Variants of subtlety in $P_{\kappa}\lambda$

— comparison of ideals defined by combinatorial principles —

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概要

A type of subtlety for $P_{\kappa}\lambda$ called "A-subtle" is presented and compared with Menas' notion "M-subtle". Using it we prove almost ineffability is consistencywise stronger than Shelah property although $P_{\kappa}\lambda$ is A-subtle for every $\lambda \geq \kappa$ if $\kappa$ is subtle. The following are also shown (i) \(\{x \in P_{\kappa}\lambda : x \cap \kappa < |x|\}\) is A-subtle" has rather strong consequences. (ii) The subtle ideals are not $\lambda$-saturated, and completely ineffable ideal is not precipitous. (iii) $NAIn_{\kappa,\lambda} = NIn_{\kappa,\lambda}$ and $I_{\kappa,\lambda}$ does not have the partition property if $\lambda^{<\kappa} = 2^\lambda$. (iv) We can not prove "$\kappa$ is $\lambda^{<\kappa}$-ineffable whenever $\kappa$ is $\lambda$-ineffable".

1 Notations and basic facts

Throughout this paper $\kappa$ denotes a regular uncountable cardinal and $\lambda$ a cardinal $\geq \kappa$. Let $A$ be a set and $a$ a set of ordinals with $|a| \leq |A|$. For any such pair $(a, A)$, $P_{a}A$ denotes the set $\{x \subset A : |x| < |a|\}$. Thus $P_{\kappa}\lambda$ denotes the set $\{x \subset \lambda : |x| < \kappa\}$ and, for $x \in P_{\kappa}\lambda$, $P_{x\cap\kappa}x = \{s \subset x : |s| < |x \cap \kappa|\}$.

Combinatorial properties originated for regular uncountable cardinals have been translated into $P_{\kappa}\lambda$ in [10], [6] and [7]. For $x \in P_{\kappa}\lambda$, $\hat{x}$ denotes the set $\{y \in$\(^{1}\)2000 Mathematical Subject Classification: Primary 03E. Reserach partially supported by "Grant-in-Aid for Scientific research (C), The Ministry of Education, Science, Sports and Culture of Japan 09640299, and Japan Society for the Promotion of Science 14540142"
$P_{\kappa \lambda} : x \subset y \}$. We say $X \subset P_{\kappa \lambda}$ is unbounded if $X \cap \dot{x} \neq \emptyset$ for all $x \in P_{\kappa \lambda}$. Let $I_{\kappa \lambda} = \{X \subset P_{\kappa \lambda} : X$ is not unbounded\}.

We say $I$ is an ideal on $P_{\kappa \lambda}$ if the following hold:

1. $I \subset PP_{\kappa \lambda}$,
2. $\emptyset \in I$ and $P_{\kappa \lambda} \notin I$,
3. $I_{\kappa \lambda} \subset I$,
4. $I$ is $\kappa$-complete; $\cup X \in I$ for any $X \subset I$ with $|X| < \kappa$.

Thus $I_{\kappa \lambda}$ is the minimal ideal on $P_{\kappa \lambda}$.

We say $X \subset P_{\kappa \lambda}$ is closed if $\cup D \in X$ for any $\subset$-chain $D \subset X$ with $|D| < \kappa$. $X$ is called club if it is closed and unbounded.

**Fact 1.1.** Let $X \subset P_{\kappa \lambda}$. Then, $X$ is club if and only if there exists $f : \lambda \times \lambda \rightarrow \lambda$ such that $C_f := \{x \in P_{\kappa \lambda} : f''(x \times x) \subset x \text{ and } x \cap \kappa \in \kappa\} \subset X$.

We say $X$ is stationary if $X \cap C \neq \emptyset$ for any club $C$. Let $NS_{\kappa \lambda} = \{X \subset P_{\kappa \lambda} : X$ is nonstationary\}.

Let $I^+ = PP_{\kappa \lambda} \setminus I$. For $X \in I^+ I | X$ denotes the set $\{Y \subset P_{\kappa \lambda} : Y \cap X \in I\}$, which is an ideal on $P_{\kappa \lambda}$ extending $I$. We say an ideal $I$ is normal if $I$ is closed under diagonal unions; $\nabla X := \{x \in P_{\kappa \lambda} : x \in \cup \{X_\alpha : \alpha \in x\}\} \in I$ for any $X = \{X_\alpha : \alpha < \lambda\} \subset I$. Note that $I$ is normal if and only if every regressive function on $X \in I^+$ is constant on some $Y \in P(X) \cap I^+$, where a function $f$ is said to be regressive if $f(x) \in x$ for any $x$ in $\text{dom}(f) \setminus \{\emptyset\}$.

The relation $\prec$ is defined by $y \prec z$ if $y \in P_{\pi \kappa} z$. An ideal $I$ is strongly normal if for any $X \in I^+$ and $f : X \rightarrow P_{\kappa \lambda}$ such that $f(x) \prec x$ for all $x \in X$ there exists $Y \in P(X) \cap I^+$ such that $f \upharpoonright Y$ is constant. This is equivalent to the following: for any $\{X_s : s \in P_{\kappa \lambda}\} \subset I$, $\nabla_s X_s := \{x : x \in \cup \{X_s : s \prec x\}\} \in I$. Clearly every strongly normal ideal is normal. A filter $F$ on $P_{\kappa \lambda}$ and an ideal $I$ on $P_{\kappa \lambda}$ are dual to each other if the following holds:

$$X \in F \text{ if and only if } P_{\kappa \lambda} \setminus X \in I \text{ for every } X \subset P_{\kappa \lambda}.$$ 

The dual filter of $I$ will be denoted by $I^*$.

For $f : P_{\kappa \lambda} \rightarrow P_{\kappa \lambda}$, let $C_f = \{x \in P_{\kappa \lambda} : f''P_{\pi \kappa} x \subset P_{\pi \kappa} x\}$. We define an ideal $WNS_{\kappa \lambda}$ by:

$$X \in WNS_{\kappa \lambda} \text{ if and only if } X \subset P_{\kappa \lambda} \text{ and } X \cap C_f = \emptyset \text{ for some } f : P_{\kappa \lambda} \rightarrow P_{\kappa \lambda}.$$
The following are well-known [5], [8] and [1].

Fact 1.2. (1) $NS_{\kappa,\lambda}$ is the minimal normal ideal on $P_\kappa \lambda$.
(2) $WNS_{\kappa,\lambda}$ is the minimal strongly normal ideal on $P_\kappa \lambda$ and $NS_{\kappa,\lambda} \subseteq WNS_{\kappa,\lambda}$.
(3) $WNS_{\kappa,\lambda}$ is proper if and only if $\kappa$ is Mahlo or $\kappa = \nu^+$ with $\nu^{<\nu} = \nu$.
(4) If $\kappa$ is Mahlo, then $\{ x \in P_\kappa \lambda : x \cap \kappa = |x \cap \kappa| \text{ is inaccessible} \} \in WNS_{\kappa,\lambda}^*$.
(5) If $h : P_\kappa \lambda \rightarrow \lambda$ is a bijection, then $WNS_{\kappa,\lambda} = NS_{\kappa,\lambda} \mid \{ x : h''P_\kappa \kappa x = x \}$.
(6) If $\{ s_\alpha : \alpha < \lambda^{<\kappa} \}$ is an enumeration of $P_\kappa \lambda$ and $f : P_\kappa \lambda \rightarrow P_\kappa \lambda^{<\kappa}$ is defined by $f(x) = \{ \alpha : s_\alpha < x \}$, then $f_*(WNS_{\kappa,\lambda}) := \{ X \subseteq P_\kappa \lambda^{<\kappa} : f^{-1}(X) \in WNS_{\kappa,\lambda} \} = WNS_{\kappa,\lambda}^{<\kappa}$ and $\{ x \in P_\kappa \lambda : f(x) \cap \lambda = x \} \in WNS_{\kappa,\lambda}^*$.

All the notions defined above for $P_\kappa \lambda$ can be naturally translated into $P_{\kappa \cap \kappa} x$ if $x \cap \kappa$ is regular uncountable. For instance, $X \subseteq P_{\kappa \cap \kappa} x$ is unbounded if for any $y \subseteq P_{\kappa \cap \kappa} x$ there is $z \subseteq X$ such that $y \subseteq z$, and $L_{\kappa \cap \kappa} x$ denotes the set $\{ X \subseteq P_{\kappa \cap \kappa} x : X \text{ is not unbounded in } P_{\kappa \cap \kappa} x \}$ which is a $x \cap \kappa$-complete ideal on $P_{\kappa \cap \kappa} x$.

First we observe a type of subtlety for $P_\kappa \lambda$: $X \subseteq P_\kappa \lambda$ is $A$-subtle if for any $\{ S_x \subseteq P_{\kappa \cap \kappa} x : x \in P_\kappa \lambda \}$ and $C \subseteq WNS_{\kappa,\lambda}^*$, there exist $y \subseteq z$ both in $C \cap X$ such that $S_y = S_x \cap P_{\kappa \cap \kappa} y$.

The following are shown in §2:

Theorem 1.3. (1) $\kappa$ is subtle if and only if $P_\kappa \lambda$ is $A$-subtle for every $\lambda \geq \kappa$.
(2) If $X \subseteq P_\kappa \lambda$ is $A$-subtle, there exists $\{ S_x \subseteq P_{\kappa \cap \kappa} x : x \in X \}$ such that $\{ x \in X : S_x = S \cap x \} \subseteq WNS_{\kappa,\lambda}^+$ for every $S \subseteq P_\kappa \lambda$.
(3) If $\kappa$ is weakly Mahlo and $2^{<\kappa} \leq \lambda$ for every $\alpha < \lambda$, there exists $\{ S_x \subseteq P_{\kappa \cap \kappa} x : x \in P_\kappa \lambda \text{ and } x \cap \kappa \text{ is regular} \}$ such that $S_x$ is a club in $P_{\kappa \cap \kappa} x$ and $\{ x : S_x \not\subseteq C \} \in NS_{\kappa,\lambda}$ for any club $C \subseteq P_\kappa \lambda$.

The last statement is false for $\lambda = \kappa$.

In the next section we study large cardinal aspects of $A$-subtlety to prove an analogue of Baumgartner’s theorem [4] for regular uncountable cardinals:

Theorem 1.4. If $X \subseteq P_\kappa \lambda$ is $A$-subtle, then $\{ x \in X : X \cap P_{\kappa \cap \kappa} x \text{ is } \Pi^m_n\text{-indescribable} \}$ is $A$-subtle for every $m$, $n < \omega$.

Thus our subtlety takes an appropriate place in $P_\kappa \lambda$ combinatorics. The next follows immediately.
Corollary 1.5. If $cf(\lambda) \geq \kappa$ and $X \subset P_\kappa \lambda$ is almost ineffable, then \{ $x \in X : X \cap P_{\kappa \cap x} is Shelah\} is almost ineffable.

Hence "$\kappa$ is almost $\lambda$-ineffable" is an essentially stronger hypothesis than "$\kappa$ is $\lambda$-Shelah". Kamo [12] already proved:

Fact 1.6. (Kamo) If $\kappa$ is $\lambda$-ineffable, then \{ $x \in P_\kappa \lambda : P_{\kappa \cap x} is not almost ineffable\} is not ineffable.

Thus we have the same hierarchy of combinatorial properties for $P_\kappa \lambda$ as for regular uncountable cardinals. Our proof is applicable for Kamo's theorem and more simple than his.

Another corollary is:

Corollary 1.7. If \{ $x \in P_\kappa \lambda : x \cap \kappa < |x|\} is $A$-subtle, then $V \neq L[U]$.

Note that $L \models "\kappa is subtle" if $\kappa$ is subtle.

In §4 we turn to saturation of subtle ideals and show:

Theorem 1.8. (1) The ideals of non-subtle sets are not $\lambda$-saturated.

(2) The ideal of non-completely ineffable sets is not precipitous.

The last section is devoted to almost ineffability and ineffability. Our results might be surprising comparing with Kamo's theorem:

Theorem 1.9. Suppose that $\lambda^{<\kappa} = 2^\lambda$. Then, $X$ is almost ineffable if and only if $X$ is ineffable.

As a corollary we get:

Corollary 1.10. (1) We can not prove in ZFC that $\kappa$ is $\lambda^{<\kappa}$-ineffable whenever $\kappa$ is $\lambda$-ineffable.

(2) If $\lambda^{<\kappa} = 2^\lambda$, $I_{\kappa,\lambda}$ does not have the partition property.

2 The Subtle ideals on $P_\kappa \lambda$.

Menas [16] tried to introduce subtlety into $P_\kappa \lambda$ as follows (we call $M$-subtle in this paper):
Definition 2.1. Let $X \subset P_{\kappa}\lambda$. $X$ is $M$-subtle if for any $\{S_{x} \subset x : x \in P_{\kappa}\lambda\}$ and a club $C \subset P_{\kappa}\lambda$ there exist $y \subset z$ both in $C \cap X$ such that $S_{y} = S_{z} \cap y$.

This is not an essential generalization as proved in the same paper.

Fact 2.2. For every $\lambda \geq \kappa$, $P_{\kappa}\lambda$ is $M$-subtle if and only if $\kappa$ is subtle.

We present a new definition of subtlety for $P_{\kappa}\lambda$ and use the word "$A$-subtle" for it. In place of the filter of club sets we use $WNS_{\kappa,\lambda}$.

Definition 2.3. For $X \subset P_{\kappa}\lambda$, $X$ is $A$-subtle if for any $\{S_{x} \subset x : x \in P_{\kappa}\lambda\}$ and $C \in WNS_{\kappa,\lambda}^{*}$, there are $y \prec z$ both in $C \cap X$ such that $S_{y} = S_{z} \cap y$.

Set $I_{M} = \{X \subset P_{\kappa}\lambda : X$ is not $M$-subtle$\}$ and $I_{A} = \{X \subset P_{\kappa}\lambda : X$ is not $A$-subtle$\}$.

Remark 2.4. If $\lambda^{<\kappa} = \lambda$, then $\{x \in P_{\kappa}\lambda : h''P_{x\cap\kappa}x \subset x\} \in WNS_{\kappa,\lambda}^{*}$ for any bijection $h : P_{\kappa}\lambda \rightarrow \lambda$. Thus, in this case, $X \subset P_{\kappa}\lambda$ is $A$-subtle if and only if for any $\{S_{x} \subset x : x \in P_{\kappa}\lambda\}$ and $C \in WNS_{\kappa,\lambda}^{*}$ there are $y \prec z$ both in $C \cap X$ such that $S_{y} = S_{z} \cap y$.

We say $\kappa$ is $\lambda$-subtle if $P_{\kappa}\lambda$ is $A$-subtle (in $P_{\kappa}\lambda$). If $P_{\kappa}\lambda$ is $A$-subtle, then it is $M$-subtle. So $\kappa$ is assumed to be subtle in the rest of this section.

We collect several facts for subtle ideals:

Proposition 2.5. (1) $I_{M} \subset I_{A}$.

(2) $I_{M}$ is a normal ideal on $P_{\kappa}\lambda$.

(3) $I_{A}$ is a strongly normal ideal on $P_{\kappa}\lambda$

(4) $\{x \in P_{\kappa}\lambda : x \cap \kappa$ is Mahlo$\} \in I_{M}^{*}$.

(5) If $\kappa \leq \delta < \lambda$ and $X \subset P_{\kappa}\lambda$ is $A$-subtle, then $X \upharpoonright \delta := \{x \cap \delta : x \in X\} \subset P_{\kappa}\delta$ is $A$-subtle.

(6) If $\kappa$ is Mahlo and $\{x \in P_{\kappa}\lambda : x \cap P_{x\cap\kappa}x$ is $A$-subtle$\} \in WNS_{\kappa,\lambda}^{+}$, then $X$ is $A$-subtle.

(7) Let $\{s_{\alpha} : \alpha < \delta\}$ be an enumeration of $P_{\kappa}\lambda$ and $f(x) = \{\alpha : s_{\alpha} \prec x\}$ for $x \in P_{\kappa}\lambda$. Then, $f''X \subset P_{\kappa}\delta$ is $A$-subtle if and only if $X \subset P_{\kappa}\lambda$ is $A$-subtle.

(8) If $\lambda$ is regular, $X \subset P_{\kappa}\lambda$ and $\{\alpha < \lambda : X \cap P_{\kappa}\alpha$ is $M$-subtle$\} \in NS_{\lambda}^{+}$, then $X$ is $M$-subtle.
Proof. We only show (3). Let $X \subset P_{\kappa}\lambda$ be subtle and $f : X \to P_{\kappa}\lambda$ such that $f(x) \prec x$ for every $x \in X$. Suppose to the contradiction that $f^{-1}(\{a\})$ is not subtle for any $a \in P_{\kappa}\lambda$. For $a \in P_{\kappa}\lambda$, we fix $\{S^a_x \subset P_{\kappa\cap x} : x \in P_{\kappa}\lambda\}$ and $D_a \in WNS^a_{\kappa,\lambda}$ such that $S^a_x \not= S^a_z \cap P_{y\cap \kappa}y$ for any $y \prec z$ both in $D_a \cap f^{-1}(\{a\})$.

Let $h : P_{\kappa}\lambda \times P_{\kappa}\lambda \to P_{\kappa}\lambda$ be a bijection and set $T_x = h''(\{f(x)\} \times S^f_z) \cap P_{\kappa\cap x}$. Note that $C = \{x \in P_{\kappa}\lambda : h''(P_{\kappa\cap x} \times P_{\kappa\cap x}) \subset P_{\kappa\cap x}\} \in WNS^a_{\kappa,\lambda}$. Since $X$ is $A$-subtle and $E = C \cap \Delta \cap D_a \in WNS^a_{\kappa,\lambda}$, there exist $y \prec z$ both in $E \cap X$ such that $T_y = T_z \cap P_{\kappa\cap y}$. Then, $f(y) = f(z)$ and $S^f_z \cap P_{\kappa\cap y}$. Set $a = f(y) = f(z)$. We have $y \prec z$ are both in $D_a \cap f^{-1}(\{a\})$ and $S^a_y = S^a_z$, which contradicts our assumption.

A natural question arises:

**Question 2.6.** Can it be proved that $I_M \not\subseteq I_A$?

It turns out that "$A$-subtle" is neither an essential generalization.

**Theorem 2.7.** If $\kappa$ is subtle, then $P_{\kappa}\lambda$ is $A$-subtle.

**Proof.** Let $S_x \subset P_{\kappa\cap x}$ for $x \in P_{\kappa}\lambda$ and $D \in WNS^a_{\kappa,\lambda}$. Since $\kappa^{<\kappa} = \kappa$, $WNS^a_{\kappa,\lambda}$ is proper.

We first show $\{x \in P_{\kappa}\lambda : D \cap P_{\kappa}\lambda \in WNS^a_{\kappa,\lambda}\} \in WNS^a_{\kappa,\lambda}$. Let $f : P_{\kappa}\lambda \to P_{\kappa}\lambda$ such that $C_f \subset D$. If $\{x \in P_{\kappa}\lambda : f''P_{\kappa}\lambda \subset P_{\kappa}\lambda\} \not\subset WNS^a_{\kappa,\lambda}$, $X := \{x \in P_{\kappa}\lambda : \kappa \subset x \wedge \exists y \in P_{\kappa}\lambda (f(y) \not\subset P_{\kappa}\lambda)\} \in WNS^a_{\kappa,\lambda}$. Note that $\kappa = |x \cap \kappa|$ for every $x \in X$. By strong normality we have $y \in P_{\kappa}\lambda$ such that $Y := \{x \in X : y \subset x\} \in WNS^a_{\kappa,\lambda}$. $Y \cap f(y) = \emptyset$. Contradiction. Thus $Z := \{x \in P_{\kappa}\lambda : \kappa \subset x \wedge f \upharpoonright P_{\kappa}\lambda : P_{\kappa}\lambda \to P_{\kappa}\lambda\} \in WNS^a_{\kappa,\lambda}$. For $x \in Z D_z := \{s \in P_{\kappa}\lambda : f''P_{\kappa}\lambda \subset P_{\kappa}\lambda\} \in WNS^a_{\kappa,\lambda}$. For every $x \in Z D \cap P_{\kappa}\lambda \in WNS^a_{\kappa,\lambda}$ since $D_z \subset C_f \cap P_{\kappa}\lambda \subset D$.

Note that $\kappa$ is subtle if and only if $P_{\kappa}\kappa$ is $A$-subtle. Thus $P_{\kappa}\lambda$ is $A$-subtle for every $y \in Z$. Now we consider $\{S_x : x \in P_{\kappa}\lambda\}$ and $D \cap P_{\kappa}\lambda$. There exist $x_1 \prec x_2$ both in $D \cap P_{\kappa}\lambda$ such that $S_{x_1} = S_{x_2} \cap P_{\kappa\cap x_1}$. We have $y \prec z$ are both in $D_a \cap f^{-1}(\{a\})$ and $S^a_y = S^a_z$, which contradicts our assumption.

The following observations suggest two ideals may be the same. The first appears in [16].

**Fact 2.8.** If $X \subset P_{\kappa}\lambda$ is $M$-subtle and $S_x \subset x$ for each $x \in X$, then for any club $C \subset P_{\kappa}\lambda$ there exist $x, y$ both in $C \cap X$ such that $x \subset y$, $x \cap \kappa \subset y \cap \kappa$, and $S_x = S_y \cap x$. 

Proposition 2.9. If $\kappa$ is subtle, then $X = \{x \in P, \lambda : x \cap \kappa = |x|\}$ is $A$-subtle.

Proof. Note that $X \notin WNS_{\kappa, \lambda}$ ([1]). Let $f : P, \kappa \lambda \to P, \kappa \lambda$ and $S, x \subset P, x \cap \kappa \lambda$ for all $x \in P, \kappa \lambda$. We build a chain $(x, \alpha < \kappa)$ as follows:

Choose $x_0 \in X \cap C_f$ arbitrarily and $x_{\alpha+1} \in X \cap C_f$ so that $x_\alpha < x_{\alpha+1}$. For limit $\alpha$, let $x_\alpha = \bigcup\{x_\beta : \beta < \alpha\}$.

Set $x = \bigcup\{x_\alpha : \alpha < \kappa\}$. Then, $x \cap \kappa = |x| = \kappa$, $P, x \cap \kappa \lambda = \bigcup\{P_{x, \alpha \cap \kappa} : \alpha < \kappa\}$, and there exists a club $D \subset \kappa$ such that for every $\alpha \in D$, $x_\alpha \cap \kappa = \alpha = |x_\alpha|$. Note that $x_\alpha \in C_f$ if $\alpha$ is regular. Let $g : P, x \cap \kappa \lambda \to \kappa$ be any bijection. Then, we have a club $E \subset \{\alpha \in D : g"P, x_\alpha \subset \alpha\}$. Since $E$ is subtle in $\kappa$, there exist regular $\beta < \gamma$ both in $E$ such that $g"S, x_\beta = g"S, x_\gamma$ and $x_\beta$ belong to $C_f$.

S. Baldwin[3] and others observed the consistency strength of the stationarity of $x \in P, \kappa : x \cap \kappa < |x|$, the complement of the set we mentioned now.

Fact 2.10. ([3],[14],[1])

(1) If $\kappa$ is weakly inaccessible and $\lambda \text{ Ramsey} > \kappa$, then $\{x \in P, \kappa : x \cap \kappa < |x|\}$ is stationary.

(2) If $\{x \in P, \kappa^+: x \cap \kappa < |x|\}$ is stationary, then $0^+$ exists.

(3) If $P, \kappa \lambda$ is Shelah, then $\{x \in P, \kappa : x \cap \kappa < |x|\} \in NSh^\kappa, \lambda$.

Corollary 2.11. Suppose that $0^+$ does not exist and $S = \{x \in P, \kappa : h"P, x \cap \kappa \lambda = x\}$ where $h : P, \kappa \lambda \rightarrow \lambda$ is a bijection. Then, $I_M \upharpoonright S = I_A$. Thus $I_M = I_A$ if $I_M$ is strongly normal.

The same relation holds between subtlety and its another weakening, which relates to "ethereal" introduced by Kunen: $X \subset \kappa$ is ethereal if for any $S_\alpha \in [\alpha]^{\alpha} : \alpha < \kappa$ and a club $C \subset \kappa$, there are $\beta < \gamma$ both in $C \cap X$ such that $|S_\beta \cap S_\gamma| = |eta|$.

Definition 2.12. Let $X \subset P, \kappa \lambda$. We say $X$ is weakly subtle if for any $S_x \in I^+_{x \cap \kappa \lambda}, x \in P, \kappa \lambda$ and a club $C \subset P, \kappa \lambda$, there are $y < z$ both in $C \cap X$ such that $S_y \cap S_z \in I^+_{y \cap \kappa \lambda}$. Let $I_W = \{X \subset P, \kappa \lambda : X$ is not weakly subtle\}.

We have the following:

Proposition 2.13. (1) $I_W$ is a normal ideal.

(2) If $P, \kappa \lambda$ is weakly subtle, then $\kappa$ is ethereal.

(3) $I_W \upharpoonright S = I_A$ where $S = \{x \in P, \kappa : h"P, x \cap \kappa \lambda = x\}$ for a bijection $h : P, \kappa \lambda \rightarrow \lambda$. 

Three subtle ideals are interesting from the view of the diamond principles for $P_{\kappa}\lambda$.
The following two cardinal version of diamond principle by Jech is wellknown.

Definition 2.14. Let $X \subset P_{\kappa}\lambda$. Then,
\[ \Diamond_0(X) : \text{there exist } \{S_x \subset x : x \in X\} \text{ such that } \{x \in X : S_x = S \cap x\} \text{ is stationary for any } S \subset \lambda. \]
We simply write $\Diamond_0$ for $\Diamond_0(P_{\kappa}\lambda)$. Let $J_0 = \{X \subset P_{\kappa}\lambda : \Diamond_0(X) \text{ does not hold}\}$.

The following are known (see [9], [10]):

Fact 2.15. (1) $J_0$ is a normal ideal on $P_{\kappa}\lambda$.
(2) $L \models \text{"\Diamond_0(X) for any } X \subset P_{\omega_1}\lambda\text{"}$.
(3) If $2^{<\kappa} < \lambda$, then $J_0$ is proper.
(4) If $\Diamond_0$ holds, then $NS_{\kappa,\lambda}$ is not $2^\lambda$ saturated.

In the context of $I_A$ and $I_W$ another version of diamond arises.

Definition 2.16. Let $X \subset P_{\kappa}\lambda$. Then,
\[ \Diamond_1(X) : \text{there exists } \{S_x \subset P_{\omega_1}\kappa x : x \in X\} \text{ such that } \{x \in X : S_x = S \cap P_{\omega_1}\kappa x\} \text{ is stationary for any } S \subset P_{\kappa}\lambda. \]
\[ \Diamond_2(X) : \text{there exists } \{S_x \subset P_{\omega_1}\kappa x : x \in X\} \text{ such that } \{x \in X : S_x = S \cap P_{\omega_1}\kappa x\} \in WNS^+_{\kappa,\lambda} \text{ for any } S \subset P_{\kappa}\lambda. \]
$J_1$ and $J_2$ are similarly defined as $J_0$.

Of course $J_0 \subset J_1 \subset J_2$ and it is easily seen:

Lemma 2.17. (1) $J_1$ is a normal ideal on $P_{\kappa}\lambda$.
(2) $J_2$ is a strongly normal ideal on $P_{\kappa}\lambda$.
(3) If $\lambda^{<\kappa} = \lambda$ and $h : P_{\kappa}\lambda \rightarrow \lambda$ is a bijection, then $J_2 = J_0 \upharpoonright S$ with $S = \{x : h'' P_{\omega_1}\kappa x = x\}$.
(4) $L \models \text{"} J_2 = WNS_{\omega_1,\lambda} \text{"}$.
(5) If $J_2$ is proper, then any ideal $\subset WNS_{\kappa,\lambda}$ is not $2^{\lambda^{<\kappa}}$ saturated.

Theorem 2.18. If $\kappa$ is subtle, then $J_2$ is proper.

Proof. We show $\Diamond_2(X)$ holds for every $A$-subtle $X \subset P_{\kappa}\lambda$.
By induction on $\prec$ we define $S_x \subset P_{\omega_1}\kappa x$ for $x \in X$ as well as $C_x \subset P_{\omega_1}\kappa x$.
If $x$ is a $\prec$ minimal element of $X$, then $S_x = C_x = \emptyset$. 

Suppose $S_y$ and $C_y$ is defined for every $y \in X \cap P_{\varphi_\kappa}(x)$. If there exist $S \subset P_{\varphi_\kappa}(x)$ and $C \in WNS_{\varphi_\kappa}(x)$ such that $S_y \neq S \cap P_{\varphi_\kappa}y$ for any $y \in C$, then let $S_z$ and $C_z$ be any such $S$ and $C$. We say this is the substantial case. Otherwise let $S_z = C_z = P_{\varphi_\kappa}(x)$. To show $\{S_z : x \in X\}$ is a witness of $\Diamond_2(X)$, let $S \subset P_{\varphi_\lambda}, D \in WNS_{\varphi_\lambda}^*$, and $S_z \neq S \cap P_{\varphi_\kappa}(x)$ for any $x \in X \cap D$. Since $X \cap D$ is $A$-subtle, we may assume that $D \cap P_{\varphi_\kappa}(x) \in WNS_{\varphi_\kappa}(x)$ for every $x \in X \cap D$. Thus $S \cap P_{\varphi_\kappa}(x)$ and $D \cap P_{\varphi_\kappa}(x)$ witness the substantial case occurs for every $x \in X \cap D$. However $X \cap D$ is subtle hence there exist $y < z$ both in $X \cap D$ such that $S_y = S_z \cap P_{\varphi_\kappa}y$. In particular $y \in D \cap P_{\varphi_\kappa}(x)$. Contradiction.

Note that $\Diamond_\kappa$ holds if $\kappa$ is ethereal and $\kappa^{<\kappa} = \kappa$ [13].

Question 2.19. (1) Does $\Diamond_1$ hold if $P_{\varphi_\lambda}$ is weakly subtle?

(2) If $\kappa$ is ethereal, then $P_{\varphi_\lambda}$ is weakly subtle?

Let $\text{Reg}=\{x \in P_{\varphi_\lambda} : x \cap \kappa$ is regular$\}$. We conclude this section by:

Proposition 2.20. Suppose $\kappa$ is weakly Mahlo and $2^{\alpha^{<\kappa}} \leq \lambda$ for every $\alpha < \lambda$. Then, there exists $\{S_x : x \in P_{\varphi_\lambda}, x \cap \kappa$ is regular$\}$ such that

1. $S_x \subset P_{\varphi_\kappa}(x)$ club;
2. for every club $C \subset P_{\varphi_\lambda}$ $\{x : S_x \not\subset C\} \in NS_{\varphi_\lambda} \uparrow \text{Reg}.$

Proof. Let $\{C_\alpha : \kappa^{+} \leq \alpha < \lambda\} = \{X : \exists \beta < \lambda X$ is a club of $P_\alpha \beta\}$ be an enumeration and $C_\alpha$ a club of $P_\alpha \beta(\alpha)$. Set $B = \{x : \forall \alpha \in x \beta(\alpha) \in x\}$. Then, $B \in NS_{\varphi_\lambda}^*$. Let $S_x = \{y \in P_{\varphi_\lambda}(x) : \forall \alpha \in x y \cap \beta(\alpha) \in C_\alpha\}$. For every $\alpha \{x \in P_{\varphi_\lambda} : C_\alpha \cap P_{\varphi_\lambda}(x \cap \beta(\alpha))$ is a club of $P_{\varphi_\lambda}(x \cap \beta(\alpha))\} \in (NS_{\varphi_\lambda} \uparrow \text{Reg})^*$. Thus, $\{x \in P_{\varphi_\lambda} : \{z \in P_{\varphi_\lambda} : z \cap \beta(\alpha) \in C_\alpha\}$ is a club of $P_{\varphi_\lambda}(x)\} \in (NS_{\varphi_\lambda} \uparrow \text{Reg})^*$. Hence $A = \{x \in P_{\varphi_\lambda} : S_x$ is a club of $P_{\varphi_\lambda}(x)\} \in (NS_{\varphi_\lambda} \uparrow \text{Reg})^*$. Pick any club $C \subset P_{\varphi_\lambda}$. We have $f : \lambda \times \lambda \rightarrow \lambda$ with $C_f \subset C$. Define $g$ by $C_f \uparrow \beta = C_{g(\beta)}$ for $\beta < \lambda$. Note that $\beta = \beta(g(\beta))$.

Let $x \in A \cap B \cap C_f$. For every $\beta \in x g(\beta) \in x$. Hence for every $y \in S_z$ and $\beta \in x$ we have $y \cap \beta \in C_{g(\beta)} = C_f \uparrow \beta$. If $\{\xi, \zeta\} \subset y$, then $\{\xi, \zeta, f(\xi, \zeta)\} \subset \beta$ for some $\beta \in x$. Choose $z \in C_f$ such that $y \cap \beta = z \cap \beta$. Then, $f(\xi, \zeta) \in z \cap \beta = y \cap \beta$. Hence $y \in C_f$. We have shown that $S_x \subset C_f \subset C$ for every $x \in A \cap B \cap C_f$. □

Remark 2.21. This is false for $\lambda = \kappa$. 


3 Subtlety and large cardinals

Recall that $X \subset \kappa$ is $\Pi_n^m$-indescribable if for any $R \subset V_\kappa$ and $\Pi_n^m$ sentence $\varphi$ such that $(V_\kappa, \in, R) \models \varphi$, there exists $\alpha \in X$ such that $(V_\alpha, \in, R \cap V_\alpha) \models \varphi$.

**Fact 3.1.** (1) Suppose that $\lambda$ is weakly compact. Then, $X \subset P_\kappa \lambda$ is $M$-subtle if and only if $\{\alpha < \lambda : X \cap P_\alpha \alpha \text{ is not } M\text{-subtle}\}$ is not $\Pi_1^1$-indescribable.

(2) $I_M$ is a normal ideal on $P_\kappa \lambda$ such that $\{x \in P_\kappa \lambda : x \cap \kappa \text{ is not } \Pi_n^m\text{-indescribable}\} \in I_M$ for every $m, n < \omega$.

Carr[7] defined $P_\kappa \lambda$-version of indescribability.

**Definition 3.2.** A sequence $(V_\alpha(\kappa, \lambda) : \alpha \leq \kappa)$ is recursively defined as follows:

\begin{align*}
V_0(\kappa, \lambda) &= \lambda \\
V_{\alpha+1}(\kappa, \lambda) &= P_\kappa(V_\alpha(\kappa, \lambda)) \cup V_\alpha(\kappa, \lambda) \\
V_\alpha(\kappa, \lambda) &= \bigcup\{V_\beta(\kappa, \lambda) : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal}
\end{align*}

This definition can be carried out for $x \in P_\kappa \lambda$ if $x \cap \kappa$ is inaccessible. For such $x$ we consider the structure $(V_{x \cap \kappa}(x \cap \kappa, \in), \in)$ in the same way as $(V_\kappa(\kappa, \lambda), \in)$.

**Definition 3.3.** We say $X \subset P_\kappa \lambda$ is $\Pi_n^m$-indescribable if for any $R \subset V_\kappa(\kappa, \lambda)$ and $\Pi_n^m$ sentence $\varphi$ such that $(V_\kappa(\kappa, \lambda), \in, R) \models \varphi$, there exists $x \in X$ such that $x \cap \kappa = |x \cap \kappa|$ and $(V_{x \cap \kappa}(x \cap \kappa, \in), \in, R \cap V_{x \cap \kappa}(x \cap \kappa, x)) \models \varphi$.

**Lemma 3.4.** If $X \subset P_\kappa \lambda$ is $A$-subtle and $S_x \subset P_{x \cap \kappa} x$ for $x \in P_\kappa \lambda$, then $\{x \in X : \{y \in X \cap P_{x \cap \kappa} x : S_y = S_x \cap P_{y \cap \kappa} y\} \text{ is not } \Pi_n^m\text{-indescribable for some } m, n\}$ is not $\Pi_n^m$-indescribable.

**Proof.** Otherwise, by $\kappa$-completeness of $I_A$, $Y := \{x \in X : \{y \in X \cap P_{x \cap \kappa} x : S_y = S_x \cap P_{y \cap \kappa} y\} \text{ is not } \Pi_n^m\text{-indescribable}\}$ is subtle for some $m, n < \omega$. We may assume that $x \cap \kappa$ is inaccessible for all $x \in Y$. For $x \in Y$ there exist $R_x \subset V_{x \cap \kappa}(x \cap \kappa, x)$ and a $\Pi_n^m$ sentence $\varphi_x$ such that $(V_{x \cap \kappa}(x \cap \kappa, \in), \in, R_x) \models \varphi_x$ while $(V_{y \cap \kappa}(y \cap \kappa, y), \in, R_x \cap V_{y \cap \kappa}(y \cap \kappa, y)) \models \neg \varphi_x$ for any $y \in X \cap P_{x \cap \kappa} x$ with $S_y = S_x \cap P_{y \cap \kappa} x$. By $\kappa$-completeness again, we can assume for all $x \in Y$ $\varphi_x = \varphi$ for some $\varphi$.

Since $Y$ is subtle, there are $y \prec z$ both in $Y$ such that $R_y = R_z \cap V_{y \cap \kappa}(y \cap \kappa, y)$ and $S_y = S_z \cap P_{y \cap \kappa} y$. Then, $y \in X \cap P_{x \cap \kappa} z$, $S_y = S_z \cap P_{y \cap \kappa} y$ and $(V_{y \cap \kappa}(y \cap \kappa, y), \in, R_z \cap V_{y \cap \kappa}(y \cap \kappa, y)) \models \varphi_z$, which is a contradiction. \qed
As a corollary we have:

**Theorem 3.5.** \( \{ x \in P_\kappa \lambda : P_{x \cap \kappa} x \text{ is not } \Pi_n^m \text{-indescribable} \} \in I_A \) for every \( m, n < \omega \).

This theorem derives strong facts.

**Lemma 3.6.** If \( \{ x \in P_\kappa \lambda : 2^{x \cap \kappa} \leq |x| \} \) is \( A \)-subtle, then \( \{ x \in P_\kappa \lambda : o(x \cap \kappa) \geq 1 \} \) is \( A \)-subtle.

**Proof.** Note that \( \kappa \) is measurable if \( P_\kappa 2^\kappa \) is \( \Pi_1^1 \)-indescribable ([6],[7]). Let \( X = \{ x \in P_\kappa \lambda : P_{x \cap \kappa} 2^{x \cap \kappa} \text{ is } \Pi_1^1 \)-indescribable \}. Then, \( X \) is \( A \)-subtle. For \( x \in X \) \( x \cap \kappa \) is measurable and \( \{ y \in P_{x \cap \kappa} 2^{x \cap \kappa} : y \cap \kappa \text{ is measurable} \} \) is \( \Pi_1^1 \)-indescribable. Hence \( o(x \cap \kappa) \geq 1 \). \( \square \)

Note that \( L[U] \models \)"there exist \( \kappa < \lambda \) such that \( \{ x \in P_\kappa \lambda : x \cap \kappa < |x| \} \in NS_{\kappa,\lambda}^+ ".

**Corollary 3.7.** (1) \( L[U] \models \)"\( \{ x \in P_\kappa \lambda : x \cap \kappa < |x| \} \) is not \( A \)-subtle".

(2) \( L[U] \models \)"\( \neg \exists \kappa (\kappa \text{ is } \kappa^+ \text{-Shelah}) \)."

Thus the existence of a cardinal \( \kappa \) such that \( \{ x \in P_\kappa \lambda : x \cap \kappa < |x| \} \) is \( A \)-subtle is rather strong in consistency strength. On the other hand subtlety is a \( \Pi_1^1 \) property of \( P_\kappa \lambda \). The following proposition says the size of \( \kappa \) is not necessarily large.

**Proposition 3.8.** The least cardinal \( \kappa \) such that \( \kappa \) is \( \kappa^+ \)-subtle is not \( \kappa^+ \)-Shelah.

In the rest of this section we compare the almost ineffability and Shelah property using \( I_A \), which reveals very useful in large cardinal hierarchy.

Carr [6] defined Shelah property as a \( P_\kappa \lambda \) generalization of weak compactness. We show: if \( P_\kappa \lambda \) is subtle then there exist many \( x \in P_\kappa \lambda \) such that \( P_{x \cap \kappa} x \) is Shelah.

**Definition 3.9.** Let \( X \subset P_\kappa \lambda \). We say \( X \) is Shelah if for any \( \{ f_x \in ^x x : x \in P_\kappa \lambda \} \) there is \( f : \lambda \rightarrow \lambda \) such that for every \( y \in P_\kappa \lambda \) the set \( \{ x \in X \cap \hat{y} : f \upharpoonright y = f_x \upharpoonright y \} \in I_{\kappa,\lambda}^+ \).

We say \( X \) is almost ineffable (ineffable) if for any \( \{ f_x \in ^x x : x \in P_\kappa \lambda \} \) there is \( f : \lambda \rightarrow \lambda \) such that \( \{ x \in X : f \upharpoonright x = f_x \} \in I_{\kappa,\lambda}^+ (NS_{\kappa,\lambda}^+) \).

Let \( NS_{\kappa,\lambda} = \{ X \subset P_\kappa \lambda : X \text{ is not Shelah} \} \) and \( NAIN_{\kappa,\lambda} = \{ X \subset P_\kappa \lambda : X \text{ is not almost ineffable} \} \).

We often say \( \kappa \) is \( \lambda \)-Shelah (almost \( \lambda \)-ineffable) if \( P_\kappa \lambda \) is Shelah (almost ineffable).
Clearly $X$ is Shelah if $X$ is almost ineffable, and $X$ is almost ineffable if $X$ is ineffable. It is known that $NSh_{\kappa\lambda}$ and $NAIn_{\kappa\lambda}$ are strongly normal ideals if $cf(\lambda) \geq \kappa$. Moreover, Kamo [12] proved the following:

**Fact 3.10.** (Kamo) If $X \subseteq P_{\kappa}\lambda$ is ineffable and $cf(\lambda) \geq \kappa$, then \{ $x \in X : X \cap P_{x \cap \kappa}x$ is almost ineffable \} is ineffable.

The following follows immediately from definition and the remark after Definition 2.1 with strong normality of $NAIn_{\kappa\lambda}$.

**Proposition 3.11.** If $cf(\lambda) \geq \kappa$ and $X \subseteq P_{\kappa}\lambda$ is almost ineffable, then $X$ is $A$-subtle.

Carr [7] proved the following:

**Fact 3.12.** Let If $X \subseteq P_{\kappa}\lambda$ is $\Pi_{1}^{1}$-indestructible, then $X$ is Shelah. The converse is also true if $cf(\lambda) \geq \kappa$.

**Corollary 3.13.** If $X \subseteq P_{\kappa}\lambda$ is subtle, then $Y = \{ x \in X : x \cap P_{x \cap \kappa}x$ is not Shelah \}$ is not $A$-subtle.

**Corollary 3.14.** Let $cf(\lambda) \geq \kappa$. If $X \subseteq P_{\kappa}\lambda$ is almost ineffable, then \{ $x \in P_{\kappa}\lambda : X \cap P_{x \cap \kappa}x$ is Shelah \} is almost ineffable. In particular, \{ $x \in P_{\kappa}\lambda : x \cap \kappa$ is $x$-Shelah \} $\in NAIn_{\kappa\lambda}^{*}$ if $\kappa$ is almost $\lambda$-ineffable.

This corollary tells that almost ineffability is much stronger hypothesis than Shelah property. For instance, suppose that $\kappa$ is almost $\kappa^{+}$-ineffable. Then \{ $x \in P_{\kappa}\lambda : o.t.(x) = (x \cap \kappa)^{+}$ \} $\in NAIn_{\kappa\lambda}^{*}$ hence \{ $x \in P_{\kappa}\lambda : x \cap \kappa$ is $(x \cap \kappa)^{+}$-Shelah \} $\in NAIn_{\kappa\lambda}^{*}$. Thus, below the least $\alpha$ that is almost $\alpha^{+}$-ineffable, stationary many $\beta$ which is $\beta^{+}$-Shelah exist.

### 4 Saturation of subtle ideals on $P_{\kappa}\lambda$

Now we turn to saturation of subtle ideals.

**Proposition 4.1.** Let $I$ be a normal $\lambda$ saturated ideal on $P_{\kappa}\lambda$ with $\lambda$ regular. Then, \{ $x \in P_{\kappa}\lambda : o.t.(x)$ is regular \} $\in I^{*}$ hence \{ $x \in P_{\kappa}\lambda : x \cap \kappa < |x|$ \} $\in I^{*}$.
Proof. Let $G$ be $P_I$ generic for $V$. By $\lambda$ saturation the generic ultrapower $Ult(V, G)$ is well-founded. Let $j : V \rightarrow M \cong Ult(V, G)$ be an generic embedding. Since $\lambda$ is regular in $V[G]$, $M \models "\lambda$ is regular" and $\langle \text{ot.}(x)\mid x \in P_\kappa \lambda \rangle \cong \text{ot.}(j "\lambda") = \lambda$. Clearly $\{x \in P_\kappa \lambda : x \cap \kappa \in \kappa \land x \setminus \kappa \neq \emptyset\} \in I^*$. □

Theorem 4.2. Neither $I_A$ nor $I_M$ is $\lambda^+$-saturated.

Proof. First suppose that $\lambda$ is regular. We know $X = \{x \in P_\kappa \lambda : x \cap \kappa = |x| \land \kappa \in x\}$ is $A$-subtle. For $x \in X$ $\text{ot.}(x)$ is singular.

Second assume $\lambda$ is singular and the subtle ideal on $P_\kappa \lambda$, say $I$, is $\lambda$ saturated. In fact $I$ is $\delta$ saturated for some regular $\delta < \lambda$. Then $I \upharpoonright \delta$ is $\delta$ saturated and $\{x \in P_\kappa \delta : x \cap \kappa = |x|\} \notin I^\delta$. Contradiction. □

Remark 4.3. By Cummings' theorem $I_A$ is not $\lambda^+$-saturated if $\text{cf}(\lambda) < \kappa$.

By definition ineffability can be seen as a strengthening of Shelah property. One of the strongest version is complete ineffability defined as follows:

Definition 4.4. An ideal $I$ on $P_\kappa \lambda$ is $(\lambda, \lambda)$-distributive if for any $X \in I^+$ and $\{W_\alpha : \alpha < \lambda\}$ which is an $I$-partition of $X$ with $|W_\alpha| \leq \lambda$ there exist $Y \in P(X) \cap I^+$ and $\{X_\alpha : \alpha < \lambda\}$ such that $X_\alpha \in W_\alpha$ and $Y \setminus X_\alpha \in I$ for every $\alpha < \lambda$. We say $\kappa$ is completely $\lambda$ ineffable if there is a normal $(\lambda, \lambda)$-distributive ideal on $P_\kappa \lambda$. When $\kappa$ is completely $\lambda$ ineffable, the minimal normal $(\lambda, \lambda)$-distributive ideal on $P_\kappa \lambda$, say $I$, is called completely ineffable ideal and $X \in I^+$ is said to be completely ineffable.

If $I$ is a normal $(\lambda, \lambda)$-distributive ideal on $P_\kappa \lambda$ and $X \in I^+$, the following hold [11]:

1. For any $\{f_x \in {}^x \lambda : x \in X\}$ there is $f : \lambda \rightarrow \lambda$ such that $\{x \in X : f_x = f \upharpoonright x\} \in I^+$

2. $I$ is strongly normal.

In the rest of this section we assume $\text{cf}(\lambda) \geq \kappa$ and $I$ is a normal $(\lambda, \lambda)$-distributive ideal on $P_\kappa \lambda$. Hence $\kappa$ is Mahlo, $\lambda^{< \kappa} = \lambda$, and $I$ is strongly normal. Let $\mathbb{P}_I = (I, \subset)$, $G \subseteq \mathbb{P}_I$ be $V$ generic, and $j : V \prec M \cong Ult(V, G)$ a generic elementary embedding defined in $V[G]$.
By normality of $I$ $P(\lambda)^V \subset M$. Moreover $[\{P_{\alpha} \cap x \mid x \in P_{\alpha} \lambda\}]_G = j''P_{\alpha} \lambda \in M$, $j''P_{\alpha} \lambda = P_{\alpha''} \lambda$, and $j(X) \cap j''P_{\alpha} \lambda = j''X \in M$ for every $X \in P(P_{\alpha} \lambda)^V$ by strong normality. Conversely we have:

**Lemma 4.5.** For every $[f]_G \in j''V \cap M$ there exists $g \in V$ such that $[f]_G = j(g) \upharpoonright j''\lambda = j''g$.

**Proof.** Suppose $f \in V$, $X \in I^+$ with $X \equiv \langle f \rangle_G : j''\lambda \rightarrow j''V$. For $Y := \{s \in X : \text{dom}(f(s)) = s\} \in G$, $\alpha < \lambda$, and $x \in V$, let $X_{\alpha,x} = \{s \in Y \cap \hat{\alpha} : f(s)(\alpha) = x\}$.

Then $W_\alpha := \{X_{\alpha,x} : x \in V\} \cap I^+$ is a disjoint $I$-partition of $Y$ and $|W_\alpha| \leq \lambda$. By $(\lambda, \lambda)$-distributivity there is $g : \lambda \rightarrow V$ such that for every $\alpha < \lambda$ $Y \setminus X_{\alpha,g(\alpha)} \in I$. Hence $Z := \Delta_{\alpha}X_{\alpha,g(\alpha)} \in I^+ \cap P(Y)$ and for every $s \in Z$ and $\alpha \in s$ $f(s)(\alpha) = g(\alpha)$, that is, $Z \equiv \langle f \rangle_G = j(g) \upharpoonright j''\lambda$. \hfill $\square$

**Remark 4.6.** By $(\lambda, \lambda)$-distributivity of $I$, $\lambda$ remains a cardinal in $V[G]$ hence in $M$.

**Corollary 4.7.** (1) If $X \in M$, $X \subset j''V$, and $|X|^M \leq \lambda$, then $X = j''Y$ for some $Y \in V$.

(2) $P_{\alpha}j''\lambda = j''(P_{\alpha} \lambda)$ and $P(P_{\alpha}j''\lambda)^M = \{j''X : X \in P(P_{\alpha} \lambda)\}$.

**Proof.** (1) Note that the collapsing map $\pi : j''\lambda \rightarrow \lambda$ is a bijection in $M$. Choose any surjection $f \in M$ from $j''\lambda$ to $X$. By the lemma $f = j(g) \upharpoonright j''\lambda$ for some $g \in V$. Then $X = j(g)(j''\lambda) = \{j(g)(j(\alpha)) : \alpha < \lambda\} = \{j(g(\alpha)) : \alpha < \lambda\}$. Hence $X = j''Y$ where $Y = g''\lambda$. Now (2) is clear. \hfill $\square$

Next we present another characterization of completely ineffable subsets of $P_{\alpha} \lambda$.

**Definition 4.8.** Let $A$ be a set of ordinals. We inductively define $In_\alpha(\kappa, A) \subset P(P_{\kappa} A)$ as follows:

(1) $X \in In_0(\kappa, A)$ if $X \in NS_{\kappa,A}^+$, that is, for every $f : A \times A \rightarrow P_{\alpha}A$ there exists $x \in X$ such that $f''(x \times x) \subset P(x)$.

(2) $X \in In_{\alpha+1}(\kappa, A)$ if for every $f : P_{\alpha}A \rightarrow P_{\alpha}A$ such that $f(x) \subset x$ for any $x \in P_{\alpha}A$ there is $S \subset A$ with $\{x \in X : f(x) = x \cap S\} \in In_\alpha(\kappa, A)$.

(3) If $\alpha$ is a limit ordinal, $In_\alpha(\kappa, A) = \bigcap_{\beta < \alpha} In_\beta(\kappa, A)$.

Thus $X \in In_1(\kappa, \lambda)$ if and only if $X \subset P_{\alpha} \lambda$ is ineffable. Clearly $In_\alpha(\kappa, A) \subset In_\beta(\kappa, A)$ if $\beta < \alpha$. Hence there is $\alpha$ such that $In_\alpha(\kappa, A) = In_{\alpha+1}(\kappa, A)$. If
$In_{\alpha}(\kappa,A) = In_{\alpha+1}(\kappa,A) \neq \emptyset$, $P(P_{\kappa}A) \setminus In_{\alpha}(\kappa,A)$ is the minimal normal $(\lambda,\lambda)$-distributive ideal, that is, completely ineffable ideal on $P_{\kappa}A$.

**Definition 4.9.** We say $I$ is precipitous if $\Vdash_{\mathcal{P}} \text{Ult}(V,G)$ is well-founded”.

**Lemma 4.10.** Let $cf(\lambda) \geq \kappa$ and $I$ be a normal $(\lambda,\lambda)$-distributive precipitous ideal on $P_{\kappa}\lambda$. For any $X \in P(P_{\kappa}\lambda)$ and $\alpha$, $X \in In_{\alpha}(\kappa,\lambda)$ if and only if $M \models \text{“}j''X \in In_{\alpha}(\kappa,j''\lambda)\text{”}$.

**Proof.** By induction on $\alpha$. Assume first $X$ is stationary, $f : j''\lambda \times j''\lambda \rightarrow P_{\kappa}j''\lambda$, and $f \in M$. Since $P_{\kappa}j''\lambda = j''(P_{\kappa}\lambda)$, $f = j(g) \downarrow j''\lambda \times j''\lambda$ for some $g \in V$ such that $g : \lambda \times \lambda \rightarrow P_{\kappa}\lambda$. There is $x \in X$ such that $g''(x \times x) \subset P(x)$. Then $M \models \text{“}j(g)(a) \subset j(x)$ for all $a \in j(x) \times j(x)\text{”}$. Since $j(x) = j''x \subset j''\lambda$, $M \models \text{“}f(a) \subset j(x)$ for all $a \in j(x) \times j(x)\text{”}$. Hence $M \models \text{“}j''X \in NS_{\kappa,j''\lambda}^{+}\text{”}$.

Assume conversely $M \models \text{“}j''X \in NS_{\kappa,j''\lambda}^{+}\text{”}$, $f \in V$, and $f : \lambda \times \lambda \rightarrow P_{\kappa}\lambda$. Since $M \models \text{“}j(f) \downarrow (j''\lambda \times j''\lambda) : j''\lambda \times j''\lambda \rightarrow P_{\kappa}j''\lambda\text{”}$, there is $y \in j''X$ such that $j(f)(y \times y) \subset P(y)$. For $x \in X$ with $j(x) = y$, $j(f)(j(a)) \subset j(x)$ for every $a \in x \times x$. Hence $f''(x \times x) \subset P(x)$, showing that $X$ is stationary.

The case $\alpha$ is a limit ordinal is clear by our definition of $In_{\alpha}(\kappa,\lambda)$ and $In_{\alpha}(\kappa,j''\lambda)$.

We prove for $\alpha = \beta + 1$. Suppose first $X \in In_{\alpha}(\kappa,\lambda)$, $M \models \text{“}f : P_{\kappa}j''\lambda \rightarrow P_{\kappa}j''\lambda \text{”}$ and $f(j(x)) \subset j(x)$ for every $x \in P_{\kappa}\lambda$. We have $g \in V$ with $f = j(g) \downarrow j''P_{\kappa}\lambda$. Since $g(x) \subset x$ for every $x \in P_{\kappa}\lambda$, there is $S \subset \lambda$ such that $Y := \{x \in X : g(x) = x \cap S\} \in In_{\beta}(\kappa,\lambda)$.

By inductive hypothesis $j''Y \in In_{\beta}(\kappa,j''\lambda)$. For every $x \in Y$ $j(g)(j(x)) = j(g(x)) = j(x \cap S) = j''(x \cap S) = j''S$ and $j''S \in M$. Now $f(y) = y \cap j''S$ for every $y \in j''Y$ hence $j''X \in In_{\alpha}(\kappa,j''\lambda)$.

Suppose second $M \models \text{“}j''X \in In_{\alpha}(\kappa,j''\lambda)\text{”}$, $f \in V$, and $f(x) \subset x$ for every $x \in P_{\kappa}\lambda$. Set $j(f) \downarrow j''P_{\kappa}\lambda = g$. Since $M \models \text{“}g(x) \subset x$ for all $x \in j''P_{\kappa}\lambda\text{”}$, there is $T \subset j''\lambda$ such that $S := \{y \in j''X : g(y) = y \cap T\} \in In_{\beta}(\kappa,j''\lambda)$. By the fact $S \in P(P_{\kappa}j''\lambda)$ we have $Y \in P(P_{\kappa}\lambda)^{V}$ with $S = j''Y$. For every $x \in Y$, $g(j(x)) = j(x) \cap T$. By the same reason for $S, T = j''T_{1}$ for some $T_{1} \in V$. For every $x \in Y$, $j(f(x)) = g(j(x)) = j(x) \cap j''T_{1} = j''(x \cap T_{1}) = j(x \cap T_{1})$. So $f(x) = x \cap T_{1}$ for all $x \in Y$. The inductive hypothesis shows $Y \in In_{\beta}(\kappa,\lambda)$. Hence $X \in In_{\alpha}(\kappa,\lambda)$. □

**Theorem 4.11.** Suppose $cf(\lambda) \geq \kappa$ and $I$ is a normal $(\lambda,\lambda)$-distributive precipitous ideal on $P_{\kappa}\lambda$. Then $\{x : x \cap \kappa$ is completely o.t.$(x)$-ineffable\} $\in I^{*}$. 
Proof. Since $V$ \( = \kappa \) is completely \( \lambda \)-ineffable, $In_\alpha(\kappa, \lambda) = In_{\alpha+1}(\kappa, \lambda) \neq \emptyset$ for some $\alpha$. $In_\alpha(\kappa, \lambda) \neq \emptyset$ implies $In_\alpha(\kappa, j''\lambda) \neq \emptyset$. To show $In_\alpha(\kappa, j''\lambda) = In_{\alpha+1}(\kappa, j''\lambda)$, assume $X \in In_\alpha(\kappa, j''\lambda) - In_{\alpha+1}(\kappa, j''\lambda)$. Since $X \in P(P_\kappa j''\lambda)$, there is $Y \in P(P_\kappa \lambda)^V$ such that $X = j''Y$. Then $Y \in In_\alpha(\kappa, \lambda) = In_{\alpha+1}(\kappa, \lambda)$ hence $X = j''Y \in In_{\alpha+1}(\kappa, j''\lambda)$. Contradiction. \( \square \)

Lemma 4.12. Let $\kappa$ be Mahlo, $X \subset P_\kappa$, and $\alpha$ an ordinal.

(1) If \{ $x \in P_\kappa : X \cap P_{x \cap \kappa} \in In_\alpha(x \cap \kappa, x)$ \} $\in In_\alpha(\kappa, \lambda)$, then $X \in In_\alpha(\kappa, \lambda)$.

(2) If $X \in In_\alpha(\kappa, \lambda)$, then \{ $x \in P_\kappa : X \cap P_{x \cap \kappa} \notin In_\alpha(x \cap \kappa, x)$ \} $\in In_\alpha(\kappa, \lambda)$.

(3) If $X \subset P_\kappa$ is completely $\lambda$ ineffable, then \{ $x \in X : X \cap P_{x \cap \kappa} \notin$ completely ineffable \} is completely ineffable.

Proof. (1) By induction on $\alpha$. Suppose that \{ $x \in P_\kappa : X \cap P_{x \cap \kappa} \in NS_{x \cap \kappa, x}^+$ \} $\in NS_{\kappa, \lambda}^+$ and $f : \lambda^2 \to P_\kappa$. There is $x$ such that $X \cap P_{x \cap \kappa} \in NS_{x \cap \kappa, x}^+$ and $f \upharpoonright x \times x : x \times x \to P(x)$. Then we can find $y \in X \cap P_{x \cap \kappa}$ such that $f''(y \times y) \subset P(y)$. Hence $X$ is stationary.

Let $\alpha$ be a limit ordinal and $Y := \{ x \in P_\kappa : X \cap P_{x \cap \kappa} \in In_\alpha(x \cap \kappa, x) \} \in In_\alpha(\kappa, \lambda)$. For any $\beta < \alpha Y \subset \{ x \in P_\kappa : X \cap P_{x \cap \kappa} \in In_\beta(x \cap \kappa, x) \} \in In_\beta(\kappa, \lambda) \subset In_\alpha(\kappa, \lambda)$. By induction hypothesis $X \in In_\alpha(\kappa, \lambda)$.

Let $\alpha = \beta + 1$ and $f(x) \subset x$ for all $x \in P_\kappa$. For each $x \in Y$ there is $S_x \subset x$ such that \{ $y \in X \cap P_{x \cap \kappa} : f(y) = y \cap S_x$ \} $\in In_\beta(x \cap \kappa, x)$. Since $Y \in In_{\beta+1}(\kappa, \lambda)$ we have $S \subset \kappa$ such that $Z := \{ x \in Y : S_x = x \cap S \} \in In_\beta(\kappa, \lambda)$. For $x \in Z$ and $y \in X \cap P_{x \cap \kappa} f(y) = y \cap S_x = y \cap x \cap S = y \cap S$. So \{ $x \in P_\kappa : \{ y \in X : f(y) = y \cap S \} \cap P_{x \cap \kappa} \in In_\beta(x \cap \kappa, x)$ \} $\in In_\beta(\kappa, \lambda)$. By induction hypothesis \{ $y \in X : f(y) = y \cap S$ \} $\in In_\beta(\kappa, \lambda)$ hence $X \in In_{\beta+1}(\kappa, \lambda)$.

(2) We prove this by induction on $\kappa$. We may assume $A := \{ x \in X : X \cap P_{x \cap \kappa} \in In_\alpha(x \cap \kappa, x) \} \in In_\alpha(\kappa, \lambda)$. For any $x \in A$, by inductive hypothesis, $B_x := \{ y \in P_{x \cap \kappa} : X \cap P_{x \cap \kappa} \cap P_{y \cap \kappa} = X \cap P_{y \cap \kappa} \notin In_\alpha(y \cap \kappa, y) \} \in In_\alpha(x \cap \kappa, x)$. By (1) and the fact $B_x = \{ y \in P_\kappa : X \cap P_{y \cap \kappa} \notin In_\alpha(y \cap \kappa, y) \} \cap P_{x \cap \kappa}$, the conclusion holds.

(3) Let $\beta$ be the least ordinal such that $In_{\beta+1}(\kappa, \lambda) = In_\beta(\kappa, \lambda)$ and $X \in In_\beta(\kappa, \lambda)$.

We may assume $W := \{ x \in X : X \cap P_{x \cap \kappa} \text{ is completely ineffable} \} \in In_\beta(\kappa, \lambda)$.

For $x \in W$ let $\beta_x$ be the least ordinal such that $In_{\beta_x+1}(x \cap \kappa, x) = In_{\beta_x}(x \cap \kappa, x) \neq \emptyset$.

By (2), for each $x \in W$ $T_x := \{ y \in P_{x \cap \kappa} : X \cap P_{x \cap \kappa} \cap P_{y \cap \kappa} = X \cap P_{y \cap \kappa} \notin In_{\beta_x}(y \cap \kappa, y) \} \in In_{\beta_x}(x \cap \kappa, x)$. Set $\gamma = \max(\beta, \bigcup_{x \in W} \beta_x)$ and $T = \{ x \in P_\kappa :$
Then $T_x \subset T \cap P_{x \cap \kappa} \kappa \subseteq T$ for every $x \in W$. Hence $T \in In_\gamma(\kappa, \lambda) = In_\beta(\kappa, \lambda)$. 

Now we conclude by the following corollary:

**Corollary 4.13.** If $cf(\lambda) \geq \kappa$, then the completely ineffable ideal on $P_\kappa\lambda$ is not precipitous.

### 5 Ineffability and almost ineffability

In this section we show $NAIn_{\kappa, \lambda} = NIn_{\kappa, \lambda}$ if $cf(\lambda) < \kappa$ and $\lambda^{<\kappa} = 2^\lambda$. Thus, two ideals are the same for “small” cofinality points if GCH holds. Then we use it to prove that for $\lambda$ with cofinality less than $\kappa$, “$\kappa$ is $\lambda$-ineffable” does not always imply “$\kappa$ is $\lambda^{<\kappa}$-ineffable”. This contrasts to the fact “$\kappa$ is $\lambda^{<\kappa}$- (super)compact if $\kappa$ is $\lambda$-(super)compact.

**Fact 5.1.** (1) $WNS_{\kappa, \lambda} \subset NAIn_{\kappa, \lambda} \subset NIn_{\kappa, \lambda}$. 
(2) If $P_\kappa\lambda \not\in NAIn_{\kappa, \lambda}$ and $cf(\lambda) \geq \kappa$, then $\lambda^{<\kappa} = \lambda$.
(3) If $cf(\lambda) \geq \kappa$, then $\{x \in P_\kappa\lambda : P_{x \cap \kappa} \kappa \in NAIn_{x \cap \kappa, \kappa} \} \in NIn_{\kappa, \lambda}$ and $NAIn_{\kappa, \lambda} \subset NIn_{\kappa, \lambda}$.

We have known quite little when $cf(\lambda) < \kappa$ while the following conjecture has seemed reasonable comparing with supercompactness:

**Conjecture 5.2.** Suppose that $cf(\lambda) < \kappa$ and $\kappa$ is (almost) $\lambda$-ineffable. Then, $\lambda^{<\kappa} = \lambda^+$ and $\kappa$ is $\lambda^+$-ineffable.

**Lemma 5.3.** If $\lambda^{<\kappa} = 2^\lambda$, then there exists $X \in WNS^*_{\kappa, \lambda}$ such that $I_{\kappa, \lambda} \upharpoonright X = NS_{\kappa, \lambda} \upharpoonright X$.

**Proof.** Let $^\lambda \kappa \lambda = \{f_s : s \in P_\kappa\lambda \}$ be an enumeration and set $X = \{x \in P_\kappa\lambda : x \in \bigcap \{C_{f_s} : s \prec x \}\}$. For every $s \in P_\kappa\lambda$, $C_{f_s} \in NS^*_{\kappa, \lambda} \subseteq WNS^*_{\kappa, \lambda}$ hence $X \in WNS^*_{\kappa, \lambda}$. Let $Y \not\in I_{\kappa, \lambda} \upharpoonright X$. For every club $C \subset P_\kappa\lambda$ there exists $s \in P_\kappa\lambda$ such that $C_{f_s} \subset C$. We have $x \in Y \cap X$ with $s \prec x$. Then, $x \in C_{f_s} \cap Y \cap X \subset C \cap Y \cap X$ hence $Y \cap X \in NS^*_{\kappa, \lambda}$. 

Since $WNS_{\kappa, \lambda} \subset NAIn_{\kappa, \lambda}$ we have:
Theorem 5.4. $NAIn_{\kappa,\lambda} = NIn_{\kappa,\lambda}$ if $\lambda^{<\kappa} = 2^\lambda$.

By lemma 4.12 we have the following:

Lemma 5.5. If $\kappa$ is $\lambda$-ineffable, then $\{x \in P_{\kappa}\lambda : P_{x \cap \kappa} x \in NIn_{x \cap \kappa, x} \} \notin NIn_{\kappa,\lambda}$.

Theorem 5.6. Suppose that $\lambda^{<\kappa} = 2^\lambda$ and $\kappa$ is $\lambda^+$-ineffable. Then, $\{x \in P_{\kappa}\lambda : x \cap \kappa$ is $o.t. (x \cap \lambda)$-ineffable, not $o.t. (x)$-ineffable, and $o.t. (x) = o.t. (x \cap \lambda)^+ \} \notin NIn_{\kappa,\lambda}^+$.

Proof. We know $A = \{x \in P_{\kappa}\lambda^+ : x \cap \kappa$ is almost $o.t. (x \cap \lambda)$-ineffable, $o.t. (x) = o.t. (x \cap \lambda)^+, cf (o.t. (x \cap \lambda)) < x \cap \kappa, and o.t. (x \cap \lambda)^{<x \cap \kappa} = 2^{o.t. (x \cap \lambda)} \} \in NIn_{\kappa,\lambda}^+ (([12],[2])$. Every $x \in A$ is almost $o.t. (x \cap \lambda)$-ineffable hence $o.t. (x \cap \lambda)$-ineffable by the previous theorem. Now the conclusion follows by lemma 5.5. $\square$

We conclude by the negation of $I_{\kappa,\lambda}^+ \rightarrow (I_{\kappa,\lambda}^+)^2$ with $\lambda^{<\kappa} = 2^\lambda$.

Definition 5.7. For $X \subset P_{\kappa}\lambda$ let $[X]^2 = \{(x,y) \in X \times X : x \subset y \}$. We say $I_{\kappa,\lambda}$ has the partition property if for every $X \notin I_{\kappa,\lambda}$ and $F : [X]^2 \rightarrow 2$ there exists $H \notin I_{\kappa,\lambda}$ such that $F \upharpoonright [H]^2$ is constant.

Theorem 5.8. If $\lambda^{<\kappa} = 2^\lambda$, then $I_{\kappa,\lambda}$ does not have the partition property.

Proof. Suppose otherwise and let $I = I_{\kappa,\lambda} \upharpoonright X = NS_{\kappa,\lambda} \upharpoonright X$ with $X$ as in [?]. For every $X \in I^+$ and $F : [X]^2 \rightarrow 2$ there exists $H \notin NS_{\kappa,\lambda}$ such that $F \upharpoonright [H]^2$ is constant. Hence $NIn_{\kappa,\lambda} \subset I$ ([15]). However $I \subset WNS_{\kappa,\lambda} \subset NIn_{\kappa,\lambda}$. Contradiction. $\square$

Remark 5.9. P. Matet proved that $I_{\kappa,\kappa^+}$ does not have the partition property if $2^\kappa = \kappa^+$. While M. Shioya [17] constructed the model in which $I_{\kappa,\lambda}$ has the partition property with $\kappa$ supercompact.

参考文献


