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Optimal stopping problem with reservation where the reserving cost for the recalled offer returns

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Abstract: The author had researched optimal stopping problems where each of offers appearing subsequently and randomly can be reserved as well as accepted or rejected in [2][3][4][5][6]. In these models, it is assumed that the cost paid for an offer reserved did not return at the time of its recalling. So, the model in this paper regards it a deposit, that is, the reserving cost for the offer recalled will be paid back at the time of recalling. A major finding is that almost all the properties in [2] keep also in the current model, that is, an offer reserved during the search process must not be recalled prior to the deadline.

1. Introduction

The author had studied optimal stopping problems where a decision maker is allowed to reserve offers as well as accept or reject them. In [2] he dealt with a basic model that every offer reserved is assumed to be recallable at any time after reservation. In [3] every reservation is restricted to expire after k periods from reserving point where k is a finite fixed number. A major common result of them is that an offer reserved during the search process must not be recalled prior to its maturity of reservation or the deadline of the process. The author felt strange because he thought it natural that we could reach a time at which we began to feel wasteful for pursuing the search process further, and decided at that time to stop the process by recalling an offer reserved.

So as to confirm whether the property is essential to our stopping problems, in [4] we studied a model where remaining periods after termination of the process is assumed to be exchangeable for some monetaly value. In [6] a decision maker got ability to decide the term of reservation as well as what offer to be reserved. In these models, however, we obtained the same property as above.

Before determinating that the property is essential, we should check an influence by the way of treatment of the reserving cost. In these models, the cost to reserve an offer is not repayed even when the offer is recalled. In many real business, however, we can draw back the cost for reservation at the time of transaction. Therefore, in the present paper, we shall deal with such a model that the reserving cost for the recalled offer returns at the time of recalling. A main result is that the property above keeps also in this new model.

The model is precisely described in section 2. Preliminaries for the analyses are introduced in section 3, and the optimal equation of the model is formulated in section 4. In section 5 we analyze the properties of the optimal decision rule, and they are summarized in section 6.

2. Model

Suppose you must find and get as good an offer as possible during a time horizon you have so as to search for offers. Offers can be found at periodic intervals, but at any point in time, if you want to ovserve one, a *search cost* $s > 0$ is to be paid at the preceding point in time. The values of offers appearing subsequently are i.i.d. random variables following a distribution F such that $F(w) = 0$ for $w < 0$, $0 < F(w) < 1$ for $0 \le w < b$, $F(w) = 1$ for $b \le w$ with $0 < b < \infty$, and the expectation is μ.

Every time you find a new offer, called a *current offer*, you must inspect it and decide how to manage it. The choices available are accepting it, passing it up, and reserving it. Reserving an offer, of course, enables you to recall it at any time after that, but it is not free. So as to reserve an offer w, you must pay a *reserving cost* $r(w)$, but it returns when you recall the offer. That is, if you recall a reserved offer x, you will obtain not only the value x but also $r(x)$ at the time of recalling. Passing up the current offer, you lose the right to recall it forever.

Although you have some offers recallable, there is only one offer to be remembered because you can accept only one offer. It is, of course, the most lucrative of all reserved offers. Let us call it *leading offer*. Then, the actions which can be taken at each time except for the deadline are summarized as the following four: accepting the current offer and stopping the search (AS), reserving the current offer and continuing the search (RC), passing up the current offer and stopping the search by recalling the leading offer (PS), and passing up the current offer and continuing the search (PC), where AS, RC, PS, and PC represent the four decisions, respectively. At the deadline, only decisions AS and PS are permitted.

In the model, we consider the value of time by a discount factor β such as $0 < \beta \leq 1$, that is, the present value of q monetary units obtained at the next time is given by βq monetary units.

The objective of this model is to find an optimal decision rule that guides you to which action should be taken at each decision point so as to maximize the total expected discounted present net profit obtainable in the process ahead. That profit is the expectation of the present discounted value you can gain at the time of stopping the search, that is, accepting or recalling an offer, minus the expectation of the amount of search costs and reserving costs paid over the periods from the present point in time to the termination of the search.

Finally, let us make the following assumptions.

(1) $\beta\mu > s$.

(2) $r(w)$ is continuous and nondecreasing in w, and $0 = r(0) \leq r(b) < \infty$.

(3) $r(w) = 0$ for $w \leq 0$ and $r(w) = r(b)$ for $b \leq w$.

Assumption(1) is to avoid trivial case where continuation of search lose meanings. Since $\beta\mu - s$ is the expected discounted net profit attainable from one more search, if $\beta\mu \leq s$, everyone hopes to quit the searching as early as possible. Althogh the proof is omitted here, it can be shown theoretically.

As for Assumption(2), nondecreasing property means that the higher is the value of offer, the more is the cost needed to reserve it. The inequality $0 = r(0) \leq r(b) < \infty$ may be intuitively clear. Continuation and Assumptions(3) are introduced only for analytic reasons.

3. Preliminaries

Let us define a function ψ as

$$\psi(x) = \int_0^b \max\{w, x\} dF(w), \tag{3.1}$$

which can be expanded as

$$\psi(x) = \mu + \int_0^x F(w)\mathrm{d}w. \tag{3.2}$$

Lemma 3.1

(a) $\psi(x) = \mu$ *for* $x \leq 0$, $x < \psi(x)$ *for* $x < b$, *and* $\psi(x) = x$ *for* $b \leq x$.

(b) $\psi(x)$ *is continuous, convex, and nondecreasing in* x, *and strictly increasing in* $x \geq 0$.

Proof: See Ikuta [1]. ∎

4. Optimal Equation and Optimal Decision Rule

Let point in time t, simply referred to as *time t* later on, be equally spaced and numbered backward from the deadline $t = 0$, thus t also represents the number of periods remaining.

Suppose that we are at time t with having an offer x as the leading offer and w as the current offer. Then, if taking decision AS, we stop searching by getting the offer w. If taking decision PS, we terminate searching by receiving not only the offer x but also $r(x)$, which is the reserving cost paid at the time of reserving the offer x.

When we choose decision PC, we are to continue the search by paying the search cost s, and the leading offer of the next point in time remains to be x. As for decision RC, we first pay the reserving cost $r(w)$ for the offer w, and then spend the search cost s. The leading offer of the next time is the larger one between w and x, that is, $\max\{w, x\}$. But it can be easily shown that at any time, we have no need to reserve an offer as long as its value is inferior to that of the leading offer of that time. Hence, if taking decision RC at time t, we can regard w as the leading offer of next time, that is, time $t-1$.

Now, let us denote $u_t(w, x)$ the maximum total expected present discounted net profit by starting time t on which we have a current offer w and the leading offer x. Then, it follows that

$$u_t(w,x) = \max \left\{ \begin{array}{lll} \text{AS} & : & w, \\ \text{RC} & : & -r(w) - s + \beta v_{t-1}(w), \\ \text{PS} & : & x + r(x), \\ \text{PC} & : & -s + \beta v_{t-1}(x) \end{array} \right\}, \quad t \geq 0, \tag{4.1}$$

where

$$v_t(x) = \int_0^b u_t(w,x) \mathrm{d}F(w), \quad t \geq 0; \qquad v_{-1}(x) = -\infty. \tag{4.2}$$

We immediately get $u_0(w, x) = \max\{w, x+r(x)\}$, which indicates that we must stop the process on the deadline by accepting either the current offer or the leading offer. Note here that the equality, (3.1) and (3.2) yield, for any $x \in \mathbb{R}$,

$$v_0(x) = \psi(x + r(x)) = \mu + \int_0^{x+r(x)} F(w)\mathrm{d}w. \tag{4.3}$$

Let us define two functions $z_t^o(x)$ and $z_t^r(w)$ as follows.

$$z_t^o(x) = \max\{x + r(x), \ -s + \beta v_{t-1}(x)\}, \qquad t \geq 0, \tag{4.4}$$

$$z_t^r(w) = \max\{w, \ -r(w) - s + \beta v_{t-1}(w)\}, \qquad t \geq 0. \tag{4.5}$$

Clearly, $z_0^o(x) = x + r(x)$ and $z_0^r(w) = w$. From (4.1) we find that $z_t^o(x)$ and $z_t^r(w)$ stand for the total expected present discounted net profits attainable by passing up or not passing up the current offer w, respectively, at time t, and then following the optimal strategy. Therefore, the set of current offers which should be accepted or reserved can be denoted by

$$W_t(x) = \{w : z_t(x) \leq z_t(w)\}, \qquad t \geq 0. \tag{4.6}$$

From above,

$$v_t(x) = \int_0^b \max\{z_t^r(w), z_t^o(x)\}\mathrm{d}F(w).$$

Finally, we introduce a function f as

$$f_t(x) = \beta v_{t-1}(x) - (x + r(x)) - s, \qquad t \geq 1. \tag{4.7}$$

And let λ_t denote a solution of equation $f_t(x) = 0$, if exists, that is,

$$f_t(\lambda_t) = \beta v_{t-1}(\lambda_t) - (\lambda_t + r(\lambda_t)) - s = 0, \qquad t \geq 1. \tag{4.8}$$

Comparing (4.4) and (4.5) with (4.7), we find that λ_t is the indifferent point of time t in terms of x between decisions PS and PC as well as the indifferent point of time t in terms of x between decisions AS and RC.

Therefore, the optimal decision rule can be described by using $W_t(x)$ and λ_t.

5. Analysis

Lemma 5.1

(a) *$v_t(x)$ is continuous and nondecreasing in x for any $t \geq 0$, and nondecreasing in t for any x.*
(b) *$v_t(x) \geq \mu$ for any x, $x + r(x) < v_t(x)$ for $x < b$, and $v_t(b) = b + r(b)$.*

Proof: (a) Since $v_0(x) = \psi(x + r(x))$, the assertion holds true for $t = 0$ due to Lemma 3.1(b).

Suppose the assertion holds true for $t-1$. Then, the expression for PS and PC in (4.1) both satisfy the two properties because $r(x)$ is assumed to be continuous and nondecreasing in x. So also does the expression for AS since it is independent of x. For any w, both $r(w)$ and $v_{t-1}(w)$ are constant in x. Hence, the expression for RC also has the properties. Since all four expressions in the braces of (4.1) satisfy the two properties, so also does $v_t(x)$.

As for the assertion with t, it can be easily shown by induction starting with $v_1(x) \geq \int_0^b \max\{w, x + r(x)\}\mathrm{d}F(w) = v_0(x)$.

(b) First, clearly we have $v_t(x) \geq \int_0^b w\mathrm{d}F(w) = \mu$ for every t. Secondly, if $x < b$, we get $x + r(x) < \psi(x + r(x)) = v_0(x) \leq v_1(x) \leq \cdots$ due to Lemma 3.1(b) and assertion (a).

Finally, it easily shown that $u_t(w, b) = b + r(b)$ for any w such that $0 \leq w \leq b$ due to assertion (a), which implies that $v_t(b) = b + r(b)$. ∎

Lemma 5.2 *For any $t \geq 0$ and $\rho \geq 0$, we have $v_t(x+\rho) - v_t(x) \leq \rho + r(x+\rho) - r(x)$.*

Proof: Fix any $\rho \geq 0$. Clearly, $\rho + r(x+\rho) - r(x) \geq 0$. It follows from (4.3) that

$$\begin{aligned} v_0(x+\rho) - v_0(x) &= \left(\mu + \int_0^{x+\rho+r(x+\rho)} F(w)\mathrm{d}w\right) - \left(\mu + \int_0^{x+r(x)} F(w)\mathrm{d}w\right) \\ &= \int_{x+r(x)}^{x+\rho+r(x+\rho)} F(w)\mathrm{d}w \leq \int_{x+r(x)}^{x+\rho+r(x+\rho)} \mathrm{d}w = \rho + r(x+\rho) - r(x), \end{aligned}$$

which implies that the assertion holds true for $t = 0$.

Suppose the assertion holds for $t-1$, that is, $v_{t-1}(x+\rho) - v_{t-1}(x) \leq \rho + r(x+\rho) - r(x)$. For any real numbers A_1, A_2, B_1 and B_2 we generally have

$$\max\{A_1, A_2\} - \max\{B_1, B_2\} \leq \max\{A_1 - B_1, A_2 - B_2\}. \tag{5.1}$$

From these two relations and (4.4), we obtain

$$\begin{aligned} z_t^o(x+\rho) - z_t^o(x) &\leq \max\{\rho + r(x+\rho) - r(x), \beta(v_{t-1}(x+\rho) - v_{t-1}(x))\} \\ &\leq \max\{\rho + r(x+\rho) - r(x), \beta(\rho + r(x+\rho) - r(x))\} = \rho + r(x+\rho) - r(x). \end{aligned}$$

By using this, (??) and (5.1),

$$\begin{aligned} v_t(x+\rho) - v_t(x) &= \int_0^b \Big(\max\{z_t^r(w), z_t^o(x+\rho)\} - \max\{z_t^r(w), z_t^o(x)\}\Big)\mathrm{d}F(w) \\ &\leq \int_0^b \max\{0, z_t^o(x+\rho) - z_t^o(x)\}\mathrm{d}F(w) \\ &\leq \int_0^b \max\{0, \rho + r(x+\rho) - r(x)\}\mathrm{d}F(w) = \rho + r(x+\rho) - r(x). \end{aligned}$$

Since we have also confirmed the truth of the assertion for t, the inductive proof is completed. ∎

Let x_b denote the root of equation $x + r(x) = b$, which exists uniquely with satisfying $0 < x_b < b$. Because $x + r(x)$ is continuous and strictly increasing, and there holds $0 + r(0) = 0 < b < b + r(b)$.

We then have

$$x + r(x) \begin{cases} < b & \text{for } x < x_b, \\ = b & \text{for } x = x_b, \\ > b & \text{for } x > x_b. \end{cases} \tag{5.2}$$

Lemma 5.3

(a) $f_t(x)$ *is nonincreasing on* $\mathbb{R}$*, and strictly decreasing in* $x \le x_b$.

(b) $f_t(x) < 0$ *for* $x_b \le x$.

(c) $v_t(x) \le b$ *for* $x \le x_b$.

Proof: (a) Choose any x_1 and x_2 such that $x_1 < x_2$. For $t = 1$, it follows from (4.3) and (4.7) that

$$\begin{aligned} f_1(x_2) - f_1(x_1) &= \beta \int_0^{x_2+r(x_2)} F(w)\mathrm{d}w - \beta \int_0^{x_1+r(x_1)} F(w)\mathrm{d}w - (x_2 + r(x_2)) + (x_1 + r(x_1)) \\ &= \beta \int_{x_1+r(x_1)}^{x_2+r(x_2)} F(w)\mathrm{d}w - \int_{x_1+r(x_1)}^{x_2+r(x_2)} \mathrm{d}w \;=\; \int_{x_1+r(x_1)}^{x_2+r(x_2)} (\beta F(w) - 1)\,\mathrm{d}w. \end{aligned} \tag{5.3}$$

By noting that $\beta F(w) - 1 \le 0$ for any w such that $x_1 + r(x_1) < w < x_2 + r(x_2)$, from (5.3) we have $f_1(x_2) - f_1(x_1) \le 0$. Furthermore, if $x_2 < x_b$, then $x_1 + r(x_1) < x_2 + r(x_2) < b$, thus $\beta F(w) - 1 < 0$ for any w such that $x_1 + r(x_1) < w < x_2 + r(x_2)$. Hence, (5.3) implies $f_1(x_2) - f_1(x_1) < 0$ when $x_1 < x_2 < x_b$. Therefore, we conclude the truth of the assertion for $t = 1$.

Next, suppose the assertion is true for t, that is, $f_t(x_1) \ge f_t(x_2)$ when $x_1 < x_2$, and especially $f_t(x_1) > f_t(x_2)$ when $x_1 < x_2 < x_b$.

We shall now show that $v_t(x) - (x + r(x))$ is nonincreasing on $\mathbb{R}$ and strictly decreasing in $x \le x_b$. Applying (5.1) we gain

$$v_t(x_i) - (x_i + r(x_i)) = \int_0^b \max \left\{ \begin{array}{l} w - (x_i + r(x_i)), \\ -r(w) - s + \beta v_{t-1}(w) - (x_i + r(x_i)), \\ 0, \\ -s + \beta v_{t-1}(x_i) - (x_i + r(x_i)) \quad (= f_t(x_i)) \end{array} \right\} \mathrm{d}F(w), \quad i = 1, 2. \tag{5.4}$$

From this, (5.1) and the assumption $f_t(x_2) - f_t(x_1) \le 0$, we obtain

$$v_t(x_2) - (x_2 + r(x_2)) - v_t(x_1) + (x_1 + r(x_1)) \tag{5.5}$$

$$\le \int_0^b \max \left\{ \begin{array}{l} -x_2 - r(x_2) + x_1 + r(x_1), \\ -x_2 - r(x_2) + x_1 + r(x_1), \\ 0, \\ f_t(x_2) - f_t(x_1) \end{array} \right\} \mathrm{d}F(w) = \int_0^b 0 \; \mathrm{d}F(w) = 0, \tag{5.6}$$

which indicates that $v_t(x) - (x + r(x))$ is nonincreasing on $\mathbb{R}$.

For a while from here, assign the four expressions in the braces of (5.4) A_i, B_i, C_i, and D_i, respectively, in order.

Suppose $x_2 < x_b$. Then $x_1 + r(x_1) < x_2 + r(x_2) < b$, thus $0 < w - (x_2 + r(x_2)) < w - (x_1 + r(x_1))$, that is, $C_2 = C_1 < A_2 < A_1$ for any w such that $x_2 + r(x_2) < w \le b$. Hence,

$$\begin{aligned} &v_t(x_i) - (x_i + r(x_i)) \\ &= \int_0^{x_2+r(x_2)} \max\{A_i, B_i, C_i, D_i\}\mathrm{d}F(w) + \int_{x_2+r(x_2)}^b \max\{A_i, B_i, D_i\}\mathrm{d}F(w), \quad i = 1, 2. \end{aligned} \tag{5.7}$$

In a similar fashion to (5.6), we have

$$\int_0^{x_2+r(x_2)} \max\{A_2, B_2, C_2, D_2\}\mathrm{d}F(w) - \int_0^{x_2+r(x_2)} \max\{A_1, B_1, C_1, D_1\}\mathrm{d}F(w) \le 0. \tag{5.8}$$

Lemma 5.1(a) indicates

$$\begin{aligned} D_2 - D_1 = f_t(x_2) - f_t(x_1) &= \beta v_{t-1}(x_2) - \beta v_{t-1}(x_1) - x_2 - r(x_2) + x_1 + r(x_1) \\ &\geq -x_2 - r(x_2) + x_1 + r(x_1) = A_2 - A_1 = B_2 - B_1. \end{aligned}$$

From this, the assumption $f_t(x_2) - f_t(x_1) < 0$, and $F(x_2 + r(x_2)) < 1$, we get

$$\begin{aligned} &\int_{x_2+r(x_2)}^{b} \max\{A_2, B_2, D_2\} \mathrm{d}F(w) - \int_{x_2+r(x_2)}^{b} \max\{A_1, B_1, D_1\} \mathrm{d}F(w) \\ &= \int_{x_2+r(x_2)}^{b} (D_2 - D_1) \mathrm{d}F(w) \;=\; (f_t(x_2) - f_t(x_1))\,(1 - F(x_2 + r(x_2))) < 0. \end{aligned} \tag{5.9}$$

Owing to (5.7), (5.8) and (5.9), we deduce $v_t(x_2) - (x_2 + r(x_2)) - v_t(x_1) + (x_1 + r(x_1)) < 0$, implying that $v_t(x) - (x + r(x))$ is strictly decreasing in $x < x_b$.

Finally, by noting that $f_{t+1}(x) = \beta\,(v_t(x) - (x + r(x))) + (\beta - 1)(x + r(x)) - s$, we arrive at the truth of the assertion.

(b) and (c) shall be proven together by induction on t. Easily we have, for $x \leq x_b$,

$$v_0(x) = \int_0^b \max\{w, x + r(x)\} \mathrm{d}F(w) \leq \int_0^b \max\{w, b\} \mathrm{d}F(w) = b,$$

which implies $f_1(x_b) = \beta v_0(x_b) - (x_b + r(x_b)) - s \leq \beta b - b - s = (\beta - 1)b - s < 0$. This inequality and assertion (a) lead us to $f_1(w) < 0$ for $w \geq x_b$. Up to here the first step of the proof is completed.

Suppose $v_{t-1}(x) \leq b$ for $x \leq x_b$ and $f_t(x) < 0$ for $x_b \leq x$, where note that the latter inductive assumption is equivalent to $-r(w) - s + \beta v_{t-1}(w) < w \leq b$ for $x_b \leq w$. Then, for $x \leq x_b$,

$$\begin{aligned} v_t(x) &\leq \int_0^b \max \left\{ \begin{array}{l} b, \\ -r(w) - s + \beta v_{t-1}(w), \\ b, \\ -s + \beta b \end{array} \right\} \mathrm{d}F(w) \\ &= \int_0^b \max\{b, -r(w) - s + \beta v_{t-1}(w)\} \mathrm{d}F(w) \\ &= \int_0^{x_b} \max\{b, -r(w) - s + \beta v_{t-1}(w)\} \mathrm{d}F(w) + \int_{x_b}^{b} \max\{b, -r(w) - s + \beta v_{t-1}(w)\} \mathrm{d}F(w) \\ &\leq \int_0^{x_b} \max\{b, -r(w) - s + \beta b\} \mathrm{d}F(w) + \int_{x_b}^{b} \max\{b, b\} \mathrm{d}F(w) \\ &= \int_0^{x_b} b\, \mathrm{d}F(w) + \int_{x_b}^{b} b\, \mathrm{d}F(w) \;=\; b, \end{aligned}$$

from which we get $f_{t+1}(x_b) = \beta v_t(x_b) - (x_b + r(x_b)) - s \leq \beta b - b - s = (\beta - 1)b - s < 0$. This inequality and assertion (a) show $f_{t+1}(x) < 0$ for $x \geq x_b$. We have thus completed the proof. ∎

Lemma 5.4 *λ_t exists uniquely with satisfying* $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_t \leq \cdots < x_b$.

Proof: By remembering assumptions $r(0) = 0$ and $\beta\mu - s > 0$ and using (4.3) and (4.7) we obtain $f_1(0) = \beta v_0(0) - (0 + r(0)) - s = \beta\mu - s > 0$. From this and Lemma 5.3(a,b) we conclude that λ_1 exists uniquely with $0 < \lambda_1 < x_b$.

For any $t \geq 1$, suppose λ_t exists uniquely. Then, it follows from (4.7) and Lemma 5.1(a) that $0 = f_t(\lambda_t) \leq f_{t+1}(\lambda_t)$. Owing to this inequality and Lemma 5.3(a,b) we conclude that λ_{t+1} also exists uniquely with satisfying $\lambda_t \leq \lambda_{t+1} < x_b$. We have thus completed the proof. ∎

Corollary 5.5 *For $t \geq 1$:*

(a) (1) *if $x < \lambda_t$, then $x + r(x) < -s + \beta v_{t-1}(x)$, thus $z_t^o(x) = -s + \beta v_{t-1}(x)$.*
(2) *if $x = \lambda_t$, then $x + r(x) = -s + \beta v_{t-1}(x)$, thus $z_t^o(x) = x + r(x) = -s + \beta v_{t-1}(x)$.*
(3) *if $x > \lambda_t$, then $x + r(x) > -s + \beta v_{t-1}(x)$, thus $z_t^o(x) = x + r(x)$.*

(b) (1) *if $w < \lambda_t$, then $w < -r(w) - s + \beta v_{t-1}(w)$, thus $z_t^r(w) = -r(w) - s + \beta v_{t-1}(x)$.*
(2) *if $w = \lambda_t$, then $w = -r(w) - s + \beta v_{t-1}(w)$, thus $z_t^r(w) = w = -r(w) - s + \beta v_{t-1}(x)$.*
(3) *if $w > \lambda_t$, then $w > -r(w) - s + \beta v_{t-1}(w)$, thus $z_t^r(w) = w$.*

Proof: Obvious due to Lemmas 5.3(a) and 5.4. ∎

Lemma 5.6 *$z_t^o(x)$ is continuous and nondecreasing in x.*

Proof: Since r is assumed to be continuous and nondecreasing, it follows from Lemma 5.1(a) that both $x + r(x)$ and $-s + \beta v_{t-1}(x)$ are nondecreasing in x. Hence, due to (4.4) we have confirmed the truth of the assertion. ∎

Lemma 5.7 *For $t \geq 1$, we have $z_t(w) \leq \lambda_t + r(\lambda_t)$ for $w \leq \lambda_t + r(\lambda_t)$, and $z_t(w) = w$ for $\lambda_t + r(\lambda_t) \leq w$.*

Proof: We first show an assertion $z_t(w) \leq \lambda_t + r(\lambda_t)$ for $w \leq \lambda_t$ and $z_t(w) = w$ for $\lambda_t \leq w$.

If $w < \lambda_t$, it follows from (4.8) and Lemma 5.1(a) that $-r(w) - s + \beta v_{t-1}(w) \leq -s + \beta v_{t-1}(\lambda_t) = \lambda_t + r(\lambda_t)$, implying

$$z_t^r(w) = \max\{w, -r(w) - s + \beta v_{t-1}(w)\} \leq \max\{\lambda_t, \lambda_t + r(\lambda_t)\} = \lambda_t + r(\lambda_t).$$

If $w > \lambda_t$, then $f_t(w) < 0$ by Lemma 5.3(a), thus $z_t^r(w) = \max\{w, -r(w) - s + \beta v_{t-1}(w)\} = w$.

We have finished the proof of the assertion. This assertion immediately enables us to confirm the truth of this lemma. ∎

Lemma 5.8 *For $t \geq 1$ we have $\lambda_t = \lambda_{t+1}$ if and only if $v_{t-1}(\lambda_t) = v_t(\lambda_t)$.*

Proof: If $\lambda_t = \lambda_{t+1}$, due to (4.8) we get $f_t(\lambda_t) = 0$ and $f_{t+1}(\lambda_t) = f_{t+1}(\lambda_{t+1}) = 0$, thus $f_t(\lambda_t) = f_{t+1}(\lambda_t)$, that is, $\beta v_{t-1}(\lambda_t) - (\lambda_t + r(\lambda_t)) - s = \beta v_t(\lambda_t) - (\lambda_t + r(\lambda_t)) - s$. Hence, $v_{t-1}(\lambda_t) = v_t(\lambda_t)$.

Conversely, if $v_{t-1}(\lambda_t) = v_t(\lambda_t)$, it follows from (4.8) that

$$f_{t+1}(\lambda_t) = \beta v_t(\lambda_t) - (\lambda_t + r(\lambda_t)) - s = \beta v_{t-1}(\lambda_t) - (\lambda_t + r(\lambda_t)) - s f_t(\lambda_t) = 0.$$

Since λ_{t+1} is an unique root of $f_{t+1}(x) = 0$ as seen in Lemma 5.4, we deduce $\lambda_t = \lambda_{t+1}$. ∎

Let us define a number λ as a root of equation

$$\beta\mu - s - \beta \int_0^{x+r(x)} F(w)\mathrm{d}w - (x + r(x)) = 0. \tag{5.10}$$

Theorem 5.9 *$\lambda_t = \lambda$ for every $t \geq 1$.*

Proof: We have $0 < \lambda_t + r(\lambda_t) < x_b + r(x_b) = b$ because of $0 < \lambda_t < x_b$ due to Lemma 5.4.

It follows from (4.1) and (4.2) that

$$\begin{aligned} v_0(\lambda_1) &= \int_0^b \max\{w, \lambda_1 + r(\lambda_1)\}\mathrm{d}F(w) \\ &= \int_0^{\lambda_1 + r(\lambda_1)} (\lambda_1 + r(\lambda_1))\mathrm{d}F(w) + \int_{\lambda_1 + r(\lambda_1)}^b w\mathrm{d}F(w). \end{aligned} \tag{5.11}$$

From Corollary 5.5(a2) we get $z_1^o(\lambda_1) = \lambda_1 + r(\lambda_1)$. Hence, by (??) and Lemma 5.7,

$$\begin{aligned} v_1(\lambda_1) &= \int_0^b \max\{z_1^r(w), \lambda_1 + r(\lambda_1)\}\mathrm{d}F(w) \\ &= \int_0^{\lambda_1+r(\lambda_1)} \max\{z_1^r(w), \lambda_1 + r(\lambda_1)\}\mathrm{d}F(w) + \int_{\lambda_1+r(\lambda_1)}^b \max\{z_1^r(w), \lambda_1 + r(\lambda_1)\}\mathrm{d}F(w) \\ &= \int_0^{\lambda_1+r(\lambda_1)} \big(\lambda_1 + r(\lambda_1)\big)\mathrm{d}F(w) + \int_{\lambda_1+r(\lambda_1)}^b w\mathrm{d}F(w) \end{aligned} \tag{5.12}$$

Since $v_0(\lambda_1) = v_1(\lambda_1)$ from (5.11) and (5.12), we get $\lambda_1 = \lambda_2$ owing to Lemma 5.8.

Next, assume $\lambda_t = \lambda_{t+1}$, or equivalently, $v_{t-1}(\lambda_t) = v_t(\lambda_t)$. Then, in exactly the same way as in (5.12) we obtain

$$\begin{aligned} v_t(\lambda_{t+1}) = v_t(\lambda_t) &= \int_0^b \max\{z_t^r(w), \lambda_t + r(\lambda_t)\}\mathrm{d}F(w) \\ &= \int_0^{\lambda_t+r(\lambda_t)} \big(\lambda_t + r(\lambda_t)\big)\mathrm{d}F(w) + \int_{\lambda_t+r(\lambda_t)}^b w\mathrm{d}F(w) \\ &= \int_0^{\lambda_{t+1}+r(\lambda_{t+1})} \big(\lambda_{t+1} + r(\lambda_{t+1})\big)\mathrm{d}F(w) + \int_{\lambda_{t+1}+r(\lambda_{t+1})}^b w\mathrm{d}F(w) \\ &= \int_0^b \max\{z_{t+1}^r(w), \lambda_{t+1} + r(\lambda_{t+1})\}\mathrm{d}F(w) = v_{t+1}(\lambda_{t+1}), \end{aligned} \tag{5.13}$$

from which $\lambda_{t+1} = \lambda_{t+2}$. Therefore, by induction we conclude $\lambda_t = \lambda_{t+1}$ for all $t \geq 1$.

Finally, let us consider λ_1. According to (4.3) and (4.7) we have

$$f_1(x) = \beta\Big(\mu + \int_0^{x+r(x)} F(w)\mathrm{d}w\Big) - (x - r(x)) - s,$$

which indicates $\lambda_1 = \lambda$, thus the truth of the assertion is confirmed. ∎

Since this theorem tells that λ_t, the value of the indifferent points both between **AS** and **RC** and between **PS** and **PC**, is independent of time, we use λ instead of λ_t from now on.

By adding this fact and Corollary 5.5 we can prescribe an optimal decision rule as follows:

Optimal decision rule: Suppose that you are at time t on which you already have the leading offer x and draw an offer w. Then, the optimal choices are:

(a) if $w \in W_t(x)$, then:
- if $\lambda < w$, AS (accept current offer w and stop the search)
- if $w \leq \lambda$, RC (reserve current offer w and continue the search)

(b) if $w \notin W_t(x)$, then:
- if $\lambda < x$, PS (pass up current offer w and stop the search by accepting the leading offer x)
- if $x \leq \lambda$, PC (pass up current offer w and continue the search)

Theorem 5.10 *For $t \geq 0$:*

(a) *For any x, if $w \in W_t(x)$, then $x + r(x) \leq w$.*

(b) *If $x^1 < x^2$, then, $W_t(x^1) \supseteq W_t(x^2)$.*

(c) *$W_t(x) = \{w : x + r(x) \leq w\}$ for $x \geq \lambda$.*

Proof: (a) Fix any x. Since $W_0(x) = \{w : x + r(x) \leq w\}$ by definition, the assertion holds true for

To complete the proof, we assume the assertion to be true for $t-1$ and then show the contrapositive, that is, if $w < x + r(x)$, then $z_t^r(x) < z_t^o(x)$.

First, choose any w so that $w \leq x$. Then, it follows from Lemma 5.1(a) that $v_{t-1}(w) \leq v_{t-1}(x)$, thus $-r(w) - s + \beta v_{t-1}(w) < -s + \beta v_{t-1}(x)$, hence

$$z_t^r(w) = \max\{w, -r(w) - s + \beta v_{t-1}(w)\} < \max\{x + r(x), -s + \beta v_{t-1}(x)\}.$$

Next, choose any w so that $x < w < x + r(x)$ and let ρ be a number such that $w = x + \rho$. Then, $0 < \rho < r(x)$. By using Lemma 5.2 we deduce

$$\begin{aligned}
\big(-r(w) - s + \beta v_{t-1}(w)\big) - \big(-s + \beta v_{t-1}(x)\big) &= \beta\big(v_{t-1}(w) - v_{t-1}(x)\big) - r(w) \\
&= \beta\big(v_{t-1}(x+\rho) - v_{t-1}(x)\big) - r(x+\rho) \\
&\leq \beta\big(\rho + r(x+\rho) - r(x)\big) - r(x+\rho) \\
&= \beta(\rho - r(x)) + (\beta - 1)r(x+\rho) < 0,
\end{aligned}$$

which also yields $z_t^r(w) < z_t^o(x)$. As seen above, we get $z_t^r(w) < z_t^o(x)$ for both $w \leq x$ and $x < w < x + r(x)$, hence $w \notin W_t(x)$ for $w < x + r(x)$. We have thus completed the proof.

(b) Choose any x^1 and x^2 such that $x^1 < x^2$. Furthermore, choose any $w \in W_t(x^2)$. Then, by (4.6) we have $z_t^o(x^2) \leq z_t^r(w)$. From this inequality and Lemma 5.6 we get $z_t^o(x^1) \leq z_t^r(w)$, which means $w \in W_t(x^1)$. Accordingly, any $w \in W_t(x^2)$ is also an element of $W_t(x^1)$.

(c) Assertion (a) means $W_t(x) \subseteq \{w : x + r(x) \leq w\}$, thus what remains to prove assertion (c) is only to show $\{w : x + r(x) \leq w\} \subseteq W_t(x)$.

Choose an x so that $\lambda \leq x$, and fix a w so that $x + r(x) \leq w$. Since $z_t^o(x) = x + r(x)$ for the x due to Corollary 5.5(a3), we have $z_t^o(x) \leq z_t^r(w)$, that is, $w \in W_t(x)$ for the w.

We now find that any $w \geq x + r(x)$ belongs to $W_t(x)$ for $\lambda \leq x$, and then the proof is finished. ∎

6. Conclusions

In this section we check three main results. What to be emphasized first is that almost the same results as these in this research had gained, respectively, in the past model where the reserving cost had never returned.

1. An offer reserved during the search process must not be accepted prior to the deadline.

The reason is as follows. At time t, if we find an offer with value $\tilde{w}$ such that $\tilde{w} \in W_t(x)$ and $\tilde{w} < \lambda_t$, then we reserve it according to the optimal decision rule. But at any time t' after time t, inequality $\lambda_{t'} \leq \tilde{w}$ never holds because Theorem 5.9 tells us that $\lambda_t = \lambda_{t'} = \lambda$. Therefore, up to the deadline, we cannot be in the situation where decision PS is the best, which means that at only the deadline we are to take the decision.

In our model, every time you decide to proceed the search once more, a positive search cost must be spent and chances to meet offers superior to reserved ones up to the deadline decrease. So, the author think it natural that an idea "it seems waste of time for pursuing the search further, I now decide to stop the search by recalling the offer reserved before" comes at a time prior to the deadline.

He has thought so from his start of studying optimal stopping problems with reservation. This idea, however, proves to be illogical not only in past models where the reservation is effective for a finite fixed periods[3], the remaining periods have some value[4], the reserving period is also determinable[6] but also in the present model where the reserving cost for the recalled offer returns. As a consequence, the Result 1 must be an essential property to optimal stopping problems with reservation.

2. An offer to be either accepted or reserved must have a value superior to the value you can obtain at the time of recalling the leading offer.

This is from Theorem 5.10(a), that is, $W_t(x) \subseteq \{w : x + r(x) \leq w\}$. We can think it is very reasonable. In [2] we got a similar result $W_t(x) \subseteq \{w : x \leq w\}$.

3. After you renew the leading offer, you must choose an offer to be either accepted or reserved more severely.

This comes from Theorem 5.10(b), that is, $W_t(x_1) \supseteq W_t(x_2)$ for $x_1 < x_2$. As well as Result 2, this is very reasonable result, and the same result had obtained in [2].

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