数理解析研究所講究録 1306 巻 2003 年 34-39

BEST-CHOICE GAMES WHERE ARBITRATION COMES IN

坂口 実 (Minoru Sakaguchi)

1. Multistage Poker——Simultanious-Move

We first consider a simple n-round poker. Each of two players I and I receives a hand x and y, respectively, in [0,1], according to a uniform distribution, and chooses one of two alternatives Reject or Accept. If choice-pair is R-R,the game proceeds to the next round and both players are dealt new hands x and y. If the choice-pair is A-A showdown occurs and the game ends with I's reward sgn (x − y). If players choose different choices, then arbitration comes in, and forces them to take the same choices as I's (II's) with probability p ($\bar{p}$). This zero-sum game is played in n-rounds, and player I (II) aims to maximize (minimize) the expected reward to I.

Each of two players must make one choice among two. Players know that arbitration comes in, if their choices are different, and the arbitration is not necessarily fair. Arbitration is made at most in n times, within which the final decision must be reached. Most of papers [1,2,3,5] analyse a model where arbitration is absent, or fair even if it comes in. We consider in the present article and Ref.[4,6,7] a version of arbitration problemes with active and unfair arbitrator who is under the influence of the relative powor of the players.

2000 Mathematics subject classification. 62L15, 90C39, 90D40

Keywords and phrases. Optimal stopping game, poker, optimal strategy.

Let $\phi_n(x)$ $(\psi_n(y))$ be the probability that player I (II) chooses A on the hand x (y).

Also let v_n be the value (for I) of the n-round game. Then we have

(1.1) $$v_n = \max_{\phi_n(\cdot)} \min_{\psi_n(\cdot)} E_{x,y}\left[(\bar{\phi}_n(x), \phi_n(x)) M_n(x,y) (\bar{\psi}_n(y), \psi_n(y))^T\right]$$

where

(1.2) $$M_n(x,y) = \begin{array}{c c} & \begin{array}{cc} R & A \end{array} \\ \begin{array}{c} R \\ A \end{array} & \begin{bmatrix} v_{n-1} & \bar{p}\,sgn(x-y) + p v_{n-1} \\ p\,sgn(x-y) + \bar{p} v_{n-1} & sgn(x-y) \end{bmatrix} \end{array}$$

It is clear that

(1.3) $$v_n - v_{n-1} = \max_{\phi} \min_{\psi} A_n(\phi, \psi), \quad \text{with}$$

(1.4) $$A_n(\phi, \psi) \equiv E\left[(\bar{\varphi}, \varphi)(sgn(x-y) - v_{n-1}) \begin{bmatrix} 0 & \bar{p} \\ p & 1 \end{bmatrix} (\bar{\psi}, \psi)^T\right]$$
$$= E\left[(sgn(x-y) - v_{n-1})(p\phi + \bar{p}\psi)\right],$$

where $\phi_n(x)$ and $\psi_n(y)$ in (1.3)-(1.4) are abreviated by ϕ and ψ. We repeatedly use this simplification throughout this paper.

Theorem 1. The solution to the n-round poker (1.1)-(1.2) is :

(1.5) $$\phi_n^*(x) = I(x > a_n), \quad \psi_n^*(y) = I(y > \bar{a}_n), \quad \text{and} \quad v_n = 2a_{n+1} - 1,$$

where $\{a_n\}$ satisfies the recursion

(1.6) $$a_{n+1} = a_n + \frac{1}{2}(p\bar{a}_n^2 - \bar{p} a_n^2) \qquad (n \geq 1, \ a_1 = 1/2)$$

As n tend to ∞, we have

(1.7) $$a_n \uparrow \alpha = \frac{\sqrt{p}}{\sqrt{p} + \sqrt{\bar{p}}} \quad \text{and} \quad v_n \uparrow 2\alpha - 1 = \frac{\sqrt{p} - \sqrt{\bar{p}}}{\sqrt{p} + \sqrt{\bar{p}}}$$

Proof is omitted.

The limit $v_\infty = \dfrac{\sqrt{p} - \sqrt{\bar{p}}}{\sqrt{p} + \sqrt{\bar{p}}}$ is, as a function of $p \in [\frac{1}{2}, 1]$, convex and increasing with values 0 at $p = 1/2$, and 1 at $p = 1$.

Corollary 1.1 If $p = 1/2$ then $a_n = 1/2$ and $v_n = 0$, $\forall n \geq 1$.

If $p = 1$, then $a_{n+1} = \frac{1}{2}(1 + a_n^2)$ (i.e., Moser's sequence) and $v_n = a_n^2$

In Table 1, the values of the game for various p and n are given.

Table 1. Values of the game for various p and n.

	$p=0.5$	$p=0.6$		$p=0.8$		$p=1$	
		a_n	v_n	a_n	v_n	a_n	$v_n\ (=a_n^2)$
$n=1$		0.5	0.05	0.5	0.15	0.5	0.25
2		.525	.0752	.575	.2284	.625	.3906
3	$a_n=\frac{1}{2}$	.5376	.0878	.6142	.2720	.6953	.4834
4	$v_n=0$	.5439	.0942	.6360	.2971	.7417	.5501
5	$\forall n\geq 1$	.5471	.0975	.6486	.3119	.7751	.6008
6		.5488	.0993	.6559	.3205	.8004	.6406
⋮		⋮	⋮				

2. Multistage Poker —— Bilateral-move.

We investigate in this section the bilateral-move variant of the poker discussed in the previous section. `Players act not simultaniously but consecutively,i.e, player I acts first and then player II acts, after knowing his rival's choice of either R or A. Denote this n-round poker by G_n , and let w_n be the value of G_n. The game would be conveniently described by

Player	Hand	1 st step	2nd step	Payoff to I
I	x	R	R	w_{n-1}
		R	A	$pw_{n-1}+\bar{p}\,sgn(x-y)$
II	y	A	R	$p\,sgn(x-y)+\bar{p}\,w_{n-1}$
		A	A	$sgn(x-y)$

Let $\phi_n(x)$ be the probability that I chooses A when his hand is x. Also let $\psi_{Rn}(y)$ ($\psi_{An}(y)$) be the probability that II chooses A when his hand is y and his rival's choice has been R (A). The expected payoff to I when players use the strategy-triple ϕ_n - ψ_{Rn} - ψ_{An} is

$$(2.1)\quad M_n(\phi, \psi_R, \psi_A) = E_{x,y}\Big[w_{n-1}\bar\phi\bar\psi_R + (p w_{n-1} + \bar p\,\mathrm{sgn}(x-y))\bar\phi\psi_R + (p\,\mathrm{sgn}(x-y) + \bar p w_{n-1})\phi\bar\psi_A + \mathrm{sgn}(x-y)\phi\psi_A\Big]$$

$$= w_{n-1} + E\Big[(\mathrm{sgn}(x-y) - w_{n-1})\{p\phi + \bar p(\bar\phi\psi_R + \phi\psi_A)\}\Big],$$

where $\phi_n(x)$, $\psi_{Rn}(y)$ and $\psi_{An}(y)$ are simply written by ϕ, ψ_R and ψ_A, respectively.

The Optimality Equation we want to discuss is

$$(2.2)\quad w_n = \max_{\phi}\ \min_{\psi_R \cdot \psi_A} M_n(\phi, \psi_R, \psi_A), \quad (n \geq 1, w_0 = 0).$$

Theorem 2. The case where $\frac{1}{2} < p < 1$. Determine the three sequences by

$$(2.3)\quad b_n = \frac{1}{2}\left[1 + \frac{2p w_{n-1}}{\bar p(w_{n-1})^2 + 3p - 1}\right],$$

$$(2.4)\quad r_n = \frac{1}{2}\bar w_{n-1} b_n,$$

$$(2.5)\quad \bar a_n = \frac{1}{2}(1 + w_{n-1})\bar b_n \qquad (n \geq 1;\ w_0 = 0)$$

Then $0 < r_n < b_n < a_n < 1$, $\forall n \geq 1$, and the optimal strategy-triple is

$$\phi_n^*(x) = I(x > b_n), \quad \psi_{Rn}^*(y) = I(y > r_n), \quad \psi_{An}^*(y) = I(y > a_n).$$

The value w_n of the game G_n is given by the recursion (2.13) given below.

Proof is omitted.

The two extreme cases are discussed in the following. Proofs are omitted.

Theorem 3. The case $p = \frac{1}{2}$. Let $c_{n-1} = \frac{1 + w_{n-1}}{1 - w_{n-1}} \in (0, 1)$. Determine the three sequences $\{r_n\}, \{a_n\}$ and $\{w_n\}$ by

$$(2.14)\quad \bar a_n = \frac{1 + w_{n-1}}{2(1 + c_{n-1}^2)}, \quad r_n = c_{n-1}\bar a_n \qquad (0 < r_n < a_n < 1, \forall n)$$

and

$$(2.15)\quad w_n = w_{n-1} - \frac{1}{2}\{(w_{n-1} + r_n)\bar r_n - (\bar a_n)^2\}.$$

The solution to the game is

$$\phi_n^*(x) = \begin{cases} 0 & \text{if } x \leq r_n \\ \text{arbitrary in } [0,1], \text{ but satisfies } \int_{r_n}^{a_n}\phi_n^*(x)dx = \bar a_n / c_{n-1}, & \text{if } r_n \leq x \leq a_n \\ 1, & \text{if } x \geq a_n \end{cases}$$

$$\psi_{Rn}^*(y) = I(y > r_n), \quad \text{and} \quad \psi_{An}^*(y) = I(y > a_n),$$

and the value of the game is w_n. Moreover $w_n \downarrow w_\infty$ as $n \to \infty$, and w_∞ is given by (2.22).

[c.f. (2.22) is $w_\infty = -\left[1 - \left(\frac{\sqrt5 - 1}{2}\right)^{1/2}\right] / \left[1 + \left(\frac{\sqrt5 - 1}{2}\right)^{1/2}\right] \doteqdot -0.1197.$]

Theorem 4. The solution to the game for p = 1. The optimal strategy for I is "Choose A (R), if $x>(<)\frac{1}{2}(1+w_{n-1})$", where $\{w_n\}$ is determined by the recursion

$$w_n=\frac{1}{4}(1+w_{n-1})^2, \qquad (n\geqq 1;\quad w_0=0)$$

Player II always chooses R until the game is stopped by I.

Remark 1. We can show after some algebra that (2.4)-(2.5), in Theorem 2, when $p\downarrow 1/2$, become (2.14) in Theorem 3. Also, it is easy to find, from (2.15) and (2.21), that

$$w_n=-\frac{1+c_{n-1}^2-c_{n-1}^4}{2(1+c_{n-1})^2(1+c_{n-1}^2)} \qquad (n\geqq 1,\ w_0=0,\ c_0=1) \tag{2.23}$$

It is a surprizing result that $w_n\downarrow w_\infty$, and the limit is different from -1 and 0. The disadvantage for the first-mover which is unavoidable from leaking some information about his private hand x, does not increase to its extreme and stops at an intermediate point by making a skillful bluff.

Computation of the values which appear in Theorem 3 is shown in Table 2.

Table 2. Solution of the bilateral poker for $p=1/2$.

	r_n	a_n	B_n	w_n	c_n
n=1	1/4	3/4	1/4	−0.0625(=−1/16)	0.8824
2	0.2326	0.7364	0.2973	−0.0930	.8298
3	.2229	.7314	.3237	−0.1074	.8060
4	.2181	.7295	.3357	−0.1141	.7952
5	.2158	.7286	.3412	−0.1171	.7903
6	.2148	.7283	.3438	−0.1185	.7881
7	.2143	.7281	.3450	−0.1192	.7870
8	.2140	.7280	.3456	−0.1195	.7865
9	.2139	.7280	.3458	−0.1196	.7863
10	.2139	.7280	.3459	−0.1197	.7862
11	.2139	.7280	.3459	−0.1197	.7862
⋮					
Limit	0.2139	0.7280	0.3460	−0.1197	0.7862

Remark 2. When p = 1 the game value w_n is equal to the game value v_n in the simultanious-move version of this poker. Because, as was shown by Corollary 1.1, ······.

3. Final Remark.

From Theorem 2 we show a computation of the solution of the bilateral poker for $p = 0.55$ (0.8) in Table 3 (4) for very small n. It is instructive to compare these values with those appearing in Tables 1 and 2. (Tables 3 and 4 are omitted.)

Theorem 2 gives a result that if $p = 0.6$, then

$$r_n = \frac{1}{4}, \quad b_n = \frac{1}{2}, \quad a_n = \frac{3}{4}, \quad w_n = 0, \qquad \text{for all } n \geqslant 1.$$

This implies that the disadvantage for I, which is inevitable due to his "moving-first" disappears if $p = 0.6$.

The values of bilateral poker for various $p \in [\frac{1}{2}, 1]$ are compared in Figure 2. Curves (a),(b),(c), (d),and (e) are for $p = 0.5, 0.55, 0.6, 0.8$, and 1, respectively.

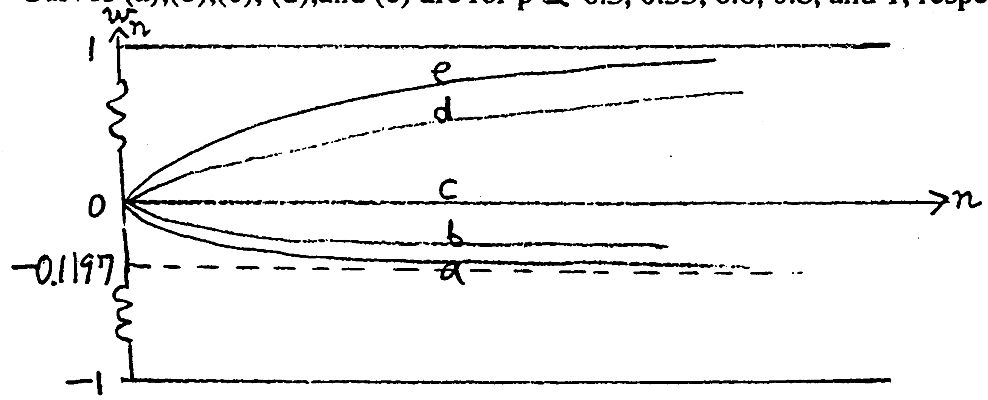

Figure 2. Values of bilateral poker for various p

REFERENCES

[2] Garnaev. A.Y., Values of information on optimal stopping games, Game Th. Appl.5(2000), 55-64.

[4] Mazalov,V.V, Sakaguchi,M. and Zabelin,A.A. Multistage arbitration game with random payoffs, to appear in Game Th. Appl.8 (2002). 75-83.

[7] Sakaguchi, M., Optimal stopping games where players have weighted privilege, Game Th. Appl. 6 (2000),116-131.

* 3-26-4 Midorigaoka, Toyonaka, Osaka 560-0002, Japan.

Fax : 81-6-6856-2314

E-mail: smf@mc.kcom.ne.jp

Full paper is to appear in Game Th. Appl. 8 (2002).