数理解析研究所講究録 1303 巻 2003 年 20-23

Topologies of Hyperspaces
(巾空間のトポロジー)

筑波大学・数理物質科学研究科 数学専攻
矢口雅人 (Masato Yaguchi)
Graduate School of Pure and Applied Sciences,
University of Tsukuba

1 Introduction

In this note, we survey recent results on hypespaces with the Wijsman topology and the Attouch-Wets topology.

For a metric space $X=(X,d)$, let $\mathrm{Cld}(X)$ be the hyperspace of non-empty closed sets. By $\mathrm{Fin}(X)$, $\mathrm{Comp}(X)$ and $\mathrm{Bdd}(X)$, we denote the subspaces of $\mathrm{Cld}(X)$ consisting of finite sets, compact sets and bounded closed sets, respectively. Let $C(X)$ be the set of all continuous real-valued functions on X. By identifying each $A\in\mathrm{Cld}(X)$ with the map

$$X\ni x\mapsto d(x,A)=\inf_{a\in A}d(x,a)\in\mathbb{R},$$

we can regard $\mathrm{Cld}(X)\subset C(X)$, whence $\mathrm{Cld}(X)$ has various topologies inherited from $C(X)$. The *Hausdorff metric topology* on $\mathrm{Cld}(X)$ is the topology of uniform convergence, the *Atouch-Wets topology* on $\mathrm{Cld}(X)$ is the topology of uniform convergence on bounded sets, and the *Wijsman topology* on $\mathrm{Cld}(X)$ is the topology of point-wise convergence, which depend on the metric d for X.

It should be remarked that the Attouch-Wets topology and the Wijsman topology are equal to the *Fell topology* on $\mathrm{Cld}(X)$ if X is a finite-dimensional normed linear space (cf. [2, p.142 & p.144]).

2 The Wijsman Topology

When we consider hyperspaces with the Wijsman topology, we denote $\mathrm{Cld}_W(X)$, $\mathrm{Fin}_W(X)$, $\mathrm{Bdd}_W(X)$, etc. It is well-known that $\mathrm{Cld}_W(X)$ is metrizable if and only if X is separable, whence we can define an admissible metric d_W for $\mathrm{Cld}_W(X)$ by using a countable dense set $\{x_i \mid\in i \in \mathbb{N}\}$ in X as follows:

$$d_W(A,B) = \sup_{i\in\mathbb{N}} \min\{2^{-i},\ |d(x_i,A) - d(x_i,B)|\}.$$

In [4], the following theorem is proved:

Theorem 2.1. *If X is an infinite-dimensional separable Banach space, then $\mathrm{Cld}_W(X)$ is homeomorphic to ($\approx$) the separable Hilbert space ℓ_2.*

Also, for $\mathrm{Fin}_W(X)$ and $\mathrm{Bdd}_W(X)$, similar results are proved in [4]:

Theorem 2.2. *If X is an infinite-dimensional separable Banach space, then*

$$\mathrm{Fin}_W(X) \approx \mathrm{Bdd}_W(X) \approx \ell_2 \times \ell_2^f,$$

where $\ell_2^f = \{(x_i)_{i\in\mathbb{N}} \in \ell_2 \mid x_i = 0$ except for finitely many $i \in \mathbb{N}\}$.

To prove Theorems 2.1 and 2.2, we need characterizations of ℓ_2 and $\ell_2 \times \ell_2^f$. The following characterization of ℓ_2 is due to Toruńczyk [7] (cf. [8]):

Theorem 2.3. *In order that $X \approx \ell_2$, it is necessary and sufficient that X is a separable completely metrizable AR which has the discrete approximation property, that is,*

> *Given a map $f : \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n \to X$, there exist maps $g : \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n \to X$ arbitrarily close to f such that $\{g(\mathbf{I}^n) \mid n \in \mathbb{N}\}$ is discrete in X.*

□

To state the characterization of $\ell_2 \times \ell_2^f$ due to Bestvina and Mogilski [3], we need some notions. A metrizable space X is *σ-completely metrizable* if X is a countable union of completely metrizable closed subsets. A closed set $A \subset X$ is a (*strong*) *Z-set* in X if there are maps $f : X \to X \setminus A$ arbitrarily close to id (such that $A \cap \mathrm{cl}\, f(X) = \emptyset$). A countable union of (strong) Z-sets is called a (*strong*) *Z_σ-set.* When X itself is a (strong) Z_σ-set in X, we call X a (*strong*) *Z_σ-space.* For a class $\mathcal{C}$ of spaces, X is *strongly universal* for $\mathcal{C}$ if the following condition is satisfied:

Given a map $f : A \to X$ of $A \in \mathcal{C}$ such that $f|B$ is a Z-embedding of a closed set $B \subset A$, there exist Z-embeddings $g : A \to X$ arbitrarily close to f such that $g|B = f|B$.

In these definitions, the phrase '*arbitrarily close*' is understood with respect to the *limitation topology*. In case $X = (X, d)$ is a metric space, given a collection $\mathcal{M}$ of maps from a space Y to X, a map $f : Y \to X$ is *arbitrarily close* to maps in $\mathcal{M}$ if for each $\alpha : X \to (0,1)$ there is $g \in \mathcal{M}$ such that $d(f(y), g(y)) < \alpha(f(y))$ for every $y \in Y$. The following is Corollary 6.3 in [3].

Theorem 2.4. *In order that $X \approx \ell_2 \times \ell_2^f$, it is necessary and sufficient that X is a separable σ-completely metrizable AR which is a strong Z_σ-space and is strongly universal for separable completely metrizable spaces.*

3 The Attouch-Wets Topology

When we consider hyperspaces with the Attouch-Wets topology, we denote $\mathrm{Cld}_{AW}(X)$, $\mathrm{Fin}_{AW}(X)$, $\mathrm{Bdd}_{AW}(X)$, etc. Without the separability of X, $\mathrm{Cld}_{AW}(X)$ is always metrizable and has an admissible metric d_{AW} defined as follows:

$$d_{AW}(A, B) = \sup_{n \in \mathbb{N}} \min \left\{ 1/n, \sup_{x \in X_n} \{ |d(x, A) - d(x, B)| \} \right\},$$

where $x_0 \in X$ is fixed and $X_r = \{x \in X \mid d(x_0, x) \leqslant r\}$ for each $r \in \mathbb{R}$.

In [1], Banakh, Kurihara and Sakai showed the following theorem:

Theorem 3.1. *If X is an infinite-dimensional Banach space with weight τ, then $\mathrm{Cld}_{AW}(X) \approx \ell_2(2^\tau)$, where $\ell_2(\gamma)$ is the Hilbert space with weight γ.*

In [6], we have a following result which is analogous to Theorem2.2:

Theorem 3.2. *For every infinite-dimensional Banach space X with weight τ,*

$$\mathrm{Fin}_{AW}(X) \approx \mathrm{Comp}_{AW}(X) \approx \ell_2(\tau) \times \ell_2^f \text{ and}$$
$$\mathrm{Bdd}_{AW}(X) \approx \ell_2(2^\tau) \times \ell_2^f.$$

Theorem 3.2 is based on the following theorem, which is obtained in [5] as the non-separable version of Bestvina-Mogilski's characterization.

Theorem 3.3. *In order that $X \approx \ell_2(\tau) \times \ell_2^f$, it is necessary and sufficient that X is a σ-completely metrizable AR with weight τ which is a strong Z_σ-space and is strongly universal for $\mathfrak{M}_1(\tau)$, where $\mathfrak{M}_1(\tau)$ is the space of all completely metrizable spaces with weight τ.*

References

[1] T. Banakh, M. Kurihara and K. Sakai, *Hyperspaces of normed linear spaces with the Attouch-Wets topology,* Set-Valued Anal. , in press.

[2] G. Beer, Topologies on Closed and Closed Convex Sets, MIA **268**, Kluwer Acad. Publ., Dordrecht, 1993.

[3] M. Bestvina and J. Mogilski, *Characterizing certain incomplete infinite-dimensional absolute retracts,* Michigan Math. J. **33** (1986), 291–313.

[4] W. Kubiś, K. Sakai and M. Yaguchi, *Hyperspaces of separable Banach spaces with the Wijsman topology,* preprint.

[5] K. Sakai and M. Yaguchi, *Characterizing manifolds modeled on certain dense subspaces of non-separable Hilbert spaces,* Tsukuba J. Math., to appear.

[6] K. Sakai and M. Yaguchi, *Hyperspaces of infinite-dimensional Banach spaces with the Attouch-Wets topology,* preprint.

[7] H. Toruńczyk, *Characterizing Hilbert space topology,* Fund. Math. **111** (1981), 247–262;

[8] H. Toruńczyk, *A correction of two papers concerning Hilbert manifolds,* Fund. Math. **125** (1985), 89–93.