* (resp. *q-sequence*) is a decreasing sequence $\{A_n : n \in N\}$ such that $A = \bigcap\{A_n : n \in N\}$ is compact (resp. countably compact), and any open set $U \supset A$ contains some A_n ([M3]).

Let us recall that a space X is of *pointwise countable type* (resp. *q-space*) if each point has nbds $\{V_n : n \in N\}$ which is a k-sequence (resp. q-

[1] Following [GMT], we shall use "X is determined by $\mathcal{P}$" instead of the usual "X has the ***weak topology*** with respect to $\mathcal{P}$".

sequence). Also, a space is an *M-space* if and only if it is the inverse image of a metric space under a quasi-perfect map. The following diagrams hold.

(a) Locally compact spaces, or first countable spaces $\rightarrow$ spaces of pointwise countable type $\rightarrow$ bi-k-spaces $\rightarrow$ countably bi-k-spaces $\rightarrow$ k-spaces.
(b) Locally countably compact spaces, or M-spaces $\rightarrow$ q-spaces $\rightarrow$ bi-quasi-k-spaces $\rightarrow$ countably bi-quasi-k-spaces.

A space X is called a *Tanaka space* [My2], if X satisfies the following condition (C) in [T2].
(C) Let $\{A_n : n \in N\}$ be a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in N$. Then there exist $x_n \in A_n$ such that $\{x_n : n \in N\}$ converges to some point $y \in X$. If $y = x$, then such a space X is called *countably bi-sequential* [M3] (= *strongly Fréchet* [S]).
Sequentially compact spaces, or sequential countably bi-quasi-k-spaces are Tanaka spaces. But, every Tanaka space (actually, sequentially compact space) need not be sequential, not even a k-space[2].
A space X is *strongly sequential* [M1] if, whenever $\{A_n : n \in N\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in N$, then the point x belongs to the (idempotent) *sequential closure* of A, where A is the set of all limit points of convergent sequences $\{x_n : n \in N\}$ with $x_n \in A_n$. Namely, a space X is strongly sequential if and only if it is a sequential space such that if $\{A_n : n \in N\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in N$, then the point x belongs to the (usual) closure of the above set A. Strongly Fréchet spaces are strongly sequential. Every strongly sequential space is precisely a sequential Tanaka space ([My2]).

A map $f : X \rightarrow Y$ is called *bi-quotient* [M2] if, whenever $y \in Y$ and $\mathcal{U}$ is a cover of $f^{-1}(y)$ by open subsets of X, then finitely many $f(U)$, with $U \in \mathcal{U}$, cover some nbd of y in Y. If $\mathcal{U}$ is countable, then such a map f is called *countably bi-quotient* [S]. Open maps, or perfect maps are bi-quotient. Every product of bi-quotient maps is bi-quotient, hence quotient ([M2]). A map $f : X \rightarrow Y$ is called a *compact* (resp. *s-map*) if every $f^{-1}(y)$ is compact (resp. separable).

[2]This is pointed out by Z. Dolecki or P. Nyikos.

In the following characterizations, (1) is well-known, (2) is routinely shown, and (3) is due to [M3].

Characterization: (1) X is a k-space (resp. sequential space) $\Leftrightarrow$ X is the quotient image of a locally compact (resp. locally compact, metric) space.

(2) (a) X is a *k_ω-space* (resp. *s_ω-space*) $\Leftrightarrow$ X is the quotient image of a locally compact Lindelöf (resp. locally compact, separable metric) space.

(b) X is a space determined by a point-finite cover of compact (resp. compact metric) subsets $\Leftrightarrow$ X is the quotient compact image of a locally compact paracompact (resp. locally compact metric) space. Here, we can replace "point-finite cover" by "point-countable cover", but change "quotient compact image" to "quotient s-image".

(3) (a) X is a bi-k-space (resp. bi-quasi-k-space) $\Leftrightarrow$ X is the bi-quotient image of a paracompact M-space (resp. M-space).

(b) X is a countably bi-k-space (resp. countably bi-quasi-k-space) $\Leftrightarrow$ X is the countably bi-quotient image of a paracompact M-space (resp. M-space).

In the following results, (1) is well-known (see [M1], for example). (2) (resp. (3)) is due to [M3] (resp. [M2]). (4) holds in view of [My1] and [M2], here note that every product of a first countable space with a strongly sequential space is strongly sequential ([M1]). (5) is due to [T1].

Result: (1) Every product of a locally compact space (resp. locally countably compact, sequential space) with a k-space (resp. sequential space) is a k-space (resp. sequential space).

(2) Every product of bi-k-spaces is a bi-k-space, hence a k-space.

(3) Every product of k_ω-spaces is a k_ω-space, hence a k-space.

(4)[3] Every product of a first countable space with a sequential Tanaka space is a sequential space.

(5) For sequential spaces X and Y, $X \times Y$ is sequential if and only if

[3]This is an afirmative answer to the author's question (when he prepared [T2]). F. Mynard obtained this result by use of categorical method ([My1] & [My2]). The result is also proved by use of *multisequences* method ([D]), or directly shown without these methods ([L]).

it is a k-space.

2. Questions and Comments

Question 1. ([T5]) Every product of sequentially compact (or countably compact) k-spaces X and Y is a k-space ?

Comment: (1.1) Question 1 is affirmative if X or Y is sequential ([T1]). But, not every product of a countably compact first countable space with a k-space is a k-space.

(1.2) Every product of a k-and-q-space with a bi-k-space (or sequential q-space) is a k-space by (2.2) below. If Question 1 is affirmative, then every product of k-and-q-spaces is a k-space.

(1.3) Let X be sequentially compact (countably compact; q), and let Y be sequentially compact (resp. countably compact k; q-and-k), then $X \times Y$ is sequentially compact (resp. countably compact; q). Note that every sequentially compact space need not be a k-space.

Question 2. Let X be a k-space which is bi-quasi-k. Let Y be a sequential space. Then the following are equivalent ?

(a) $X \times Y$ is a k-space.

(b) X is locally countably compact, or Y is a Tanaka space ?

Comment: (2.1) Question 2 is affirmative if X is a bi-k-space by (2.2) & (2.4) below.

(2.2) In Question 2, (b) $\Rightarrow$ (a) holds. In general, the following case (c_1) or (c_2) implies that $X \times Y$ is a k-space ([T5]).

(c_1) X is a k-space which is bi-quasi-k, and Y is a sequential Tanaka space (in particular, a sequential countably bi-quasi-k-space).

(c_2) X is a bi-k-space, and Y is a k-space which is countably bi-quasi-k.

(2.3) Every product of sequential countably bi-k-spaces (actually, countably bi-sequential, countable spaces) need not be a k-space (not a Tanaka space) under ($2^{\aleph_0} < 2^{\aleph_1}$) ([O]).

(2.4) In Question 2, (a) $\Rightarrow$ (b) holds if X is a first countable space ([T2]), more generally, a bi-k-space ([TS], etc.).

(2.5) Every product of sequential Tanaka spaces (actually, countably bi-sequential, countable spaces) need not be a Tanaka space (hence, not strongly sequential). (Also, cf. (2.3)). But, every product $X \times Y$ of

Tanaka spaces is a Tanaka space if X is bi-quasi-k. Thus, for sequential spaces X and Y, (c_1) or (c_2) in (2.2) implies that $X \times Y$ is a Tanaka space which is sequential by means of (2.2) and *Result* (5) . In view of this and (2.3), the author has following question: For sequential spaces X and Y, if $X \times Y$ is a Tanaka space, then $X \times Y$ is sequential ?

Let $S = \{\infty\} \cup \{p_n : n \in N\} \cup \{p_{nm} : n, m \in N\}$ be an infinite countable space such that each p_{nm} is isolated in S, $K = \{p_n : n \in N\}$ converges to $\infty \notin K$, and each $L_n = \{p_{nm} : m \in N\}$ converges to $p_n \notin L_n$. We recall the following canonical spaces; the *Arens' space* S_2, and the *sequential fan* S_ω. S_2 is not Fréchet, but S_ω is Fréchet.

$S_2 = S$, but $\bigcup\{F_n : n \in N\}$ is closed in S for every finite $F_n \subset L_n$ $(n \in N)$.

$S_\omega = S_2/(K \cup \{\infty\})$ (i.e., the space obtained from the topological sum of countably many convergent sequences by identifying all the limit points).

Question 3. ([TS]) Let X be a bi-k-space, and let Y be a sequential space. Then the following are equivalent ?

(a) $X \times Y$ is a k-space.

(b) X is locally countably compact, or Y contains no (closed) copy of S_ω, and no (closed) copy of S_2 ?

Let us recall that a cover $\mathcal{P}$ of a space X is a *k-network* for X if, for any compact subset K, and any open set V with $K \subset V$, $K \subset \cup \mathcal{F} \subset V$ for some finite $\mathcal{F} \subset \mathcal{P}$. If K is a single point, then such a cover $\mathcal{P}$ is called a *network*. Bases are k-networks, and k-networks are networks. Quotient s-images (or closed images) of metric spaces have point-countable k-networks. Paracompact M-spaces with point-countable k-networks are metrizable ([GMT]).

Comment: (3.1) In Question 3, (a) $\Rightarrow$ (b) holds ([TS]).

(3.2) Question 3 is reduced to the following question in view of (2.1): For a sequential space X, X is a Tanaka space if and only if it contains no (closed) copy of S_ω, and no S_2 ? (The "only if" part holds).

(3.3) Question 3 is affirmative if the sequential space Y is one of the following spaces ([TS]).

(A_1) Fréchet space.
(A_2) Space in which every point is a G_δ-set.
(A_3) Hereditarily normal space.
(A_4) Space having a point-countable k-network.
(A_5) Closed image of a countably bi-k-space.
(A_6) Closed image of an M-space.
(3.4) The author does not know whether Question 3 is affirmative when the sequential space Y is the quotient s-image of a paracompact (countably) bi-k-space ([TS]). Question 3 is affirmative if the domain is metric by (A_4).

Question 4. ([T6]) For a k-space X, X is locally countably compact if and only if $X \times Y$ is a k-space for every quotient compact image Y of a *locally compact* metric space ?

Let us recall that a space X is called *symmetric* if there exists a real valued, non-negative function d defined on $X \times X$ such that (a) $d(x,y) = 0$ iff $x = y$, (b) $d(x,y) = d(y,x)$, and (c) $F \subset X$ is closed in X iff $d(x,F) > 0$ for any $x \in X - F$. If we replace (c) by "$d(x,F) = 0$ iff $x \in \overline{F}$", then such a space X is called *semi-metric*. Semi-metric spaces, or quotient compact images of metric spaces (e.g., the space S_2) are symmetric. Symmetric spaces are sequential. Symmetric M-spaces are metrizable ([N]).

Comment: (4.1) In Question 4, the "only if" part holds.
(4.2) Question 4 is affirmative if X is one of the following spaces. For (B_1), see (5.2) below. For (B_4), we can replace "k-space" by "symmetric space" in Question 4.
(B_1) Bi-k-space.
(B_2) Space having character $\leq 2^\omega$ (in particular, locally separable space).
(B_3) Space having a point-countable k-network.
(B_4) Symmetric space.
(4.3) Question 4 is affirmative if we omit the locally compactness of the metric domain. Question 4 is also affirmative if we replace "metric space" by "Fréchet space"; or "quotient compact image" by "closed image".
(4.4) A k-space X is locally compact if and only if $X \times Y$ is a k-space for every quotient compact image Y of a locally compact, paracompact

space. Here, we can replace "quotient compact image" by "closed image".

Question 5. ([T6]) For a k-space X, X is a locally k_ω-space if and only if $X \times Y$ is a k-space for every k_ω-space Y ?

Comment: (5.1) In Question 5, the "only if" part holds by *Result* (3).

(5.2) If we replace "k_ω-space" Y by "s_ω-space" Y, then Question 5 is negative under (MA + $\neg$ CH).

(5.3) A bi-k-space X is locally compact (resp. locally countably compact) if and only if $X \times Y$ is a k-space for every k_ω-space (resp. s_ω-space) Y. Here, the space Y can be chosen to be the quotient compact (or closed) image of a locally compact Lindelöf (resp. locally compact separable metric) space.

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