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# Location Game with Degressive Weighted Voronoi Region

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**Abstract:** This paper investigates a facility location model with $n$ weighted demand points on a line segment and a plane under a competitive environment. Customers at a demand point patronize one facility according to the attractiveness and the transportation cost. The smaller the difference of the location of the facilities, the less the customers distinguish the facilities. Two companies, the leader and the follower, establish their facilities by turns in this market to get as much buying power as possible. We formulate the problems to find the optimal location for the follower and for the leader, as a medianoid problem and a centroid problem respectively, and propose a solution procedure to solve them.

**Keywords:** continuous location, noncooperative games

## 1 Introduction

Competitive facility location problem was introduced by H.Hotelling [1], who studied the Nash equilibrium problem of two sellers on a linear market. S.L.Hakimi considered the Stackelberg equilibrium problem on a network [2], that is, two companies "leader" and "follower" establish their facilities on nodes in order to capture as much buying power as possible. He showed that the problem is NP-hard. Z.Drezner studied the same kind of a competitive problem on a plane [3].

This paper investigates an alternative game by two players on a linear market and on a plane. The leader company A locate his facility on the market first, and then the follower company B locate his facility. The aim of each player is to maximize their gain, i.e., to capture as much buying power as possible.

Various models has been proposed in this field [4], considering how demands are allocated, how is the customer's preference in facilities. Most commonly used assumption is that customers utilize only the nearest facility. This is expected to derive a proper approximation for fast food restaurants, coffee chain stores, video rental shops, etc. However, in this assumption, when two facilities are mutually located in near, the property which cannot necessarily be referred to as realistic will appear. We describe the details of this aspect below.

On a simple and typical case, the optimal strategy for B is locating adjacent to A, since B can expand his domain of influence by approaching to A. Suppose the demands are distributed continuously on a linear market, and the amount of demand at $x$ is given by a non-negative integrable function $f(x)$. For the sake of simplicity, we assume that the gain of each player is equal to the amount of the capturing demand. In this case when A exists at $a$, the gain of B at

$b$ is represented by $G(b|a) = \int_{-\infty}^{\frac{a+b}{2}} f(x)\,dx$ in the case where $b < a$, and $G(b|a) = \int_{\frac{a+b}{2}}^{\infty} f(x)\,dx$ in the case where $a < b$. It is natural to assume that $G(b|a) = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$ in the case where $b = a$. Since $f(x)$ is non-negative, $G(b|a)$ becomes greater as b approaches to a, so the optimal strategy for B is locating his facility adjacent to A. This property holds not only continuous model but also discrete model. But the undesirable part of this model is that generally $G(b|a)$ is "too sensitive" around $b = a$. For example, if the demands are uniformly distributed on the interval $[0,1]$ and the leader $x$ exists at $\frac{1}{3}$, then the left-hand limit of $G(b|a)$ as $b$ approaches $a$ is equal to $\frac{1}{3}$ , while the right-hand limit of $G(b|a)$ as $b$ approaches $a$ is equal to $\frac{2}{3}$. This means that the location of B exerts an influence on his gain too sensitively on the neighborhood of A. It must be noted that B gets or loses double gain by moving slightly around A with this case.

Similar sensitivity arises in the case where $n$ competitive facilities exist on a plane. When a certain facility approaches to the neighborhood facility and pass through it to the other side, the Voronoi regions of them change places, which leads to the change of the amount of the capturing demand discontinuously at the point where the two facilities meet. These hypersensitive properties described above is not so realistic, so we propose an improved model based on two types of preferences for more reality.

It may be worth mentioning, in passing, that B's adjacency strategy is not optimal on the condition that A can locate his second facility in the same market [5]. In such a case, if B locate his facility close to A's first facility, then A locate the second one so as to narrow B's Voronoi region from both side, which results in making the gain of B close to 0.

## 2 Our Model

We consider continuous facility location problem with discrete demand. Two companies, A and B , establish their facilities by turns in the same market to get as much weight of demand point as possible. In the later part, we formulate the problems to find the optimal location for the follower and for the leader, as a medianoid problem and a centroid problem respectively, and propose a solution procedure to solve them.

| | |
|---|---|
| $p_i$ | location of demand point $P_i$ $(i = 1 \cdots n)$ |
| $w_i$ | weight of $P_i$ $(W = \sum_i w_i)$ |
| $a$ | location of the leader A |
| $b$ | location of the follower B |
| $\alpha, \beta$ | weight of A,B |
| $d(a,b)$ | Euclidean distance between A and B |
| $G_A, G_B$ | gain of A,B |

We introduce two types of preferences for customers to choose the facility. It will be appropriate to consider that attractiveness of the facility affects a customer's preference. So we define preference type 1 as follows.

**Preference Type 1** : Customers choose one facility not only by the distance to them but also by the weight (attractiveness) of them. B's domain of influence is defined by

$$D_B = \{p = (x,y) \,|\, \alpha\, d(b,p) \le \beta\, d(a,p)\} \ .$$

According to this preference, customers in domain $D_B$ choose facility B, shown in figure 1. A's domain of influence $D_A$ can be defined by complementary set of $D_B$. These forms weighted Voronoi regions.

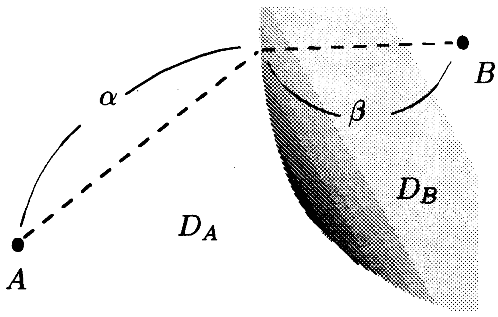


Figure 1: Proportional distribution model using distance and weight

$D_B$ is a bounded set where $\alpha > \beta$. Obviously, in the case where $\alpha \leq \beta$, B can make the area of $D_A$ close to 0 by approaching A, consequently the optimal strategy for B is always adjacent to A. So we consider only the case where $\alpha > \beta$ in the following part of this paper. Note that the area of $D_B$ becomes larger as B go away from A, but at the same time $D_B$ becomes out of the convex hull of demand points and B loses weights of demand point, so differentiation strategy is not always advantageous to B.

This preference avoids the hypersensitive properties mentioned in the introduction, but new problem arises. Even in the case where B is only slightly inferior to A, $G_B$ becomes 0 at the same location as the competitor, because the area of $D_B$ becomes 0 at that point. This property cannot be always acceptable, so we introduce following preference to avoid this.

**Preference Type 2** : Not all customers are so sensitive about the difference of the distance between A and B. Let $\lambda$ denotes the ratio of the customers who cannot ignore the magnitude of the distance between A and B. We define $\lambda$ as follows.

$$\lambda = \begin{cases} 0, & 0 \leq d(a,b) \leq d_1 \\ 1, & d_2 \leq d(a,b) \\ \dfrac{1}{d_2 - d_1}(\,d(a,b) - d_1) & \text{otherwise} \end{cases}$$

$d_1$ and $d_2$ are given constants. Since $\lambda$ is a function of $d(a,b)$, we use the notation $\lambda(d(a,b))$ in some cases.

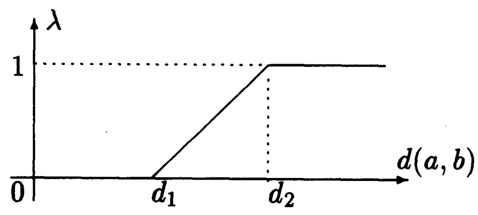


Figure 2: Ratio of the customers who cannot ignore the distance between A and B

If A and B are close to each other, no customers mind the difference of their location, which means that the influence of weighted Voronoi region is reduced close to zero. We call such Voronoi regions as the **degressive weighted Voronoi regions**.

The customers of the rate $\lambda$ of the whole market distinguish the distance between A and B, on the other hand the customers of the rate $1 - \lambda$ do not distinguish the distance but utilize a facility in proportion to the attractiveness of the facility. So we define $G_B$ at location $b$ with given $a$ as follows.

$$G_B(b\,|\,a) = (1-\lambda)\frac{\beta}{\alpha+\beta}W + \lambda \sum_{p_i \in D_B} w_i$$
$$\text{provided that } D_B = \{p = (x,y)\,|\,\alpha\, d(b,p) \leq \beta\, d(a,p)\}$$

Since $D_B$ is determined by the location of $a$ and $b$, we use the notation $D_B(a,b)$ in some cases. The first term represents the amount of weight which is obtained from the customers who feel the distance to each facility is same and utilize it according to the attractiveness. The second term represents the amount of weight which is gained from the customers who cannot ignore the distance between A and B, and utilize a facility according to the distance and the attractiveness. The ratio of these customer segments change with the distance between A and B.

In this model, $G_B$ becomes continuous function and has no hypersensitive properties, whereas $G(b|a)$ in the introduction is not continuous and too sensitive around $b = a$.

The medianoid problem is the problem to find the optimal location for the follower B which maximizes

$$\text{MP}: \max_b G_B(b\,|\,a)\ .$$

Let $b^*(a)$ denote the solution for MP with given $a$. The centroid problem is the problem to find the optimal location for the leader A which maximizes

$$\text{CP}: \max_a G_A(b^*(a)\,|\,a)\ .$$

This can be transformed into

$$\text{CP}: \min_a G_B(b^*(a)\,|\,a)$$

by using $G_A = W - G_B$.

In general, solving centroid problems are much harder than the case of medianoid problems, since the leader must take it into account that the follower locate his facility afterward with the aim of maximization of his own profit.

If B establish his facility at the location where $d(a,b) \leq d_1$, then $G_B$ takes the constant value $\frac{\beta}{\alpha+\beta}W$ irrespective of the distribution of demand points, so we can use this value as the lower bound $LB$ in searching for the optimal solution for medianoid problem.

# 3 Linear Market

In this section, we consider the case where demands are distributed discretely on the interval $[0,1]$. At first, we consider MP with given location $a$ of A.

Let $D_{P_i}$ denote the existence region of B where B can include $P_i$ in his domain of influence $D_B$. This domain is determined by the location $a$, so it is represented by

$$D_{P_i}(a) = \left[\frac{\alpha-\beta}{\alpha}p_i + \frac{\beta}{\alpha}a,\ \frac{\alpha+\beta}{\alpha}p_i - \frac{\beta}{\alpha}a\right]\ .$$

It is equivalent that $P_i$ is under the influence of B, and B is inside $D_{P_i}$. So the next duality relation holds.

$$p_i \in D_B(a,b) \iff b \in D_{P_i}(a)$$

When the set of demand points and $\alpha, \beta$ are given, $D_{P_i}(a)$ of all demand points are fixed with arbitrary $a$. So we can draw a diagram with $a$ as horizontal axis and the extreme points of $D_{P_i}(a)$ as vertical axis. An example with five demand points are shown in figure 3.

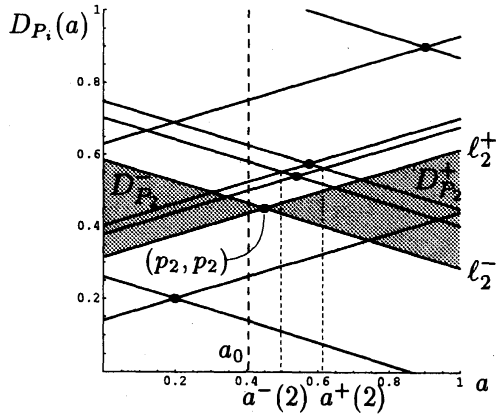


Figure 3: Diagram with 5 demand points (Shading region represents $D_{P_2}$)

In this diagram, $D_{P_i}$ is the domain that lies between the two lines passing through the point $P_i = (p_i, p_i)$ with slopes of $\frac{\beta}{\alpha}$ and $-\frac{\beta}{\alpha}$. Let $\ell_i^+, \ell_i^-$ denote these two lines respectively. For the later part, we separate $D_{P_i}$ into the left side and the right side of $P_i$, with labels of $D^-_{P_i}$ and $D^+_{P_i}$ respectively.

When A is at a certain point $a_0$, $D_{P_i}(a_0)$ becomes the interval which is the intersection of the line $a = a_0$ and $D_{P_i}$. The values of the extreme points of $D_{P_i}(a_0)$ are read from the vertical axis; from the example, if A locates his facility at $a = 0.4$, B can take $P_2$ in his domain of influence locating at $b \in D_{P_2}(0.4) \approx [0.42, 0.44]$ .

Suppose $D_{P_i}(a_0)$ and $D_{P_j}(a_0)$ have an common interval and $b$ exists on the interval, and $b \notin D_{P_k}(a_0)$ for all $k$ $(k \neq i,j)$, rearranging terms with $G_B$ yields

$$G_B(b \,|\, a_0) = (1-\lambda)\frac{\beta}{\alpha+\beta}W + \lambda(w_i + w_j) = LB + \lambda\,(w_i + w_j - LB) \ .$$

Since $LB$ is a constant and $\lambda$ is an increasing function in the wider sense with $d(a_0, b)$, there is advantage to B in going away from A in the case where $w_i + w_j > LB$, meanwhile there is advantage to B in approaching A in the case where $w_j + w_j < LB$.

Let $R_x$ denote a piece of region on a line $a = a_0$ divided by the boundaries of $D_{P_i}(a_0)$. As long as B is in a certain $R_x$, wherever B moves, the member of the capturing demand points remains unchanged. So in general, next property holds.

**Property 1** *With $b \in R_x$, if $\sum_{p_i \in D_B} w_i > LB$ then $G_B$ increases in the wider sense as B goes away from A, else if $\sum_{p_i \in D_B} w_i < LB$ then $G_B$ increases in the wider sense as B approaches to A.*

Therefore in either case the candidate solutions for MP with given $a_0$ are obtained by enumerating the extreme points of $R_x$, such points are easily calculated as the intersection points of line $a = a_0$ and lines $\ell_i^+, \ell_i^-$. The optimal solution $b^*$ for MP can be searched linearly in the order of $O(n)$ among the candidate solutions, for the point which maximizes $G_B$.

CP is the problem to find the location of $a$ which minimizes $G_B(b^*(a)\,|\,a)$. At first, we consider the example in figure 3. The interval between two thin broken lines is the region where no pairing of $D_{P_i}$ has an intersection. In such a region, no $D_{P_i}$ overlaps with each other, which means that if A locates his facility in this region then B cannot take more than 1 demand point in B's domain of influence. For details, if all $w_i$ are less than $LB$ then the optimal location for B is the point which satisfies $d(a,b) \le d_1$ and $\max G_B$ becomes $LB$. If $w_m$ which has the maximum value among $w_i$ is greater than $LB$ then the optimal location for B is the farthest point from A in the range of $D_{P_m}(a)$ and $\max G_B$ becomes $LB+\lambda(w_m - LB)$.

But this example is a special case, since such an interval of $a$ as no $D_{P_i}$ overlaps with each other does not always exist. Let $a^-(k), a^+(k)$ denote the maximum and minimum value of $a$-coordinate where $k$ domains of $D^-_{P_i}$ and $D^+_{P_i}$ overlap each other. For example, in figure 3, $a$-coordinate values of thin broken lines are $a^-(2), a^+(2)$ from left to right. If $a^-(2) > a^+(2)$ then there is no interval of $a$ where less than or equal to two $D_{P_i}$ overlap with each other.

Let $k^*$ denote minimum $k$ which satisfied $a^-(k) < a^+(k)$. The solution for CP exists in the interval $[a^-(k^*), a^+(k^*)]$, since A can reduce $\sum_{p_i \in D_B} w_i$ as small as possible by locating his facility in this interval. Simultaneously A must reduce the maximum value of $\lambda$ with $b^*$, by minimizing the maximum distance from $a$ to the extreme points of $D_{P_i}(a)$ ($p_i \in [a^-(k^*), a^+(k^*)]$). Such a point $a^*$ is calculated by

$$\begin{aligned} a^* &= \frac{1}{2}\left(\left(\frac{\alpha+\beta}{\alpha}p_i - \frac{\beta}{\alpha}a^+(k^*)\right) + \left(\frac{\alpha+\beta}{\alpha}p_j - \frac{\beta}{\alpha}a^-(k^*)\right)\right) \\ &= \left(\frac{\alpha+\beta}{2\alpha}\right)(p_i + p_j) - \frac{\beta}{2\alpha}(a^+(k^*) + a^-(k^*)) \end{aligned}$$

where $p_i, p_j \in [a^-(k^*), a^+(k^*)]$ are the nearest demand points to $a^-(k^*), a^+(k^*)$ respectively. Then $a^*$ is the solution for CP.

Now we must search for $k^*$ and the interval $[a^-(k^*), a^+(k^*)]$. The candidate solutions for $a^-(k^*), a^+(k^*)$ are the points where the number of the overlap of $D_{P_i}$ changes, i.e., the lattice points composed by $\ell_i^+$ and $\ell_i^-$. Let $P^+_{i,j}$ denote the intersection of $\ell_i^+$ and $\ell_j^-$ where $i < j$, $P^-_{i,j}$ denote the intersection of $\ell_i^-$ and $\ell_j^+$. Then $a^+(k)$ is the point which has the minimum value of $a$-coordinate among $P^+_{i,i+k-1}$, $a^-(k)$ is the maximum $a$-coordinate point among $P^-_{i,i+k-1}$.

When searching for $a^+(k)$, we can utilize the following property. If inequality

$$a_{i+k} - a_{i+k-1} < \frac{\alpha-\beta}{\alpha+\beta}(a_{i+1} - a_i)$$

holds then $a$-coordinate value of $P^+_{i,i+k-1}$ is greater than that of $P^+_{i+1,i+k}$. We omit the similar inequality for $a^-(k)$. If $a^-(k) < a^+(k)$ then $k, a^-(k), a^+(k)$ are the candidate solutions. When searching for $k^*$, binary search is available. This method changes the value of $k$ from 1 to $n-1, 2, n-2, 3, \cdots, \frac{n}{2}$ to find smallest $k$ which satisfies $a^-(k) < a^+(k)$. By this means we can find $k^*$, $a^-(k^*)$, $a^+(k^*)$ and $a^*$ which minimizes $\max G_B$ as the solution for CP.

## 4 Plane Market

This section considers MP on a plane. We use following notations for two dimensional model.

| | |
|---|---|
| $p_i = (p_{i1}, p_{i2})$ | location of demand point $P_i$ $(i = 1 \cdots n)$ |
| $a = (a_1, a_2)$ | location of A |
| $b = (b_1, b_2)$ | location of B |

The other notations are same as previous section.

The shape of B's domain of influence $D_B$ becomes a circle represented by

$$D_B = \left\{(x,y) \mid \left(x - \frac{b_1 - k^2 a_1}{1-k^2}\right)^2 + \left(y - \frac{b_2 - k^2 a_2}{1-k^2}\right)^2 \le \frac{k^2}{(1-k^2)^2}((a_1-b_1)^2 + (a_2-b_2)^2)\right\}$$

provided that $k = \frac{\beta}{\alpha}$.

Using similar way of thinking in the previous section, we start with formulating the domain $D_{P_i}$, which represent the existence region of B where B can include $P_i$ in his domain of influence $D_B$.

Let $O$ denote the center of $D_B$, $r$ denote the radius of $D_B$. Calculating the length of $\overline{AO}$ and $r$, we obtain the relation

$$r = \frac{\beta}{\alpha}\overline{AO} .$$

If we draw two tangent lines to the circles $D_B$ through A, then the angle between the lines which contain $D_A$ remains constant wherever B moves. Let $\theta$ denote the angle, then we obtain

$$\sin\frac{\theta}{2} = \frac{\beta}{\alpha} .$$

Since the radius of $D_B$ is in proportion to the distance between A and O, and the included angle of two tangent lines is constant, it is natural to use polar coordinate in this part. Without loss of generality, A is assumed to be given at origin (0,0). We use polar coordinate $b = (\gamma, d)$, $p_i = (\delta_i, \ell_i)$ , provided that $d = d(a,b)$ and $\gamma, \delta_i, \ell_i$ satisfy the relations $(b_1, b_2) = (d\cos\gamma, d\sin\gamma)$, $(p_{i_1}, p_{i_2}) = (\ell_i\cos\delta_i, \ell_i\sin\delta_i)$, $0 \le \gamma, \delta_i < 2\pi$ .

When $P_i$ is on the boundary of $D_B$, using cosine theorem, the following equation is derived on condition that $\delta_i - \frac{\theta}{2} \le \gamma \le \delta_i + \frac{\theta}{2}$.

$$2\,d^2 + (1+\cos\theta)\,\ell_i^2 - 4\,d\,\ell_i\cos(\delta_i - \gamma) = 0$$

Conversely if this relation is satisfied, then $P_i$ is on the boundary of $D_B$. Eliminating $\theta$ by using $\sin\frac{\theta}{2} = \frac{\beta}{\alpha} = k$, this equation becomes

$$d^2 + (1-k^2)\,\ell_i^2 - 2\,d\,\ell_i\cos(\delta_i - \gamma) = 0 .$$

When $\alpha, \beta$ and $P_i$ are given, we can think left side member of this equation as an implicit function $h(\gamma, d)$. Now $D_{P_i}$ can be formulated as $D_{P_i} = \{(\gamma,d) \mid h(\gamma,d) \le 0\}$. If $B$ is in this domain, then $P_i$ is in B's domain of influence as follows.

$$b \in D_{P_i} \iff p_i \in D_B$$

Let $B_{P_i}$ denote the boundary of $D_{P_i}$, then $B_{P_i}$ is formulated as $B_{P_i} = \{(\gamma,d) \mid h(\gamma,d) = 0\}$ which shape of the graph becomes ovaloid shape illustrated in figure 4.

If $D_{P_i} \cap D_{P_j} \ne \phi$ and $b \in D_{P_i} \cap D_{P_j}$, then B can get at least $\max\{\lambda(w_i + w_j - LB), LB\}$. So if we can omit the value of $\lambda$ and $w_i$, the candidate solution for MP is $b^0$ corresponding to $D_B$ which covers the maximum number of $P_i$. Such $b^0$ is on the intersection region of $D_{P_i}$, which includes the intersection points of $B_{P_i}$. For the first step to find the solution for MP, we start to examine the intersection points of $B_{P_i}$, which are potential candidate solutions.

When searching for the intersection with a certain $B_{P_i}$, we can obviously exclude $B_{P_j}$ which satisfies $|\,\delta_i - \delta_j| > \theta$ or $\ell_j(1-k) > \ell_i(1+k)$ or $\ell_j(1+k) < \ell_i(1-k)$. The existence region of demand points which boundary intersects $B_{P_i}$ is expressed by $\{(x,y) \mid (x - \frac{p_{i1}}{1-k^2})^2 +$

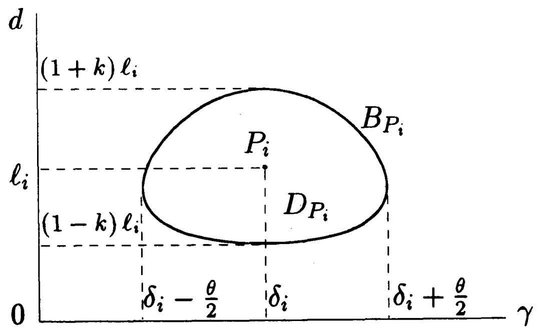


Figure 4: Boundary of existence region of B which satisfies $p_i \in D_B$

$(y - \frac{p_{i2}}{1-k^2})^2 \le \frac{(2-k^2)k^2}{(1-k^2)^2}(p_{i1}^2 + p_{i2}^2)\}$. If $P_j$ is not in this region, then $B_{P_j}$ does not intersect $B_{P_i}$. The boundary is constrained in shape, two of them intersect at most 2 times each other, and one never enclosed by the other.

We can calculate the coordinates of intersection points between $B_{P_i}$ and $B_{P_j}$, but the result becomes very complicated expression either in polar coordinates and in orthogonal coordinates. It becomes a little simpler in orthogonal coordinates, so we use orthogonal coordinates in this part. Solving the following simultaneous equations,

$$\begin{cases} (x - \frac{p_{i1}}{1-k^2})^2 + (y - \frac{p_{i2}}{1-k^2})^2 = \frac{k^2}{(1-k^2)^2}(p_{i1}^2 + p_{i2}^2) \\ (x - \frac{p_{j1}}{1-k^2})^2 + (y - \frac{p_{j2}}{1-k^2})^2 = \frac{k^2}{(1-k^2)^2}(p_{j1}^2 + p_{j2}^2) \end{cases}$$

we obtain the intersection points between $B_{P_i}$ and $B_{P_j}$ as follows.

$$\begin{aligned} x &= \frac{-2p_{i2}UV + 2p_{i1}V^2 + RU(US + VT) \pm \sqrt{Z}}{2R(U^2 + V^2)} \\ y &= \frac{V\,(2p_{i2}U^2 + V(-2p_{i1}U + RUS + RVT)) \mp U\sqrt{Z}}{2RV(U^2 + V^2)} \end{aligned}$$

provided that

$$\begin{aligned} Z &= (2V\,(p_{i1}V - p_{i2}U) + RU(US + VT))^2 \\ &\quad -R\,(U^2 + V^2)(R\,(US + VT)^2 + 4V(p_{i1}^2V + p_{i2}^2V - p_{i2}(US + VT))) \\ S &= p_{i1} + p_{j1}\ ,\ T = p_{i2} + p_{j2}\ ,\ U = p_{i1} - p_{j1}\ ,\ V = p_{i2} - p_{j2}\ . \end{aligned}$$

These solutions can be translated into polar coordinates as $d = \sqrt{x^2 + y^2}$, $\gamma = \arccos(y/d)$ where $y \ge 0$ , $\gamma = 2\pi - \arccos(y/d)$ where $y < 0$.

Let $R_x$ denote a piece of region on $\gamma$-$d$ plane divided by $B_{P_i}$. Then the same property discussed in the previous section holds, i.e., as long as $b$ remains in a fixed region $R_x$, if $\sum_{p_i \in D_B} w_i > LB$ then $G_B$ is expected to increase as B goes away from A, else if $\sum_{p_i \in D_B} w_i < LB$ then $G_B$ is expected to increase as B approaches to A. So the candidate solution for MP is the farthest or nearest point from A in the region of $R_x$.

The shading region in figure 5 shows an example of the region of $D_{P_i} \cap \overline{D_{P_j}} \cap D_{P_k} \cap \overline{D_{P_\ell}}$. As long as $b$ is included in this region, the member of demand points covered by $D_B$ is fixed.

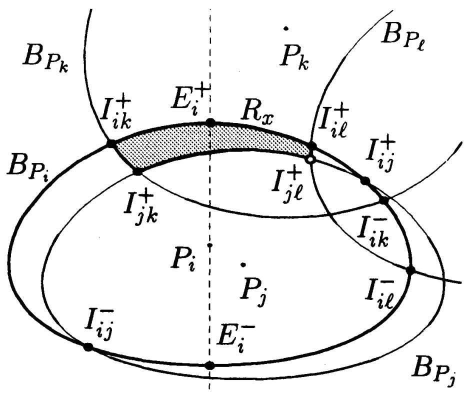


Figure 5: Labels for the extreme points of divided regions

In order to treat the extreme point of divided region $R_x$, we label each intersection of $B_{P_i}$ as shown in figure 5. When $\delta_i < \delta_j$ , we can uniquely label the intersection of $B_{P_i}$ and $B_{P_j}$ as $I^+_{ij}$ and $I^-_{ij}$, according to the descending order of the value of $d$-coordinate ($I^+_{ij} = I^-_{ij}$ when touching). The points where $B_{P_i}$ attains maximum and minimum $d$-coordinates value must be taken into consideration. These points are labeled $E^+_i$ and $E^-_i$ respectively, which coordinates are $(\delta_i, (1+k)\,\ell_i)$ and $(\delta_i, (1-k)\,\ell_i)$.

In general, tree algorithm is useful for searching for this kind of intersections, but what we want is the maximum value of $G_B$ in the region, so we must calculate it with $\lambda$ and compare with the other candidate solution, so tree algorithm is not appropriate in this case.

On $R_x$ in figure 5, $E^+_i$ is the farthest point from A, and $I^+_{jk}$ is the nearest. Therefore $E^+_i$ and $I^+_{jk}$ are listed as the temporary candidate solutions with $R_x$. Comparing $G_B(E^+_i)$ and $G_B(I^+_{jk})$, the greater one is the candidate solution. Note that strictly $I^+_{j\ell}$ is not contained in $R_x$ in this figure, but it is the extreme point of the other region, so it will be listed as an candidate solution in the other scan. In the same way, $I^+_{i\ell}, I^+_{j\ell}, I^+_{ij}, \cdots$ will be also scanned in our algorithm.

Our algorithm for solving $M_P$ is shown below.

**Step 1.** Sort $P_i$ by ascending order of $\delta_i$, and save the original order to $S_k$.

**Step 2.** $u \leftarrow 1$

**Step 3.** $G_B \leftarrow LB\,,\ v \leftarrow u+1\,,\ L \leftarrow \phi$

**Step 4.** If $\delta_v - \delta_u > \theta$ then go to **step 6**.
Check whether $B_{P_u}$ and $B_{P_v}$ intersect each other. If intersection exists, calc the points $I^+_{uv}$ and $I^-_{uv}$. $L \leftarrow L + \{I^+_{uv}\,,\ I^-_{uv}\}$.

**Step 5.** $v \leftarrow v+1$. Go to **step 4**.

**Step 6.** $L \leftarrow L + \{E^+, E^-\}$. Sort elements of $L$ by ascending order of the clockwise angles between the element and $\overline{P_u T}$. ($T$ is a contact point between $B_{P_u}$ and the line $\gamma = \delta_u - \frac{\theta}{2}$)

**Step 7.** $G_u \leftarrow 0\,,\ B_u \leftarrow (0,0)\,,\ t \leftarrow 1,\ \ V \leftarrow \{S_u\}$

**Step 8.** Choose $t$-th element of $L$ as $L_t$. If $d(a, L_t) \le d_1$ then $G_u = LB$ and go to **step 10**.
If $L_t = I^+_{um}$ then $V \leftarrow V \cup \{S_m\}$ and $V_t \leftarrow V$. If $L_t = I^-_{um}$ then $V_t \leftarrow V$ and

$V \leftarrow V \setminus \{S_m\}$.
If $L_t = E^+$ or $L_t = E^-$ then $V_t \leftarrow V$.

**step 9.** $m \leftarrow (1-\lambda(L_t))LB+\lambda(L_t)\sum_{i \in V_t} w_i$. If $m > \max\{G_u, LB\}$ then $G_u \leftarrow m$ and $B_u \leftarrow L_t$.

**step 10.** If $t <$ the number of element of $L$ then $t \leftarrow t+1$ and go to **step 8**.

**step 11.** If $G_u > G_B$ then $G_B \leftarrow G_u$ and $b \leftarrow B_u$. If $u \geq n-1$ then **stop** else $u \leftarrow u+1$. go to **step 3**.

Basic idea is selecting one candidate solution for one corresponding region. $S_k$ holds the original index $i$ of $P_k$ and $\delta_k$. $L$ holds the position of the temporary candidate solutions on $B_{P_u}$. For example, sorted $L$ becomes $\{I^+_{ik}, E^+_i, I^+_{i\ell}, I^+_{ij}, I^-_{ik}, I^-_{il}, E^-_i, I^-_{ij}\}$ with $u = i$ in figure 5. $V$ holds a set of indices of demand points which are in B's domain of influence when B is at $L_t$. If $u = i$ and $L_t = I^+_{ij}$ then $V = \{i, k, \ell, j\}$ in figure 5. When algorithm stops, the solution for MP is $b$, with the maximum value of $G_B$.

On this algorithm, step 4-5 takes $O(n-1)$ times, step 6 takes $O(2(n-1)\log 2(n-1))$ times, step 8-10 takes $O(2(n-1))$ times, step 3-11 takes $O(n)$ times , so the total computational complexity becomes $O(n^2 \log n)$.

## 5 Conclusion and Further Research

We considered a competitive facility location problem on a linear and a plane market introducing two types of preferences to avoid hypersensitive property. One of the preferences is a model for a kind of psychological distance. We formulated the alternative game by two players and proposed a method to find the optimal location for the follower and the leader on a linear market. The solution procedure for the follower on a plane was also shown.

Our further research will be on finding the optimal location for the leader on a plane market. We used Euclidean distance in this paper, but weight proportional distribution model with $\ell_1$-distance seems to be another interesting problem. But it is conjectured much more difficult, since B's domain of influence generally becomes non-convex domain.

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