Cardinal invariants associated with some combinatorial statements

Shizuo Kamo

Abstract

T. Bartoszyński [1] characterized the uniformity non(\textcal{M}) of the meager ideal on the real line as the smallest size of a family \( X \subset \omega^\omega \) such that \( \forall y \in \omega^\omega \exists x \in X \exists^\infty n < \omega y(n) = x(n) \). By replacing \( \omega^\omega \) by certain restricted subsets, we can get weaker combinatorial statements and define cardinal invariants. In this talk, we study these cardinal invariants.

0 Introduction

We use standard notion and notations in set theory (see e.g. [3]). Set
\[
\mathcal{F} = \{ f \in (\omega \setminus \{0\})^\omega \mid \text{f is non-decreasing and } \lim_{n<\omega} f(n) = \omega \}.
\]
For each \( f \in \mathcal{F} \), define the cardinal \( \theta_f \) by
\[
\theta_f = \min \{ |X| \mid X \subset \prod_{n<\omega} f(n) \text{ and } \forall y \in \prod_{n<\omega} f(n) \exists^\infty n < \omega y(n) = x(n) \}.
\]
By the Bartoszyński’s characterization of non(\textcal{M}), it holds that \( \theta_f \leq \text{non}(\textcal{M}) \) for all \( f \in \mathcal{F} \). Also, it is easy to see that \( \theta_{f_1} \leq \theta_{f_2} \) if \( f_1, f_2 \in \mathcal{F} \) and \( f_1 \leq^* f_2 \). In the next section, we show that, in a certain generic model which is obtained by adjoining random reals, \( \theta_{f_1} < \theta_{f_2} \) holds for some \( f_1, f_2 \in \mathcal{F} \). Put \( \theta = \min \{ \theta_f \mid f \in \mathcal{F} \} \). Let me introduce another cardinal invariant \( \theta^* \) which is associated with a weaker combinatorial statement. For this, we need some definitions. Set
\[
\mathcal{H} = \{ h \in \omega^\omega \mid h \text{ is strictly increasing and } \lim_{n<\omega} h(n+1) - h(n) = \omega \}.
\]
For each \( h \in \mathcal{H} \) and \( n < \omega \), \( a_n^h \) denotes the interval \([h(n), h(n+1))\) of \( \omega \). Define \( \theta^* \) by
\[
\theta^* = \min \{ |W| \mid W \subset 2^\omega \times \mathcal{H} \text{ and } \forall y \in 2^\omega \exists (x, h) \in W \exists^\infty n < \omega y \mid a_n^h = x \mid a_n^h \}.
\]
It is easy to check that \( \omega_1 \leq \theta^* \leq \theta \). Furthermore, we have:

Theorem 0.1 Assume that \( \text{cof}([d]^\omega, \subset) = d \). Then, it holds that \( \theta^* \leq d \).
Proof  Take a sufficiently large regular cardinal \( \rho \). By using the assumption, take an elementary substructure \( M \) of \( H(\rho) \) such that

\[
M \cap \omega^\omega \text{ is a dominating family and } |M| = d \text{ and } M \cap [M]^\omega \text{ is } \subset \text{-cofinal in } [M]^\omega.
\]

Since \( M \cap \omega^\omega \) is a dominating family, it holds that

\[
(\ast) \quad \forall h \in H \exists h' \in M \cap H \forall^\infty n < \omega \exists m < \omega \ a^h_m \subset a^h_n.
\]

We show that \( W = M \cap (2^\omega \times H) \) satisfy the definition of \( \theta^* \). To get a contradiction, assume that there is \( y \in 2^\omega \) such that

\[
\forall^\infty n < \omega \ y \upharpoonright a^h_n \neq x \upharpoonright a^h_n, \text{ for all } (x, h) \in W. \tag{3''}
\]

Put \( X = 2^\omega \cap M \). The next claim is easily verified by using (\ast).

Claim 0.2  \( \forall x \in X \exists k < \omega \forall^\infty m < \omega \ y \upharpoonright [m, m + k) \neq x \upharpoonright [m, m + k). \tag{\triangle} \)

By Claim 0.2, define \( \varphi : X \to \omega \) by

\[
\varphi(x) = \text{ the largest } k < \omega \text{ such that } \exists^\infty m < \omega \ x \upharpoonright [m, m + k) \subset y.
\]

It is easy to check that \( \sup \{ \varphi(x) \mid x \in X \} = \omega \). By this, since \( [M]^\omega \cap M \) is \( \subset \) -cofinal in \( [M]^\omega \), we can take \( A = \{ a_i \mid i < \omega \} \in M \) such that \( \sup \{ \varphi(a_i) \mid i < \omega \} = \omega \). Take \( \psi : \omega \times \omega \to \omega \) such that, for each \( (i, n) \in \omega \times \omega \),

\[
i + n + \varphi(a_i) \leq \psi(i, n) \text{ and } \exists m \in [n, \psi(i, n) - \varphi(a_i)) \ a_i \upharpoonright [m, m + \varphi(a_i)) \subset y.
\]

Without loss of generality, we may assume that \( \psi \in M \). Define \( \langle k_i \mid i < \omega \rangle \in M \) by

\[
\begin{cases}
k_0 & = 0 \\
k_{i+1} & = \psi(i, k_i), \text{ for } i < \omega
\end{cases}
\]

and set \( x = \bigcup_{i<\omega} a_i \upharpoonright [k_i, k_{i+1}) \in X \). Then, it holds that

\[
\forall k < \omega \exists m < \omega \ x \upharpoonright [m, m + k) \subset y.
\]

But this contradicts Claim 0.2 \( \square \)

Let \( C_\omega \) be the notion of forcing which adds a Cohen real. Then, it holds that

\[
\vdash_{C_\omega} \forall y \in 2^\omega \exists x \in 2^\omega \cap V \exists^\infty n < \omega \ x \upharpoonright [n^2, n^2 + n) = y \upharpoonright [n^2, n^2 + n).
\]

So, \( \theta^* < d \) holds in a certain Cohen generic model.

It is known that the assumption \( \text{cof}([d]^\omega, \subset) = d \) is followed from the non-existence of \( 0^\# \). So, it seems to prove Theorem 0.1 without this assumption. But I failed to find a proof.

Question 0.1  \( \text{Is } \theta^* \leq d \text{ proved in ZFC?} \)

In sections 2, 3, 4, we show that the cardinals \( \omega_1, \theta, \theta^* \) can be separated for certain generic models.
1 Generic extensions by random reals

For each infinite cardinal \( \kappa \), we denote by \( \mathcal{B}(\kappa) \) the measure algebra which adds a random function from \( \kappa \) to \( 2 \) and by \( \mu_\kappa : \mathcal{B}(\kappa) \to [0,1] \) the measure function. In this section, we prove the following theorem.

**Theorem 1.1** Assume CH. Let \( \kappa > \omega_1 \) be a regular cardinal. Then, there are \( f_1, f_2 \in \mathcal{F} \) such that
\[
\Vdash_{\mathcal{B}(\kappa)} \theta_{f_1} = \omega_1 \quad \text{and} \quad \theta_{f_2} = \kappa.
\]

Set \( f_2 = \langle 2^n \mid n < \omega \rangle \in \mathcal{F} \). The next well-known lemma guarantees that this \( f_2 \) is as required in Theorem 1.1.

**Lemma 1.2** (Forklore) \( \Vdash_{\mathcal{B}(\omega)} \exists y \in \prod_{n < \omega} f_2(n) \forall x \in \prod_{n < \omega} f_2(n) \cap \mathcal{V} \forall^\infty n < \omega x(n) \neq y(n) \).

**Proof** Define \( k_n < \omega \) (for \( n < \omega \)) by
\[
k_0 = 0 \quad \text{and} \quad k_{n+1} = k_n + n \quad \text{for} \quad n < \omega.
\]
For each \( n < \omega \), put \( I_n = [k_n, k_{n+1}) \) and take a bijections from \( I_n^2 \) to \( f_2(n) \). Using these bijections, we identify \( \prod_{n < \omega} f_2(n) \) with \( \prod_{n < \omega}^{I_n} 2 \). Let \( \dot{g} \) be the canonical \( \mathcal{B}(\omega) \)-name of generic real. Define \( \dot{y} \) by
\[
\Vdash \dot{y} = \langle j \upharpoonright I_n \upharpoonright n < \omega \rangle.
\]
It holds that, for each \( n < \omega \) and \( s : I_n \to 2 \),
\[
\mu_\omega(\| s = \dot{g} \| I_n I) = 2^{-|I_n|} = 2^{-n}.
\]
So, \( \mu_\omega(\| \exists^\infty n < \omega x \mid I_n = \dot{y}(n) \|) = 0 \) for all \( x \in 2^\omega \). This implies that
\[
\Vdash \forall^\infty n < \omega x(n) \neq \dot{y}(n), \text{ for all } x \in \prod_{n < \omega} I_n^2.
\]

**Lemma 1.3** Let \( 0 < K, M < \omega \). Suppose that \( \{ b_i^m \mid i < K \text{ and } m < M \} \subset \mathcal{B}(\omega) \) and \( b \in \mathcal{B}(\omega) \) satisfy
\[
b = \sum_{i < K} b_i^m, \text{ for all } m < M.
\]
Then, there is a function \( \varphi : M \to K \) such that
\[
\mu_\omega(\sum_{m < M} b_{\varphi(m)}^m) \geq \mu_\omega(b) - \left( \frac{K-1}{K} \right)^M \mu_\omega(b).
\]

**Proof** By induction on \( M \in [1, \omega) \). The case \( M = 1 \) is clear. Let \( M = M_0 + 1 > 1 \). Using the induction hypothesis, take \( \varphi_0 : M_0 \to K \) such that
\[
\mu_{\omega}(\sum_{m<M_{0}}b_{\varphi_{0}(m)}^{m}) \geq \mu_{\omega}(b) - (\frac{K-1}{K})^{M_{0}}\mu_{\omega}(b).
\]

Put \(c = \sum_{m<M_{0}}b_{\varphi_{0}(m)}^{m}\). Since \(b - c = \sum_{i<K}(b_{i}^{M_{0}} - c)\), there exists \(j < K\) such that \(\mu_{\omega}(b_{j}^{M_{0}}) \geq \frac{1}{K}\mu_{\omega}(b-c)\). Then, \(\varphi = \varphi_{0}(j)\) is as required. \(\square\)

For each \(n < \omega\), let
\[M_{n} = \min\{M < \omega \mid (\frac{n}{n+1})^{M} < 2^{-n}\}.\]
Define \(f_{1} \in \mathcal{F}\) by
\[\{|\{k < \omega \mid f_{1}(k) = n + 1\}\| = M_{n}, \text{ for all } n < \omega.\]
The next lemma implies that \(f_{1}\) satisfies the condition in Theorem 1.1.

Lemma 1.4 \(\models_{\mathbf{B}(\omega)} \forall y \in \prod_{k<\omega}f_{1}(k) \exists x \in \prod_{k<\omega}f_{1}(k) \cap \mathbf{V} \exists^{\infty}k < \omega x(k) = y(k)\).

Proof For each \(n < \omega\), put \(J_{n} = \{k < \omega \mid f_{1}(k) = n + 1\}\). To show this lemma, let \(\dot{y}\) be a \(\mathbf{B}(\omega)\)-name such that \(\models \dot{y} \in \prod_{k<\omega}f_{1}(k)\). For each \(n < \omega\), using Lemma 1.3, take \(s_{n} \in \prod_{k \in J_{n}}f_{1}(k)\) such that
\[\mu_{\omega}(\sum_{k \in J_{n}}|s_{n}(k) = \dot{y}(k)|) \geq 1 - (\frac{n}{n+1})^{M_{n}}\].
Put \(x = \bigcup_{n<\omega}s_{n}\). It is easy to check that
\[\mu_{\omega}(\|\forall^{\infty}n < \omega \exists k \in J_{n} x(k) = \dot{y}(k)\|) = 0.\]
So, it holds that \(\models \exists^{\infty}k < \omega x(k) = \dot{y}(k)\). \(\square\)

2 A forcing notion with the ccc which lifts up \(\theta^{*}\)

Define the forcing notion \((Q, \leq)\) by
\[Q \subset 2^{<\omega} \times [2^{\omega} \times H]^{<\omega}\]
and, for any \((s, u) \in 2^{<\omega} \times [2^{\omega} \times H]^{<\omega}\),
\[(s, u) \in Q \]
if and only if, for all \((x, h) \in u\) and all \(k < \omega\),
\[\text{if } a_{k}^{h} \setminus \text{dom}(s) \neq \phi \text{ then } |a_{k}^{h} \setminus \text{dom}(s)| \geq |u| \text{ or } \exists i \in a_{k}^{h} \cap \text{dom}(s) x(i) \neq s(i),\]
and, for any \((s, u), (s', u') \in Q\),
\[(s', u') \leq (s, u) \]
if and only if
\[s' \supset s \text{ and } u' \supset u \text{ and, for all } (x, h) \in u \text{ and all } k < \omega, \text{ if } a_{k}^{h} \cap (\text{dom}(s') \setminus \text{dom}(s)) \neq \phi \text{ then } |a_{k}^{h} \setminus \text{dom}(s')| \geq |u'| \text{ or } \exists i \in a_{k}^{h} \cap \text{dom}(s') x(i) \neq s'(i).\]
We show that a finite support iteration by the above forcing notion lifts up the value $\theta^*$. For this, we need several lemmas.

**Lemma 2.1** Let $n < \omega$. Then, for every $(s, u) \in Q$, there is $s' \in 2^{<\omega}$ such that $(s', u) \in Q$ and $(s', u) \leq (s, u)$ and $n \in \text{dom}(s)$.

**Proof** For each $j < \omega$, define $\varphi_j : \mathcal{H} \rightarrow \omega$ by
$$
\varphi_j(h) = \text{the unique } k < \omega \text{ such that } j \in a_k^h.
$$
For each $t \in 2^{<\omega}$, define $\psi_t : 2^{<\omega} \times \mathcal{H} \rightarrow \omega$ by
$$
\psi_t(x, h) = \begin{cases} 
|a_{\varphi_{\text{dom}(t)}(h)}^h \setminus \text{dom}(t)|, & \text{if } t \uparrow a_{\varphi_{\text{dom}(t)}(h)}^h \subset x, \\
|a_{\varphi_{\text{dom}(t)}(h)+1}^h|, & \text{otherwise}.
\end{cases}
$$
To show this lemma, let $n < \omega$ and $(s, u) \in Q$. Put $m = \text{dom}(s)$. Take $M < \omega$ such that $n, m < M$ and $|a_M^h \setminus M| \geq |u|$, for all $(x, h) \in u$.

By induction on $j \in [m, M]$, take $s_j : j \rightarrow 2$ as follows:

- Put $s_m = s$. Suppose that $j \in [m, M)$ and $s_j$ has been defined. Let $l_j$ be the smallest element of $\{ \psi_{s_j}(x, h) \mid (x, h) \in u \}$. Take $(x_j, h_j) \in u$ such that $\psi_{s_j}(x_j, h_j) = l_j$. Set $s_{j+1} = s_j \ominus 1 - x_j(j)$.

**Claim 2.2** $|\{(x, h) \in u \mid \psi_{s_j}(x, h) < l\}| < l$, for all $0 < l < \omega$ and $j \in [m, M]$.

By induction on $j \in [m, M]$. The case $j = m$ is followed from the fact $(s, u) \in Q$. The case $j = j_0 + 1 > m$ is followed from the fact $\psi_{s_j}(x_{j_0}, h_{j_0}) \geq |u|$. △

By Claim 2.2, it holds that $l_j > 0$, for all $j \in [m, M)$. So, it holds that $(s_M, u) \in Q$ and $(s_M, u) \leq (s, u)$.

**Lemma 2.3** For each $(x, h) \in 2^{<\omega} \times \mathcal{H}$,
$$
\{ (s, u) \in Q \mid (x, h) \in u \} \text{ is dense in } Q.
$$

**Proof** Let $(x, h) \in 2^{<\omega} \times \mathcal{H}$ and $(s, u) \in u$. Take $M < \omega$ such that
\begin{enumerate}
  \item $|s| \leq M$,
  \item if $a_k^{h'} \setminus M \neq \phi$ then $|a_k^{h'} \setminus M| > |u|$, for all $k < \omega$ and $(x', h') \in u$.
  \item if $a_k^h \setminus M \neq \phi$ then $|a_k^h \setminus M| > |u|$, for all $k < \omega$.
\end{enumerate}
Using Lemma 2.1, take $(t, u) \leq (s, u)$ such that $\text{dom}(t) = M$. Then, it holds that $(t, u \cup \{(x, h)\}) \in Q$ and $(t, u \cup \{(x, h)\}) \leq (s, u)$.

**Lemma 2.4** $Q$ satisfies the countable chain condition.
**Proof** Let $W$ be an uncountable subset of $Q$. Using Lemma 2.1, replace $W$ by certain stronger conditions if necessary, we may assume that, for all $(s, u) \in W$,

for all $(x, h) \in u$ and $k < \omega$, if $a_k^h \setminus k \neq \phi$ then $|a_k^h \setminus k| \geq 2|u|$.

Take $s_0 \in 2^{<\omega}$ and $m < \omega$ such that $W' = \{(s, u) \in W \mid s = s_0$ and $|u| = m \}$ is uncountable. Then, every elements in $W'$ are compatible. \hfill \Box

Let $\hat{G}$ be the canonical generic $Q$-name. Define $\hat{g}$ by

\[ \Vdash_Q \hat{g} = \bigcup \{ (s, u) \in \hat{G} \text{, for some } u \}. \]

**Lemma 2.5** $\Vdash_Q \hat{g} \in 2^\omega$ and $\forall x \in 2^\omega \cap V \forall h \in H \cap V \forall n < \omega \hat{g} \upharpoonright a_n^h \neq x \upharpoonright a_n^h$.

**Proof** This is directly followed from Lemmas 2.1 and 2.3. \hfill \Box

Let $\kappa$ be a regular uncountable cardinal and $P$ the $\kappa$-stage finite support iteration by the above forcing $Q$. Then, by the above arguments, it holds that $\theta^* = \kappa$ in the generic model $V^P$. Since $P$ is finite support, it adds cofinally many Cohen reals. So, in $V^P$, the covering number $\text{cov}(\mathcal{M})$ of the meager ideal on the real line lifts up to $\kappa$. Furthermore, the next lemma shows that the unbounding number $b$ of $\omega^\omega$ lifts up to $\kappa$, too.

**Lemma 2.6** There is a $Q$-name $\hat{d}$ such that

\[ \Vdash_Q \hat{d} \in \omega^\omega \text{ dominates } \omega^\omega \cap V. \]

**Proof** For each set $X$, denote by $0_X$ the constantly zero function from $X$ to $2$.

**Claim 2.7** For any $n < \omega$,

\[ \{ (s, u) \in Q \mid \exists m < \omega \ (0_{[m, m+n]} \subset s) \} \text{ is dense in } Q. \]

Let $n < \omega$ and $(s, u) \in Q$. Take $(t, u) \leq (s, u)$ such that, for all $(x, h) \in u$ and $k < \omega$,

if $a_k^h \setminus \text{dom}(t) \neq \phi$ then $|a_k^h \setminus \text{dom}(t)| \geq |u| + n$.

Define $t' : |t| + n \rightarrow 2$ by $t \subset t'$ and $t'(|t| + j) = 0$, for $j < n$. It is easy to check that $(t', u) \in Q$ and $(t', u) \leq (s, u)$. \hfill \triangle

By Claim 2.7, it holds that

\[ \Vdash_Q \forall n < \omega \exists m < \omega \hat{g} \upharpoonright [m, m+n) = 0_{[m, m+n]}. \]

So, in $V^Q$, define $\hat{d} \in \omega^\omega$ by

\[ \hat{d}(n) = \text{the smallest } m < \omega \text{ such that } n \leq m \text{ and } \hat{g} \upharpoonright [m, m+2n) = 0_{[m, m+2n]}. \]

To show $\hat{d}$ is a required one, let $f \in \omega^\omega$ and $(s, u) \in Q$. Without loss of generality, we may assume that $f$ is strictly increasing. Take $h \in H$ such that
$|a_{k}^{h}| \leq |a_{k+1}^{h}|$, for all $k < \omega$ and $|\{ k < \omega \mid |a_{k}^{h}| = n \}| \geq f(n) + 1$, for all $n < \omega$.

By Lemma 2.3, take $(t, v) \leq (s, u)$ such that $(0_{\omega}, h) \in v$. Let $k_{0}$ be the smallest $k < \omega$ such that $|t| \geq h(k)$ and set $n_{0} = |a_{k_{0}}^{h}| + |t|$. The next claim completes the proof of the lemma.

**Claim 2.8** $(t, v) \text{Q-\forall } n > n_{0} f(n) < \dot{d}(n)$

.: To get a contradiction, assume that there are $(t', v') \leq (t, v)$ and $n > n_{0}$ such that $(t', v') \text{Q-\forall } \dot{d}(n) \leq f(n)$. Replace $(t', v')$ by a stronger condition if necessary, we may assume that $(t', v')$ decides the value of $\dot{d}(n)$. Let $m < \omega$ be such that $(t', v') \text{Q-\forall } \dot{d}(n) = m$. Without loss of generality, we may assume that $m + 2n \subset \text{dom}(t')$. Let $k$ be the unique $k < \omega$ such that $m \in a_{k}^{h}$. By the choice of $h$, it holds that $|a_{k}^{h}|, |a_{k+1}^{h}| \leq n$.

.: $a_{k+1}^{h} \subset [m, m + 2n)$. Since $(t', v') \text{Q-\forall } \dot{g} \uparrow [m, m + 2n) = 0_{(m, m + 2n)}$, it holds that $t' \uparrow a_{k+1}^{h} = 0_{a_{k+1}^{h}}$. This contradicts the facts that $(t', v') \leq (t, v)$ and $(0_{\omega}, h) \in v$ and $\text{dom}(t) \cap [m, m + 2n) = \phi$. \qed

In section 4, we give a generic model in which holds $\theta^{*} = \omega_{2}$ and $\text{cov}(M) = \omega_{1}$. But I do not known whether there is a model which satisfies $b < \theta^{*}$.

**Question 2.1** Is $b < \theta^{*}$ consistent with ZFC?

### 3 A forcing notion which lifts up $\theta$

In this section, we give a forcing notion which gives a generic model of $\theta^{*} = \omega_{1}$ and $\theta = \omega_{2}$. The forcing notion which we give here is constructed by the $\omega_{2}$-stage countable support iteration. We begin with the definition of a forcing notion $\text{BT}_{f}$ for $f \in \mathcal{F}$ which will be used each stage in the iteration.

Let $f \in \mathcal{F}$. For each $n < \omega$, denote $\prod_{m < n} f(m)$ by $S_{n}^{f}$. Put $S^{f} = \bigcup_{n < \omega} S_{n}^{f}$. Note that $(S^{f}, \subset)$ is a tree. Define the forcing notion $(\text{BT}_{f}, \leq)$ by

$q \in \text{BT}_{f}$

if and only if

(1) $q$ is a subtree of $S^{f}$.

(2) there is a function $f' \in \mathcal{F}$ such that $|\text{succ}_{q}(s)| \geq f'(|s|)$ for every $s \in q$.

$q' \leq q$ if and only if $q' \subset q$.

For each $q \in \text{BT}_{f}$, define $\pi_{q} \in \omega^{\omega}$ by

$\pi_{q}(n) = \max\{ k < \omega \mid \forall n' \geq n \forall s \in q \cap S_{n'}^{f} |\text{succ}_{q}(s)| \geq k \}$.
Note that $\pi_q \in \mathcal{F}$ for all $q \in \mathcal{B}\mathcal{T}_f$. For each $k < \omega$, define the ordering $\leq_k$ on $\mathcal{B}\mathcal{T}_f$ by

$\quad q' \leq_k q$ if and only if $q' \leq q$ and $\pi_q \upharpoonright m_k = \pi_{q'} \upharpoonright m_k$,

where $m_k$ denotes the smallest $m < \omega$ such that $\pi_q(m) > k$.

In [2], Bartoszyński, Judah and Shelah have used similar but more complicated forcing notions $Q_{f,g}$. The proof of the next lemma is similar to, but quite easier than the proof of Claim 2.6 in [2].

**Lemma 3.1** Let $\dot{e}$ be a $\mathcal{B}\mathcal{T}_f$-name such that $\Vdash \dot{e} \in \mathcal{V}$. Then, for each $k < \omega$ and $q \in \mathcal{B}\mathcal{T}_f$, there are $q' \leq_k q$ and a finite set $E$ such that $q' \Vdash \dot{e} \in E$.

**Proof** Let $\dot{e}$, $k < \omega$, $q \in \mathcal{B}\mathcal{T}_f$ be as in the lemma. For each $s \in q$, denote by $q[s]$ the condition $\{ t \in q \mid s \subset t \text{ or } t \subset s \}$. Take $M < \omega$ such that $\pi_q(M) \geq 2k$. Set

$T = \{ s \in q \mid |s| \geq M \text{ and } \exists q' \leq_k q[s] \exists E \ (E \text{ is finite and } q' \Vdash \dot{e} \in E \} \}$.

Note that, whenever $s \in q \setminus T$ and $|s| \geq M$, $|\text{succ}_q(s) \cap T| < k$.

**Claim 3.2** $q \cap S_M^{f} \subset T$.

$\therefore$ To get a contradiction, assume that $s \in q \setminus S_M^{f} \setminus T$. Let $U = \{ t \in q \setminus T \mid s \subset t \}$. Then, it holds that

$\forall t \in U \ (\cup \{ u \in U \mid t \subset u \text{ and } |u| = |t| + 1 \} > \pi_q(|u| - k)).$

This implies that $r = \{ s \cap j \mid j < |s| \} \cup U \in \mathcal{B}\mathcal{T}_f$ and $r \leq_k q[s]$. Take $r' \leq r$ such that $r'$ decides $\dot{e}$. Take $t \in r'$ such that $\pi_{r'}(|t|) \geq k$. Since $r'[t] \leq_k q[t]$, we have that $t \in T$. This contradicts that $U \cap T = \phi$. \triangle

By Claim 3.2, for each $s \in q \cap S_M^{f}$, take $q_s \leq_k q[s]$ and a finite set $E_s$ such that $q_s \Vdash \dot{e} \in E_s$. Then $q' = \bigcup_{s \in q \cap S_M^{f}} q_s$ and $E = \bigcup_{s \in q \cap S_M^{f}} E_s$ satisfy this lemma. \square

**Corollary 3.3** $(\mathcal{B}\mathcal{T}_f, (\leq_k)_{k<\omega})$ satisfies Axiom A and $\mathcal{B}\mathcal{T}_f$ is $\omega^\omega$-bounding. \square

Let $\mathcal{G}$ be the canonical generic $\mathcal{B}\mathcal{T}_f$-name. Define $\mathcal{B}\mathcal{T}_f$-name $\dot{g}$ by

$\Vdash \dot{g} = \bigcup (\cap \mathcal{G}) \in \prod_{n < \omega} f(n)$.

Then, it is easy to check that

$\Vdash \forall x \in \prod_{n < \omega} f(n) \cap \mathcal{V} \forall n < \omega \ \dot{g}(n) \neq x(n)$.

Now we can describe how to construct a model which satisfies $\theta = \omega_2$ and $\theta^* = \omega_1$. Start with a ground model with CH. Let $\{ f_{\alpha} \mid \alpha < \omega_2 \} \subset \mathcal{F}$ be such that

$\{ \alpha < \omega_2 \mid f_{\alpha} = f \}$ is cofinal in $\omega_2$ for each $f \in \mathcal{F}$. 

Define the $\omega_2$-stage countable support iteration $P_\alpha$ (for $\alpha \leq \omega_2$), $\dot{Q}_\alpha$ (for $\alpha < \omega_2$) by
\[ \Vdash_\alpha \dot{Q}_\alpha = \text{BT}_{f_\alpha}. \]
Let $P = P_{\omega_2}$. Then, by the above arguments, it holds that, in $V^P$, $\theta = \omega_2$ and $d = \omega_1$. Since $\text{cof}([\omega_1]^\omega, \subset) = \omega_1$ does always hold, it holds that, in $V^P$, $\theta^* \leq d = \omega_1$.

4 A generic model of $\theta = \omega_2$ and $\text{cov}(M) = \omega_1$

In the previous section, we show that $\text{BT}_f$ does not lift up $\theta^*$. But, if we first add a dominating real then we get a certain function $f \in F$ such that $\text{BT}_f$ lifts up $\theta^*$. In this section, we show that $\theta^*$ can be separated from $\text{cov}(M)$ by using it.

Lemma 4.1 Let $V$, $W$ be transitive models of ZFC such that $V \subset W$. Assume that $d \in W \cap \omega^\omega$ dominates $V \cap \omega^\omega$. In $W$, define $h \in H$ by
\[ |a_k^h| \leq |a_{k+1}^h|, \text{ for all } k < \omega \text{ and } |\{ k < \omega \mid |a_k^h| = n \}| = d(n) + 1, \text{ for all } n < \omega. \]
Then, it holds that $\forall^\infty m < \omega \exists k < \omega \ a_k^h \subset a_m^h$ for all $h' \in V \cap H$.

Proof Let $h' \in V \cap H$. In $V$, define $f_0, f_1 \in \omega^\omega$ by
\[ f_0(n) = \text{the smallest } m < \omega \text{ such that } \forall m' \geq m \ a_{m'}^{h'} \geq 2n, \text{ and } \]
\[ f_1(n) = \max a_{f_0(n) + 1}^{h'}. \]
Since $d$ dominates $f_0, f_1$, there is $n_0 < \omega$ such that $\forall n \geq n_0 \ f_0(n), f_1(n) < d(n)$. Put $k_0 = f_0(n_0)$. To show that $\forall k \geq k_0 \exists j < \omega \ a_j^h \subset a_k^h$, let $k \geq k_0$. Take $n < \omega$ such that $f_0(n) \leq k < f_0(n + 1)$. Then, it holds that $|a_k^h| \geq 2n$ and $\max a_k^h = \max a_{f_0(n+1)}^{h'} = f_1(n) \leq d(n)$. Since $[0, d(n))$ is covered by $\{ a_j^h \mid j < d(n) \}$ and $|a_j^h| \leq n$ for all $j < d(n)$, there is $j < d(n)$ such that $a_j^h \subset a_k^h$.

Lemma 4.2 Let $V$, $W$, $d$ and $h$ be as in Lemma 4.1. Working in $W$. Define $f \in F$ by
\[ f(k) = 2^{|a_k^h|}, \text{ for all } k < \omega. \]
Then, there is a $\text{BT}_f$-name $\dot{y}$ such that
\[ \Vdash \dot{y} \in 2^\omega \text{ and } \forall^\infty k < \omega \ y \upharpoonright a_k^h \neq x \upharpoonright a_k^h, \text{ for all } x \in 2^\omega \cap V \text{ and } h' \in H \cap V. \]

Proof Working in $W$. Considering bijections from $f(k)$ to $\omega^2$ for $k < \omega$, we may identify $\Pi_{k<\omega} f(k)$ with $\prod_{k<\omega} \omega^2$. Let $\dot{G}$ be the canonical generic $\text{BT}_f$-name. Define $\text{BT}_f$-names $\dot{g}$ and $\dot{y}$ by
\[ \Vdash \dot{g} = \bigcup (\cap \dot{G}) \text{ and } \dot{y} = \bigcup \dot{g}(k). \]
Note that $\vdash \dot{y} \in \prod_{k<\omega} a_k^h 2$ and $\dot{y} \in 2^\omega$. It is easy to check that

$$\vdash \forall x \in 2^\omega \cap W \forall^\infty k < \omega \ \dot{y} \upharpoonright a^h_k \neq x \upharpoonright a^h_k.$$ 

To show $\dot{y}$ is as required, let $x \in V \cap 2^\omega$ and $h' \in V \cap H$. Since it holds that $x \in W$ and $\forall^\infty m < \omega \exists k < \omega a^h_k \subset a^h_m$, we have that

$$\vdash \forall^\infty m < \omega \ \dot{y} \upharpoonright a^h_{m'} \neq x \upharpoonright a^h_m.$$ 

\[ \square \]

**Corollary 4.3**  Assume that CH holds. There are a forcing notion $R$ and $R$-name $\dot{y}$ such that

1. $R$ is proper and does not add a Cohen real and $|R| = \omega_1$.
2. $\vdash_R \dot{y} \in 2^\omega$ and $\forall x \in 2^\omega \cap V \forall h \in H \cap V \forall^\infty k < \omega \ \dot{y} \upharpoonright a^h_k \neq x \upharpoonright a^h_k$.  

\[ \square \]

Using Corollary 4.3, we can construct a generic model which satisfies $\text{cov}(M) = \omega_1$ and $\theta^* = \omega_2$. Start with a ground model with CH. Take an $\omega_2$-stage countable support iteration by the forcing notion as in Corollary 4.3. Since the iteration does not add a Cohen real, $\text{cov}(M)$ remains $\omega_1$. On the other hand, since functions $\dot{y} \in 2^\omega$ which satisfy (2) in the corollary is added cofinally, $\theta^*$ must be lifted up.

**References**

