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An existence result for some semi-linear elliptic equation in bent strip-like unbounded domains

東京工業大学大学院 理工学研究科 柴田 将敬 (Masataka Shibata)
Department of Mathematics,
Tokyo Institute of Technology

1 Introduction and Main Result

Let $N \geq 2$ and Ω be an unbounded domain in $\mathbf{R}^N$. We consider the following equation

$$\begin{cases} -\Delta u + \lambda u = u_+^p & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1}$$

where $\lambda \geq 0$ and $1 < p < \infty$ if $N = 2$, $1 < p < (N+2)/(N-2)$ if $N \geq 3$ are given constants. It is well-known that (1) has a positive solution if Ω is bounded. In general, the existence of a positive solution of (1) is unknown if Ω is unbounded. Esteban and Lions showed in [4] that if Ω satisfies following condition (EL) then there is no nontrivial solution.

(EL) There exists a vector $X \in \mathbf{R}^N$ such that $\nu(x) \cdot X \geq 0$ and $\nu(x) \cdot X \not\equiv 0$ on $x \in \partial\Omega$, where $\nu(x)$ is the outer unit normal vector of Ω.

On the other hand, many authors showed existence result. (cf. [1, 3, 5, 7] and references therein). In this paper, we will give an existence result in bent strip-like unbounded domains. We use following notations.

$$S_d := \{x = (x', x_N) \in \mathbf{R}^N; |x'| < d\},$$
$$\hat{S}_d := \{x = (x', x_N) \in S_d; x_N > 0\}.$$

In [6], we conjectured that if $\lambda \geq 0$ and Ω satisfying the following condition (Ω1) then there is a nontrivial solution.

(Ω1) Ω is a domain in $\mathbf{R}^N$ and $\partial\Omega$ is Lipschitz continuous. There are $K \in N \setminus \{1\}$, a bounded set A and congruent transformations Λ_j $(1 \leq j \leq K)$ such that $\Omega = A \cup \Lambda_1(\hat{S}_d) \cup \cdots \cup \Lambda_K(\hat{S}_d)$ and $\Lambda_i(\hat{S}_d) \cap \Lambda_j(\hat{S}_d) = \emptyset$ if $i \neq j$.

This conjecture is still open. In this paper, we consider the following stronger conditions (Ω2), (Ω3) in two dimensional case. Here after, we assume $N = 2$.

($\Omega2$) There are $d > 0$, a smooth curve $\{c(s)\}_{s\in\mathbf{R}}$ parameterized by arc length with the curvature $\kappa(s)$ such that $\mathrm{supp}\{\kappa\}$ is compact and $\Phi : S_d \to \Omega$ is bijective, where Φ is defined by $\Phi(y) := c(y_2) + y_1 e(y_2)$ and $e(s)$ is the unit normal vector of $c(s)$.

($\Omega3$) Ω satisfies ($\Omega1$), $\exists\Omega_0 \subset \Omega$ s.t. Ω_0 satisfies ($\Omega2$).

Remark. If Ω satisfies ($\Omega2$) then

$$\Omega = \{x \in R^2; \mathrm{dist}(x, \{c(s)\}) < d\}.$$

So Ω is a bent strip-like domain.

Remark. Ω satisfies ($\Omega2$) then Ω satisfies ($\Omega3$) with $\Omega = \Omega_0$. Ω satisfies ($\Omega3$) then Ω satisfies ($\Omega1$).

Now we state our main theorem.

Theorem A. *Suppose that* $N = 2$, $\lambda \geq 0$ *and the following equation has unique nontrivial solution up to* x_2 *transformation.*

$$\begin{cases} -\Delta v + \lambda v = v_+^p & \text{in } S, \\ v \in H_0^1(S). \end{cases} \tag{2}$$

If $(\|\kappa\|_{L^\infty} d)^2 < 1 - 2^{(1-p)/(1+p)}$ *then* (1) *has a nontrivial solution.*

Remark. If $\lambda = 0$, (2) has unique nontrivial solution up to x_2 transformation by [2].

2 Preliminaries

At first, we state notations. For a domain D, we define following notations.

$$I[u] := \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + \lambda u^2\,dx - \frac{1}{p+1}\int_{\mathbf{R}^N} u_+^{p+1}\,dx \qquad \text{for } u \in H_0^1(D) \subset H_0^1(\mathbf{R}^N),$$

$$M(D) := \{u \in H_0^1(D) \setminus \{0\}; \int_D |\nabla u|^2 + \lambda u^2\,dx = \int_D u_+^{p+1}\},$$

$$\alpha(\Omega) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_D[\gamma(t)],$$

$$\Gamma := \{\gamma \in C([0,1]; H_0^1(D)); \gamma(0) = 0, I_D[\gamma(1)] \leq 0\}.$$

It is well-known that the mountain pass energy $\alpha(D)$ is well-defined and is equal to a least energy. i.e.

Lemma 2.1. *Let D be a domain. Suppose that D satisfying Poincare's inequality or $\lambda > 0$. Then*

$$\alpha(D) = \inf_{u \in M(D)} I_D[u]$$

and all nontrivial critical point v of I_D satisfies $I_D[v] \geq \alpha(D)$.

(cf. [9]).

Lemma 2.2. *If Ω satisfies ($\Omega 1$). Then Poincare's inequality holds. i.e. There exists a constant $C > 0$ such that*

$$\int_\Omega u^2 \, dx \leq C \int_\Omega |\nabla u|^2 \, dx.$$

By Lemma 2.2, we can use the norm

$$\|v\|^2_{H^1_0(\Omega)} = \int_\Omega |\nabla v|^2 \, dx.$$

Lemma 2.3. *Let K be a complete metric space, $K_0 \subset K$ be a closed set, X be a Banach space and $\chi \in C(K_0, X)$. Define Γ by*

$$\Gamma := \{\gamma \in C(K, X); \gamma(s) = \chi(s) \text{ if } s \in K_0\}.$$

For $I \in C^1(X, \mathbf{R})$, put

$$c := \inf_{\gamma \in \Gamma} \max_{s \in K} I[\gamma(s)], \qquad c_1 := \max_{v \in K_0} I[\chi(v)].$$

If $c > c_1$ then for all $\epsilon > 0$ and $\gamma \in \Gamma$ with $\max_{s \in K} I[\gamma(s)] \leq c + \epsilon$, there exists $v \in X$ such that

$$c - \epsilon < I[v] < \max_{s \in K} I[\gamma(s)], \qquad \operatorname{dist}(v, g(K)) \leq \epsilon^{\frac{1}{2}}, \qquad \|I'[v]\| \leq \epsilon^{\frac{1}{2}}.$$

Especially, there is a Palais-Smale sequence.

For the proof of this Lemma, see [8, Theorem 4.3].

Proposition 2.4 (Concentration Compactness). *Suppose* ($\Omega 1$). *Let $\{u_n\}_{n=1}^\infty$ be nonnegative Palais-Smale β-sequence for I_Ω in $H^1_0(\Omega)$. i.e.*

$$I_\Omega[u_n] = \beta + o(1), \qquad I'_\Omega[u_n] = o(1) \qquad \text{as } n \to \infty.$$

Then there exist a non-negative number l, $k_1, \dots .k_l \in \{1, \dots, k\}$, $\{z_n^i\}_{n=1}^{\infty} \subset \Lambda_{k_i}(\{x = (x', x_N); x' = 0\})$, $u^0 \in H_0^1(\Omega)$ *with* $u \geq 0$, $u^i \in H_0^1(\Lambda_{k_i}(S))$ *with* $u^i > 0$ *for* $1 \leq i \leq l$ *such that*

$$\begin{aligned}
&u_n(x) = u^0(x) + u^1(x - z_n^1) + \cdots + u^l(x - z_n^l) + o(1) \\
&\quad \text{as } n \to \infty \text{ in } H_0^1(\mathbf{R}^N), \\
&I_\Omega[u_n] = I_\Omega[u^0] + I_{\mathbf{R}^N}[u^1] + \cdots + I_{\mathbf{R}^N}[u^l] + o(1) \text{ as } n \to \infty, \\
&\begin{cases} -\Delta u^0 + \lambda u^0 = (u^0)^p & \text{in } \Omega, \\ -\Delta u^i + \lambda u^i = (u^i)^p & \text{in } \Lambda_{k_i}(S), \end{cases} \\
&|z_n^i| \to \infty \text{ as } n \to \infty.
\end{aligned}$$

We can give the proof of Lemma 2.4 by using same argument as in [7]. For reader's convenience, we give the proof in Appendix. To prove theorem A, we use the following functional. Take $\phi \in C(\mathbf{R}^N, [-1, 1])$ satisfying

$$\phi(x) = \begin{cases} 1 & x \in \Lambda_i(S_0), i\text{:odd}, \\ -1 & x \in \Lambda_i(S_0), i\text{:even}, \\ 0 & \textit{otherwise}. \end{cases}$$

Define the functional $h : L^2(\mathbf{R}^N) \setminus \{0\} \to [-1, 1]$ by

$$h[u] := \frac{1}{\|u\|_{L^2(\mathbf{R}^N)}^2} \int_{\mathbf{R}^N} \phi(x)|u(x)|^2 \, dx \qquad \text{for } u \in L^2(\mathbf{R}^N) \setminus \{0\}.$$

h is a continuous function in the following sense.

Lemma 2.5. *There is a constant* $C > 0$ *such that*

$$|h[u + v] - h[u]| \leq \frac{C(\|u\|_{L^2(\mathbf{R}^N)} + \|v\|_{L^2(\mathbf{R}^N)})}{\|u\|_{L^2(\mathbf{R}^N)}^2} \|v\|_{L^2(\mathbf{R}^N)}$$

for all $u, v \in L^2(\mathbf{R}^N)$ *with* $u \neq 0$ *and* $u+v \neq 0$. *Especially,* $|h[u+v] - h[u]| \leq C\|v\|_{L^2(\mathbf{R}^N)}/\|u\|_{L^2(\mathbf{R}^N)}$ *if* $\|v\|_{L^2(\mathbf{R}^N)} < \|u\|_{L^2(\mathbf{R}^N)}$.

We can show Lemma 2.5 by elementary calculus. We omit the proof of

3 Proof of Theorem A and Theorem B

To prove Theorem A, we consider the following mountain-pass value $\alpha_0(\Omega)$. Put

$$H = \{u \in H_0^1(\Omega); h[u] = 0\} \cup \{0\},$$
$$\alpha_0(\Omega) := \inf_{\gamma \in \Gamma_0} \sup_{t \in [0,1]} I[\gamma(t)],$$
$$\Gamma_0 := \{\gamma \in C([0,1], H); g(0) = 0, I[g(1)] \leq 0\}.$$

Here, it is easy to see that H is a closed subspace of $H_0^1(\Omega)$. By the definition of $\alpha_0(\Omega)$, $\alpha(\Omega) \leq \alpha_0(\Omega)$ holds. It is well-known that $0 < \alpha(\Omega) \leq \alpha(S_d)$ if Ω satisfies (Ω1) because of $\alpha(\hat{S}_d) = \alpha(S_d)$. So one of following cases holds.

(a) $\alpha(\Omega) < \alpha(S_d)$.

(b) $\alpha(\Omega) = \alpha(S_d)$ and $\alpha_0(\Omega) = \alpha(S_d)$.

(c) $\alpha(\Omega) = \alpha(S_d)$ and $\alpha_0(\Omega) > \alpha(S_d)$.

Proposition 3.1. *Suppose that* (Ω1). *If the case (a) or (b) holds then* (1) *has a positive solution.*

Proposition 3.1 is proved by standard arguments by using concentration compactness principle. We omit the proof of it. By Proposition 3.1, it is enough to show that Theorem A in the case (c). Hereafter, we suppose (c) and $N = 2$. For the proof of Theorem A, the least energy solution on S_d plays important role. Let $v \in H_0^1(S_d)$ be a least energy solution on S_d. i.e.

$$\begin{cases} -\Delta v + \lambda v = v_+^p & \text{in } S_d, \\ v > 0 & \text{on } \partial S_d, \end{cases}$$
$$I[v] = \alpha(S_d).$$

The existence of such solution is well-known. By the moving plain method, we can assume that

$$v(x) = v(x_1, x_2) = v(|x_1|, |x_2|) \qquad \text{for all } x \in S_d.$$

By the equation, we see

$$\int_{S_d} |\nabla v|^2 + \lambda v^2 \, dx = \int_{S_d} v_+^{p+1} \, dx. \tag{3}$$

Since $(\Omega 3)$, $\Psi := \Phi^{-1}$ is well-defined. Define v_t, u_t by

$$v_t(x) := v(\Psi_1(x), \Psi_2(x) - t), \qquad u_t(x) = s(t)v_t(x),$$

where $s(t)$ is uniquely determined positive constant satisfying $u_t(x) \in M(\Omega)$ for each t. (see Lemma 4.1.)

Lemma 3.2. *If* $(d\|\kappa\|_{L^\infty(\mathbf{R})})^2 < 1 - 2^{(1-p)/(1+p)}$ *then there exist constants* $t_0, s_0 > 0$ *such that*

$$I[u_{\pm t_0}] < \frac{1}{2}(\alpha(S) + \alpha_0(\Omega)), \tag{4}$$

$$h[u_{t_0}] > \frac{1}{2}, \qquad h[u_{-t_0}] < -\frac{1}{2}, \tag{5}$$

$$I[sv_t] \leq 0 \qquad \text{if } s \geq s_0, \tag{6}$$

$$I[u_t] < 2\alpha(S) \qquad \text{for all } t \in \mathbf{R}. \tag{7}$$

Proof. By elementally calculation for Φ,

$$\begin{aligned} I[sv_t] =& \frac{s^2}{2}\int_{S_d} \frac{1}{1 - y_1\kappa(y_2)} v_{y_2}^2(y_1, y_2 - t) + (1 - y_1\kappa(y_2))v_{y_1}^2(y_1, y_2 - t) \\ &+ \lambda(1 - y_1\kappa(y_2))v^2(y_1, y_2 - t)\, dy \\ &- \int_{S_d} (1 - y_1\kappa(y_2))F(sv(y_1, y_2 - t))\, dy. \end{aligned}$$

Since v is even function with respect to y_1 and $1/(1+t)+1/(1-t) = 2/(1-t^2)$, we have

$$I[sv_t] = \frac{s^2}{2}\int_{S_d} \frac{1}{1 - (y_1\kappa(y_2+t))^2} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2\, dy - \frac{1}{p+1}\int_{S_d} (sv)_+^{p+1}\, dy.$$

Since $\frac{d}{ds}I[sv_t]|_{s=s(t)} = 0$, we obtain

$$\int_{S_d} \frac{1}{1 - (y_1\kappa(y_2+t))^2} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2\, dy = s(t)^{p-1}\int_{S_d} v_+^{p+1}\, dy \tag{8}$$

Here, the right hand side is increasing with respect to s and

$$\int_{S_d} \frac{1}{1 - (y_1\kappa(y_2+t))^2} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2\, dy > \int_{S_d} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2\, dy = \int_{S_d} v^{p+1}\, dy \tag{9}$$

by (3). So we have

$$s(t) \geq 1. \tag{10}$$

By using Lesbergue's convergence theorem, the left hand side of (9) tends to $\int_{S_d} |\nabla v|^2 + \lambda v^2\, dy$ as $t \to \pm\infty$. It and (3) mean $s(t) \to 1$ as $t \to \pm\infty$. It asserts $I[u_t] \to \alpha(S)$ as $t \to \pm\infty$. So (4) holds for sufficiently large t_0.

(8) and (3) assert

$$s(t)^{p-1} \le \frac{1}{1-(d\|\kappa\|_{L^\infty(\mathbf{R})})^2}. \tag{11}$$

By (8), (11) and the assumption of Theorem A, we can obtain

$$\begin{aligned} I[u_t] =& \left(\frac{1}{2} - \frac{1}{p+1}\right) s(t)^2 \int_{S_d} \frac{1}{1-(y_1\kappa(y_2+t))^2} v_{y_2}^2 + v_{y_1}^2 + \lambda v^2\, dy \\ \le & s(t)^2 \frac{1}{1-(d\|\kappa\|_{L^\infty(\mathbf{R})})^2} \alpha(S_d) \\ \le & \left(\frac{1}{1-(d\|\kappa\|_{L^\infty(\mathbf{R})})^2}\right)^{\frac{p+1}{p-1}} \alpha(S_d) \\ < & 2\alpha(S_d). \end{aligned}$$

It means (7) holds for any $t \in \mathbf{R}$. It is easy to see that

$$I[sv_t] \le \frac{s^2}{2} \frac{1}{1-(d\|\kappa\|_{L^\infty(\mathbf{R})})^2} \int_{S_d} |\nabla v|^2 + \lambda v^2\, dy - \frac{s^{p+1}}{p+1} \int_{S_d} v_+^{p+1}\, dy$$

The right hand side is independent of t and tends to $-\infty$ as $s \to \infty$. So we obtain (6) for sufficiently large s_0.

By the assumption ($\Omega 3$) and the definition of v_t, we have

$$\|\chi_{\Lambda_1(\hat{S}_d)} v_t - v_t\|_{L^2(\mathbf{R}^2)} \to 0 \text{ and } \|\chi_{\Lambda_1(\hat{S}_d)} v_t\|_{L^2(\mathbf{R}^2)} \to \|v\|_{L^2(\mathbf{R}^2)} \ne 0.$$

Since $h[\chi_{\Lambda_1(\hat{S}_d)} v_t] = -1$ and Lemma 2.5, we obtain

$$h[v_t] \to -1 \qquad \text{as } t \to -\infty.$$

Similarly,

$$h[v_t] \to 1 \qquad \text{as } t \to \infty$$

holds. It completes the proof of Lemma 3.2. □

Put $K := [0, s_0] \times [-t_0, t_0]$ and define β by

$$\begin{aligned} \beta :=& \inf_{\gamma \in \Gamma} \max_{(s,t) \in K} I[g(s,t)], \\ \Gamma_1 :=& \{\gamma \in C(S, H_0^1(\Omega)); g(s,t) = sv_t \text{ if } (s,t) \in \partial K\}. \end{aligned}$$

Then the following Lemma 3.4 and Lemma 3.3 hold.

Lemma 3.3. *Suppose same assumptions as in Lemma 3.2 then*

$$\alpha(S) < \beta < 2\alpha(S).$$

Lemma 3.4. *Suppose same assumptions as in Lemma 3.2. Then there is a Palais-Smale β-sequence $\{u_n\}_{n=1}^{\infty}$. i.e.*

$$I[u_n] = \beta + o(1), \qquad \|I[u_n]\| = o(1) \qquad \textit{as } n \to \infty.$$

Proof of Lemma 3.3. Put $\gamma_0(s,t) = sv_t$ for $(s,t) \in K$ then $\gamma_0 \in \Gamma_1$. By the assumption of f, we have $I[sv_t] \le I[u_t]$. Lemma 3.2 asserts that $I[\gamma_0(s,t)] \le 2\alpha(S)$ for all $(s,t) \in K$. Hence $\beta < 2\alpha(S)$.

Fix any $\gamma \in \Gamma_1$, Lemma 3.2 and similar argument as in [10] show that there is a curve $\tau : [0,1] \to K$ such that $\gamma \circ \tau \in \Gamma_0$. So we have

$$\max_{(s,t)\in K} I[\gamma(s,t)] \ge \max_{t\in(0,1)} I[\gamma \circ \tau(t)] \ge \alpha_0(\Omega).$$

It means $\alpha(S) < \beta$ by the condition (c). □

Proof of Lemma 3.4. Put $\gamma_0(s,t) = sv_t$ for $(s,t) \in K$ then $\gamma_0 \in \Gamma_1$, Lemma 3.2 asserts

$$\max_{(s,t)\in\partial K} I[\gamma_0(s,t)] \le \frac{1}{2}(\alpha_0(\Omega) + \alpha(S)) < \beta.$$

So we can apply Lemma 2.3 to obtain the existence of Palais-Smale β sequence. □

Now we can prove Theorem B in the following Proposition.

Proposition 3.5. *Suppose that same assumption as in Theorem A. Then there is a positive solution.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a Palais-Smale β sequence in Lemma 3.4. By Proposition 2.4, by passing to a subsequence if necessary, there is a nonnegative number l such that

$$\begin{aligned} u_n(x) &= u^0(x) + u^1(x - x_n^1) + \cdots + u^l(x - x_n^1) + o(1) \qquad \text{as } n \to \infty \text{ in } H_0^1(\mathbf{R}^2), \\ I[u_n] &= I[u^0] + I[u^1] + \cdots + I[u^l] + o(1) \qquad \text{as } n \to \infty. \end{aligned}$$

If $u^0 \not\equiv 0$ then u^0 is a positive solution. So it is enough to show that $u^0 \not\equiv 0$.

Suppose $u \equiv 0$ then $l \ge 1$ and

$$I[u_n] = I[u^1] + \cdots + I[u^l] + o(1) \ge l\alpha(S) + o(1) \qquad \text{as } n \to \infty.$$

Since Lemma 3.4, we have $\beta < 2\alpha(S)$. So we can obtain $l = 1$. It mean that

$$I[u_n] = I[u^1] + o(1) \qquad \text{as } n \to \infty.$$

Hence $I[u_1] = \beta$. So, wee see that $u_1(\Lambda_{k_1}(x))$ is a critical point of I in $H_0^1(\Lambda_{k_1}(\hat{S}_d))$ with $I[u_1(\Lambda_{k_1}(x))] = \beta$. It contradicts to the uniqueness of nontrivial solutions on $\Lambda_{k_1}(S_d)$. Consequently, there exists a positive solution u^0. □

4 Appendix

In this section, we note well-known facts and give the proof of Proposition 2.4. First, we note some properties for f.

Lemma 4.1. *Suppose that D is a domain in $\mathbf{R}^N$. Fix $v \in H_0^1(D)$ with $v_+ \neq 0$ in $H_0^1(D)$. Then there is an uniquely determined constant $s_0 > 0$ such that*

$$\frac{d}{ds} I[sv] \Big|_{s=s_0} = 0.$$

Moreover,

$$\max_{s>0} I[sv] = I[s_0 v].$$

Proof. We see

$$\frac{1}{s}\frac{d}{ds} I_D[sv] = \int_D |\nabla v|^2 + \lambda v^2\, dx - s^{p-1} \int_D v_+^{p+1}\, dy$$

if $s > 0$. Second term of the right hand side is strictly decreasing with respect to s on $(0, \infty)$. Moreover, second term equals to 0 if $s = 0$ and tends to $-\infty$ as $s \to \infty$. Consequently, we obtain this Lemma. □

Proof of Proposition 2.4. By the assumption of u_n, we have

$$< I'[u_n], u_n > = ||u_n||^2_{H_0^1(\Omega)} + \lambda ||u_n||^2_{L^2(\Omega)} - \int_\Omega (u_n)_+^{p+1}\, dx = o(1) ||u_n||_{H_0^1(\Omega)} \quad \text{as } n \to \infty. \tag{12}$$

So we have

$$C \geq I[u_n] = (\frac{1}{2} - \frac{1}{p+1})(||u_n||^2_{H_0^1(\Omega)} + \lambda ||u_n||^2_{L^2(\Omega)}) + o(1) ||u_n||_{H_0^1(\Omega)} \quad \text{as } n \to \infty. \tag{13}$$

So we see that u_n is bounded in $H_0^1(\Omega)$. By using weak compactness for Hilbert space and Rellich's compactness, there exists $u^0 \in H_0^1(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u^0 && \text{weakly in } H_0^1(\Omega) \text{ as } n \to \infty, \\ u_n &\to u^0 && \text{in } L^p_{\text{loc}}(\Omega) \text{ as } n \to \infty, \\ u_n &\to u^0 && \text{a.e. in } \Omega \text{ as } n \to \infty, \end{aligned}$$

by passing to a subsequence if necessary. So we obtain

$$I'[u_n] \rightharpoonup I'[u^0] \qquad \text{weakly in } H^{-1}(\Omega).$$

It means u^0 is a critical point of I. Put $\phi_n^1 := u_n - u_0$ then

$$\phi_n^1 \rightharpoonup 0 \qquad \text{weakly in } H_0^1(\Omega) \text{ as } n \to \infty, \tag{14}$$

$$\phi_n^1 \to 0 \qquad \text{in } L^p_{\text{loc}}(\Omega) \text{ as } n \to \infty. \tag{15}$$

Moreover, we have

$$\|\phi_n^1\|^2_{H_0^1(\Omega)} = \|u_n\|^2_{H_0^1(\Omega)} - \|u_0\|^2_{H_0^1(\Omega)} + o(1) \qquad \text{as } n \to \infty.$$

We can apply Brezis-Lieb's theorem to obtain

$$\int_\Omega (\phi_n^1)^{p+1}\,dx = \int_\Omega (u_n)^{p+1}\,dx - \int_\Omega (u^0)^{p+1}\,dx.$$

By using Vitali's Lemma, we have

$$I'[\phi_n^1] = I'[u_n] - I'[u^0] + o(1) = o(1) \qquad \text{in } H^{-1}(\Omega) \text{ as } n \to \infty. \tag{16}$$

Suppose $\phi_n^1 \to 0$ in $H_0^1(\Omega)$ as $n \to \infty$, by passing to a subsequence if necessary. Then the proof is complete since $u_n \to u^0$ in $H_0^1(\Omega)$ as $n \to \infty$. So, hear-after, we can assume ϕ_n^1 is not convergence to 0 in $H_0^1(\Omega)$ for any subsequence. Put

$$\begin{aligned} Q_0 &:= \Omega \setminus (\Lambda_1(\hat{S}_d) \cup \dots \Lambda_k(\hat{S}_d)), \\ Q_m &:= \{x = (x', x_N) \in S; m - 1 < x_N \leq m\}, \\ Q_m^j &:= \Lambda_j(Q_m) \end{aligned}$$

for $m \geq 1$, $1 \leq j \leq k$. Define d_n and $\hat{d}_n$ by

$$d_n := \max_{m \in \mathbf{N}, 1 \leq j \leq k} \|\phi_n^1\|_{L^2(Q_m^j)}, \quad \hat{d}_n := \max\{d_n, \|\phi_n^1\|_{L^2(Q_0)}\}$$

and show that

$$\liminf_{n \to \infty} \hat{d}_n > 0.$$

Since Q_n^j is congruence we can apply Sobolev's inequality to obtain

$$\|\phi_n^1\|_{L^r(Q_m^j)} \le C(r)\|\phi_n^1\|_{H_0^1(Q_m^j)}$$

for $q+1 < r \le 2^*$ where $C(q)$ is a positive constant independent of n, j. By using interpolation it holds that

$$\|\phi_n^1\|_{L^{q+1}(Q_m^j)}^{q+1} \le C(r)\|\phi_n^1\|_{L^2(Q_m^j)}^{(1-\theta)(q+1)}\|\phi_n^1\|_{H_0^1(Q_m^j)}^{\theta(q+1)}$$

where $1/(q+1) = (1-\theta)/2+\theta/r$. Since $\theta \to 1$ as $r \to q+1$, $\theta(q+1)-2 > 0$ for r near $q+1$. Fix such r then we have

$$\int_{Q_m^j} |\phi_n^1|^p\, dx \le C d_n^{(1-\theta)(q+1)}\|\phi_n^1\|_{H_0^1(\Omega)}^{\theta(q+1)-2} \int_{Q_m^j} |\nabla\phi_n^1|^2\, dx.$$

Similarly for Q_0, we have

$$\int_{Q_0} |\phi_n^1|^p\, dx \le C \hat{d}_n^{(1-\theta)(q+1)}\|\phi_n^1\|_{H_0^1(\Omega)}^{\theta(q+1)-2} \int_{Q_0} |\nabla\phi_n^1|^2\, dx.$$

By taking sum, we obtain

$$\int_\Omega |\phi_n^1|^p\, dx \le C \hat{d}_n^{(1-\theta)(q+1)}\|\phi_n^1\|_{H_0^1(\Omega)}^{\theta(q+1)-2} \int_\Omega |\nabla\phi_n^1|^2\, dx$$

If $\hat{d}_n \to 0$ as $n \to \infty$ for some subsequence then $\|\phi_n^1\|_{L^q(\Omega)} \to 0$ as $n \to \infty$. On the other hand, by (16),

$$o(1) = I'_\Omega[\phi]\phi = \|\phi_n^1\|_{H_0^1(\Omega)}^2 + \lambda\|\phi_n^1\|_{L^2(\Omega)}^2 - \int_\Omega (\phi_n^1)_+^{p+1}\, dx.$$

By Sobolev's inequality,

$$\int_\Omega (\phi_n^1)^{p+1}\, dx \le \epsilon C\|\phi_n^1\|_{H_0^1(\Omega)}^2 + C(\epsilon)\|\phi_n^1\|_{L^{q+1}(\Omega)}^{q+1}.$$

So, for sufficiently small ϵ, we have

$$\|\phi_n^1\|_{H_0^1(\Omega)}^2 \le C\|\phi_n^1\|_{L^{q+1}(\Omega)}^{q+1} = o(1) \qquad \text{as } n \to \infty.$$

It is contradiction. So we obtain $\liminf_{n\to\infty} \hat{d}_n > 0$.

Here, by passing to a subsequent if necessary, there is $j(n) \in \{1, \ldots, k\}$ and $m(n) \in \mathbf{N} \cup \{0\}$ such that $\|\phi_n^1\|_{Q_{m(n)}^{j(n)}}$, where $Q_{m(n)}^j(n) = Q_0$ if $m(n) = 0$. We can assume $j(n) \equiv j$ by passing to a subsequence if necessary. We show that $m(n) \to \infty$ as $n \to \infty$. Suppose that there is a contant m_0 such that $m(n) \leq m_0$ for all n. Then

$$d_n^2 \leq \sum_{0 \leq m \leq m_0} \|\phi_n^1\|^2_{L^2(Q^j_m)} = \|\phi_n^1\|_{L^2(Q)},$$

where $Q = \cup_{0 \leq m \leq m_0} Q^j_m$. As $n \to \infty$, it contradicts to (15). We can assume that $m(n)$ is increasing without loss of generality.

Define the map Λ by

$$\Lambda(x) := \Lambda_j(x', x_n + m(n) - 1).$$

Then $\Lambda(Q_1) = Q^j_{m(n)}$, $\Lambda(\hat{S}_d) = \sum_{m \geq m(n)} Q^j_m$. Put $\hat{\phi}^1_n := \phi^1_n \circ \Lambda$ then we have

$$\|\hat{\phi}^1_n\|_{H^1(\mathbf{R}^N)} < C, \qquad \|\hat{\phi}^1_n\|_{L^2(Q_1)} \geq d_n.$$

By the weak compactness of $H^1(\mathbf{R}^N)$, there exists $\hat{u}^1 \in H^1(\mathbf{R}^N)$ such that

$$\hat{\phi}^1_n \rightharpoonup \hat{u}^1 \qquad \text{weakly in } H^1(\mathbf{R}^N)$$

by passing to a subsequence if necessary. Here, we can assume parallel transformation to Λ_j are $\Lambda_{j+1}, \ldots, \Lambda_{j+\hat{j}}$ for some $\hat{j} \in \mathbf{N} \cup \{0\}$. So there is a cone V such that $V \cap \Omega \subset V \cap (\Lambda_j(S_d) \cup \Lambda_{j+\hat{j}}(S_d))$. It means that for $n_0 \in \mathbf{N}$,

$$\begin{aligned}\hat{\phi}^1_n = 0 \text{ on } &\Lambda_j^{-1}(V \setminus (\Lambda_j(S_d) \cup \cdots \cup \Lambda_{j+\hat{j}}(S_d)) - (0, m(n_0) - 1) \\ &= (\Lambda_j^{-1}V - (0, m(n_0) - 1)) \setminus (S_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d)) \\ &\text{if } n \geq n_0.\end{aligned}$$

As $n \to \infty$, we obtain

$$\hat{u}^1 = 0 \text{ on } (\Lambda_j^{-1}V - (0, m(n_0) - 1)) \setminus (S_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d))$$

As $n_0 \to \infty$, we have

$$\hat{u}^1 = 0 \text{ on } \mathbf{R}^N \setminus (S_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d))$$

It means that there is $\hat{u}^{1,0} \in H_0^1(S_d)$, $\hat{u}^{1,1} \in H_0^1(\Lambda_j^{-1} \circ \Lambda_{j+1}(S_d)), \ldots, \hat{u}^{1,\hat{j}} \in H_0^1(\Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d))$ such that $\hat{u}^1 = \hat{u}^{1,1} + \cdots + \hat{u}^{1,\hat{j}}$.

Fix any $\psi \in C_0^\infty(S_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d))$. Since $m(n) \to \infty$ as $n \to \infty$, $\Lambda(\mathrm{supp}\psi) \subset \Omega$ for large n. So we have

$$\begin{aligned}&\left| \int_{\mathbf{R}^N} \nabla \hat{\phi}^1_n \nabla \psi + \lambda \hat{\phi}^1_n \psi - (\hat{\phi}^1_n)^p_+ \psi \, dx \right| \\ =&\left| \int_{\mathbf{R}^N} \nabla \phi^1_n \nabla (\psi \circ \Lambda) + \lambda \phi^1_n (\psi \circ \Lambda) - (\phi^1_n)^p_+ \psi \circ \Lambda \, dx \right| \\ =&| < I'[\phi^1_n], \psi \circ \Lambda > | \leq o(1) \|\psi \circ \Lambda\|_{H^1(\mathbf{R}^N)} = o(1) \|\psi\|_{H^1(\mathbf{R}^N)}.\end{aligned}$$

As $n \to \infty$, we obtain

$$\int_{\mathbf{R}^N} \nabla \hat{u}^1 \nabla \psi + \lambda \hat{u}^1 \psi - (\hat{u}^1)_+^p \psi \, dx = 0.$$

It means

$$I'[\hat{u}^1] = 0 \qquad \text{in } H^{-1}(S_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d)).$$

Hence $\hat{u}^{1,i}$ is a weak solution of

$$\begin{cases} -\Delta \hat{u}^{1,i} + \lambda \hat{u}^{1,i} = (\hat{u}^{1,i})_+^p & \text{in } \Lambda_j^{-1} \circ \Lambda_{j+i}(S_d), \\ \hat{u}^{1,i} \in H_0^1(\Lambda_j^{-1} \circ \Lambda_{j+i}(S_d)) & \end{cases}$$

for $0 \le i \le \hat{j}$. Put $u^{i+1}(x) := \hat{u}^{1,i} \circ \Lambda_j^{-1}$ and $z_n^{i+1} := \Lambda_j(x', m(n) - 1)$ with $\Lambda_j(x', 0) \in \Lambda_{j+i}(\{y' = 0\})$. for $0 \le i \le \hat{j}$. Then

$$\begin{cases} -\Delta u^{i+1} + \lambda u^{i+1} = (u^{i+1})_+^p, u^{i+1} > 0 & \text{in } \Lambda_{j+i}(S), \\ u^{i+1} = 0 & \text{on } \partial \Lambda_{j+i}(S), \end{cases}$$

$$\begin{aligned} &\phi_n^1(x) \rightharpoonup u^1(x - z_n^1) + \cdots + u^{1+\hat{j}}(x - z_n^{1+\hat{j}}) && \text{weakly in } H^1(\mathbf{R}^N), \\ &\phi_n^1(x) \to u^1(x - z_n^1) + \cdots + u^{1+\hat{j}}(x - z_n^{1+\hat{j}}) && \text{in } L_{\text{loc}}^p(\mathbf{R}^N), \\ &\phi_n^1(x) \to u^1(x - z_n^1) + \cdots + u^{1+\hat{j}}(x - z_n^{1+\hat{j}}) && \text{a.e. in } \mathbf{R}^N \quad \text{as } n \to \infty \end{aligned}$$

for $0 \le i \le \hat{j}$. If $\phi_n^1 \to u^1(x - z_n^1) + \cdots + u^{1+\hat{j}}(x - z_n^{1+\hat{j}})$ strongly in $H_0^1(\mathbf{R}^N)$ for some subsequence then the proof is complete.

If not, by using the argument above, inductively, by passing to a subsequence if necessary, we have

$$\begin{aligned} &\phi_n^l(x) = u_n(x) - u^0(x) - u^1(x - z_n^1) - \cdots - u^l(x - z_n^l) + o(1) \text{ weakly in } H_0^1(\mathbf{R}^N), \\ &\|\phi_n^l\|_{H_0^1(\mathbf{R}^N)} = \|u_n\|_{H_0^1(\mathbf{R}^N)} - \|u^0\|_{H_0^1(\mathbf{R}^N)} - \|u^1\|_{H_0^1(\mathbf{R}^N)} - \cdots - \|u^l\|_{H_0^1(\mathbf{R}^N)} \qquad \text{as } n \to \infty. \end{aligned}$$

Since $\|u^1\|_{H_0^1(\mathbf{R}^N)}, \ldots, \|u^l\|_{H_0^1(\mathbf{R}^N)} \ge c\alpha(S)$ and $\|u_n\|_{H_0^1(\mathbf{R}^N)}$ is uniformly bounded, there is some $l \ge 1$ such that $u_n(x) = u^0(x) + u^1(x - z_n^1) + \cdots + u^l(x - z_n^l) + o(1)$ strongly in $H_0^1(\mathbf{R}^N)$. It completes the proof. □

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