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Sequential estimation of the ratio of two exponential scale parameters

Chikara UNO

Department of Mathematics, Akita University

Akita 010-8502, Japan

(秋田大学・教育文化学部　宇野 力)

1. Introduction

Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be independent observations from the populations Π_1 and Π_2 respectively, where Π_i is according to an exponential distribution having the probability density function

$$f_{\sigma_i}(x) = \sigma_i^{-1} \exp(-x/\sigma_i), \quad x > 0$$

with $0 < \sigma_i < \infty$ for $i = 1, 2$. We assume that the scale parameters σ_1 and σ_2 are both unknown and two populations Π_1 and Π_2 are independent. We want to estimate the ratio σ_1/σ_2 of scale parameters. Taking samples of sizes n and m from Π_1 and Π_2 respectively, we estimate $\theta = \sigma_1/\sigma_2$ by

$$\hat{\theta}_{(n,m)} = \overline{X}_n/\overline{Y}_m$$

where $\overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ and $\overline{Y}_m = m^{-1}\sum_{i=1}^{m} Y_i$. As the loss function, we consider

$$L(\hat{\theta}_{(n,m)}) = (\hat{\theta}_{(n,m)} - \theta)^2 + c(n+m)$$

where $c > 0$ is the known cost per unit sample in each population, and the risk is given by $R(\hat{\theta}_{(n,m)}) = E\{L(\hat{\theta}_{(n,m)})\}$ which is finite if $m > 2$.

As for two-sample case, sequential estimation of the difference of the means under the above loss structure has been considered in the literature. Ghosh and Mukhopadhyay (1980) and Mukhopadhyay and Chattopadhyay (1991) considered the normal and the exponential cases, respectively and gave second order approximations to the risks as $c \to 0$. Mukhopadhyay and Purkayastha (1994) treated the same problem in the case of unspecified distributions. For the present exponential distributions, it is sometimes of interest to estimate the ratio of scale parameters rather than the difference of the means and hence, in this paper we propose a sequential procedure for estimating σ_1/σ_2. In Section 2, we present a fully sequential procedure and give second order asymptotic expansions for the expected sample size and the regret of the sequential procedure. Bias-corrected procedure is also proposed to reduce the risk. These procedures can be applied to the estimation of the ratio of two normal variances, which are stated in Section 3 and proved in Section 4.

2. Main results

In this section, we propose a fully sequential procedure and give second order asymptotic properties of the procedure. Let $m > 2$. Estimating $\theta = \sigma_1/\sigma_2$ by $\hat{\theta}_{(n,m)}$, the risk is given by

$$R(\hat{\theta}_{(n,m)}) = E(\overline{X}_n/\overline{Y}_m - \theta)^2 + c(n+m) = (\tfrac{1}{n} + \tfrac{1}{m})\theta^2 + r_{n,m}\theta^2 + c(n+m),$$

where $r_{n,m} = (\frac{1}{n} + \frac{1}{m})\frac{3m-2}{(m-1)(m-2)} + \frac{2}{(m-1)(m-2)}$. Since $r_{n,m} = O\left((\frac{1}{n} + \frac{1}{m})^2\right)$ as n and m tend to infinity, we have

$$R(\hat{\theta}_{(n,m)}) = (\tfrac{1}{n} + \tfrac{1}{m})\theta^2 + c(n+m) + O\left((\tfrac{1}{n} + \tfrac{1}{m})^2\right).$$

If we ignore the order term above, then the risk $R(\hat{\theta}_{(n,m)})$ is (approximately) minimized by taking

$$n = m = c^{-1/2}\theta = n^* \quad \text{(say)} \tag{2.1}$$

(in practice, one of the two integers closest to this value) with $R(\hat{\theta}_{(n^*,n^*)}) \approx 4cn^*$ for sufficiently small c. But σ_1 and σ_2 are unknown, so is n^*. Since fixed sample size procedures are not available, we propose the following sequential sampling procedure motivated by (2.1). As the starting sample sizes, we take $X_1, \ldots, X_k$ and $Y_1, \ldots, Y_k$ from Π_1 and Π_2 respectively, where $k > 2$. If $k < c^{-1/2}\overline{X}_k/\overline{Y}_k$, then we take one observation in addition from each population, that is, X_{k+1} and Y_{k+1} are taken from Π_1 and Π_2 respectively. The resulting stopping time is defined by

$$N = N_c = \inf\{n \geq k : \; n \geq c^{-1/2}\overline{X}_n/\overline{Y}_n\}.$$

Then, by the strong law of large numbers, $P(N < \infty) = 1$ for all $c > 0$. Once the sampling stops, using the total $2N$ samples $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$, we estimate $\theta = \sigma_1/\sigma_2$ by $\hat{\theta}_N \equiv \hat{\theta}_{(N,N)} = \overline{X}_N/\overline{Y}_N$. The risk $R(\hat{\theta}_N)$ associated with $\hat{\theta}_N$ is

$$R(\hat{\theta}_N) = E(\overline{X}_N/\overline{Y}_N - \theta)^2 + cE(2N).$$

The performance of the sequential procedure is assessed by the regret $R(\hat{\theta}_N) - 4cn^*$.

We shall now give the main results concerning second order asymptotic expansions of the expected sample size and the risk of the procedure. Let $\{\cdot\}^-$ denote negative part such that $x^- \equiv \max(-x, 0)$.

Theorem 2.1. *(i) If $k > 3$, then as $c \to 0$,*

$$E(N) = n^* + \rho - 1 + o(1),$$

where

$$\rho = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{1}{n} E\left[\left\{\sum_{i=1}^{n}\left(1 - \frac{X_i}{\sigma_1} + \frac{Y_i}{\sigma_2}\right)\right\}^-\right],$$

and so $0 \le \rho \le \frac{3}{2}$.
(ii) If $k > 12$, then as $c \to 0$,

$$R(\hat{\theta}_N) - 4cn^* = 4c + o(c).$$

We shall propose another procedure to reduce the risk. The following theorem concerns the bias of the sequential procedure $\hat{\theta}_N$.

Theorem 2.2. *If $k > 6$, then as $c \to 0$,*

$$E(\hat{\theta}_N) - \theta = -\sqrt{c} + o(\sqrt{c}).$$

Taking account of Theorem 2.2, we propose a bias-corrected procedure

$$\hat{\theta}_N^* = \overline{X}_N/\overline{Y}_N + \sqrt{c}.$$

Then, from Theorem 2.2, if $k > 6$, $E(\hat{\theta}_N^*) = \theta + o(\sqrt{c})$ as $c \to 0$. The risk associated with $\hat{\theta}_N^*$ is given by $R(\hat{\theta}_N^*) = E(\hat{\theta}_N^* - \theta)^2 + cE(2N)$ and its second order asymptotic expansion is given below.

Theorem 2.3. *If $k > 12$, then as $c \to 0$,*

$$R(\hat{\theta}_N^*) - 4cn^* = 3c + o(c).$$

Proofs of Theorems 2.1–2.3 are omitted (see Uno (2003)). We have, from Theorems 2.1 (ii) and 2.3, if $k > 12$, then $R(\hat{\theta}_N^*) - R(\hat{\theta}_N) = -c + o(c)$ as $c \to 0$, which says that the risk of the bias-corrected procedure $\hat{\theta}_N^*$ is asymptotically one cost less than the one of the original procedure $\hat{\theta}_N$.

For two exponential populations Π_1 and Π_2, Mukhopadhyay and Chattopadhyay (1991) considered sequential point estimation of the difference $\sigma_1 - \sigma_2$ and showed that the regret of their sequential procedure was $4c + o(c)$ as $c \to 0$. Thus, for two exponential populations, it seems from Theorem 2.3 that as compared with the estimation of the difference $\sigma_1 - \sigma_2$, estimating the ratio σ_1/σ_2 by $\hat{\theta}_N^*$ is more efficient in the regret.

3. Estimation of the ratio of two normal variances

We shall apply the sequential procedures proposed in the previous section to the estimation of the ratio of two normal variances. Let Π_1 and Π_2 be according to normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ respectively, where the parameters $-\infty < \mu_1$, $\mu_2 < \infty$ and $0 < \sigma_1$, $\sigma_2 < \infty$ are all unknown. We assume that $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are independent observations from the populations Π_1 and Π_2 respectively and two populations Π_1 and Π_2 are independent. We want to estimate the ratio of the variances σ_1^2/σ_2^2. Taking samples of sizes n and m from Π_1 and Π_2 respectively, we estimate $\theta = \sigma_1^2/\sigma_2^2$ by

$$\hat{\theta}_{(n,m)} = \frac{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}{\frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \overline{Y}_m)^2} \quad \text{where} \quad \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \overline{Y}_m = \frac{1}{m}\sum_{i=1}^{m} Y_i.$$

As the loss function, we consider

$$L(\hat{\theta}_{(n,m)}) = (\hat{\theta}_{(n,m)} - \theta)^2 + c(n+m)$$

where $c > 0$ is the known cost per unit sample in each population. Let $m > 4$. Then the risk is given by

$$\begin{aligned} R(\hat{\theta}_{(n,m)}) = E\{L(\hat{\theta}_{(n,m)})\} &= E(\hat{\theta}_{(n,m)} - \theta)^2 + c(n+m) \\ &= 2\left(\tfrac{1}{n} + \tfrac{1}{m}\right)\theta^2 + r_{n,m}\theta^2 + c(n+m), \end{aligned}$$

where

$$r_{n,m} = \left(\frac{1}{n} + \frac{1}{m}\right)\frac{4(3m-4)}{(m-2)(m-4)} + \frac{8}{(m-2)(m-4)}.$$

By the same argument as Section 2, if we ignore the term $r_{n,m}\theta^2$, the risk $R(\hat{\theta}_{(n,m)})$ is (approximately) minimized by taking

$$n = m = (2/c)^{1/2}\theta = n^* \quad \text{(say)} \tag{3.1}$$

with $R(\hat{\theta}_{(n^*,n^*)}) \approx 4cn^*$ for sufficiently small c. But σ_1 and σ_2 are unknown, so is n^*. Taking account of (3.1) and the sequential procedure presented in Section 2, we propose a stopping rule

$$N = N_c = \inf\left\{n \geq k_0 : \ n \geq (2/c)^{1/2}\, l_n\, \hat{\theta}_{(n,n)}\right\} \tag{3.2}$$

where $X_1, \ldots, X_{k_0}$ and $Y_1, \ldots, Y_{k_0}$ are the initial samples from Π_1 and Π_2 respectively, with $k_0 > 4$ and $l_n = \frac{n}{n-1}\,(> 1)$ which plays the role to avoid underestimating n^* as seen in Chapter 10 of Woodroofe (1982). Then $P(N < \infty) = 1$ for all $c > 0$. Once the sampling stops, we estimate $\theta = \sigma_1^2/\sigma_2^2$ by

$$\hat{\theta}_N \equiv \hat{\theta}_{(N,N)} = \frac{\sum_{i=1}^N (X_i - \overline{X}_N)^2}{\sum_{i=1}^N (Y_i - \overline{Y}_N)^2}.$$

The risk $R(\hat{\theta}_N)$ associated with $\hat{\theta}_N$ is given by $R(\hat{\theta}_N) = E(\hat{\theta}_N - \theta)^2 + cE(2N)$.

The following theorem assesses the performance of the sequential procedure $\hat{\theta}_N$.

Theorem 3.1. *(i) If $k_0 > 7$, then as $c \to 0$,*

$$E(N) = n^* + \rho_0 - 1 + o(1),$$

where ρ_0 is a constant given in (4.11) *and $0 \leq \rho_0 \leq \frac{5}{2}$.*
(ii) If $k_0 > 25$, then as $c \to 0$,

$$R(\hat{\theta}_N) - 4cn^* = 10c + o(c).$$

The theorem below concerns the bias of the sequential procedure $\hat{\theta}_N$.

Theorem 3.2. *If $k_0 > 13$, then as $c \to 0$,*

$$E(\hat{\theta}_N) - \theta = -\sqrt{2c} + o(\sqrt{c}).$$

Taking account of Theorem 3.2, we propose a bias-corrected procedure $\hat{\theta}_N^* = \hat{\theta}_N + \sqrt{2c}$. Theorem 3.1 (ii) and the following theorem say that the risk of the bias-corrected procedure $\hat{\theta}_N^*$ is asymptotically two costs less than the one of the original procedure $\hat{\theta}_N$.

Theorem 3.3. *If $k_0 > 25$, then as $c \to 0$,*

$$R(\hat{\theta}_N^*) - 4cn^* = 8c + o(c).$$

4. Proofs of the results in Section 3

We shall prove all results given in Section 3. Throughout this section, let M be a generic positive constant and $c_0 > 0$ be chosen such that $n^* \geq 1$ for $0 < c \leq c_0$. Considering the transformation

$$U_i = \frac{1}{i(i+1)\sigma_1^2}\left\{\sum_{j=1}^{i}(X_j - X_{i+1})\right\}^2, \quad i \geq 1$$

and

$$V_i = \frac{1}{i(i+1)\sigma_2^2}\left\{\sum_{j=1}^{i}(Y_j - Y_{i+1})\right\}^2, \quad i \geq 1,$$

$U_1, U_2, \ldots$ and $V_1, V_2, \ldots$ are i.i.d. χ_1^2 random variables, and

$$\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 = \sigma_1^2 \sum_{j=1}^{n-1} U_j \quad \text{and} \quad \sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2 = \sigma_2^2 \sum_{j=1}^{n-1} V_j \quad \text{for all } n \geq 2.$$

We use the following notation:

$$D_n = \sum_{i=1}^{n}(U_i - 1), \quad Q_n = \sum_{i=1}^{n}(V_i - 1), \quad \overline{U}_n = \frac{1}{n}\sum_{i=1}^{n} U_i \quad \text{and} \quad \overline{V}_n = \frac{1}{n}\sum_{i=1}^{n} V_i .$$

Then, the stopping time N defined by (3.2) becomes

$$N = \inf\left\{n \geq k_0\,(> 4) : \; (n-1)\frac{\overline{V}_{n-1}}{\overline{U}_{n-1}} \geq n^*\right\},$$

from which, we have $N = t + 1$, where

$$t = t_c = \inf\left\{n \geq k\,(> 3) : \; n\frac{\overline{V}_n}{\overline{U}_n} \geq n^*\right\} \quad \text{with } k = k_0 - 1. \tag{4.1}$$

The stopping variable t is written in the form

$$t = \inf\{n \geq k : \; Z_n \geq n^*\},$$

where

$$Z_n \equiv n\frac{\overline{V}_n}{\overline{U}_n} = n - D_n + Q_n + \xi_n$$

and by Taylor's Theorem,

$$\xi_n \equiv Z_n - (n - D_n + Q_n) = -n(\overline{U}_n - 1)(\overline{V}_n - 1) + n\overline{V}_n(\overline{U}_n - 1)^2\eta_n^{-3}, \tag{4.2}$$

in which η_n is a random variable lying between 1 and $\overline{U}_n$.

Lemma 4.1. Let $q > 0$. $E(\overline{U}_t)^q \leq M$ and if $k > 2q$, then $E(\overline{U}_t)^{-q} \leq M$. These assertions hold for $\overline{V}_t$ instead of $\overline{U}_t$.

Proof. For a real number s,

$$E(\overline{U}_k)^s = \frac{\Gamma(\frac{k}{2} + s)}{k^s\Gamma(\frac{k}{2})} < \infty \quad \text{if} \quad k > -2s, \tag{4.3}$$

where $\Gamma(x)$ is the gamma function. For $q > 1$, from the Doob's maximal inequality,

$$E(\overline{U}_t)^q \le E\left\{\sup_{n\ge 1}(\overline{U}_n)^q\right\} \le \left(\tfrac{q}{q-1}\right)^q E(U_1)^q < \infty. \tag{4.4}$$

For $0 < q \le 1$, we have from the Hölder inequality, for $q' > 1$, $E(\overline{U}_t)^q \le \{E(\overline{U}_t)^{q'}\}^{q/q'}$ which is finite from (4.4). Thus, the first assertion holds. We shall show the second assertion. For $q > 1$, from the Doob's maximal inequality and (4.3),

$$E(\overline{U}_t)^{-q} \le E\left\{\sup_{n\ge k}(\overline{U}_n)^{-q}\right\} \le \left(\tfrac{q}{q-1}\right)^q E(\overline{U}_k)^{-q} < \infty \quad \text{if } k > 2q. \tag{4.5}$$

For $0 < q \le 1$, it follows from the Hölder inequality and (4.5) that for $1 < q' < \frac{3}{2}$, $E(\overline{U}_t)^{-q} \le \{E(\overline{U}_t)^{-q'}\}^{q/q'} < \infty$ if $k > 2q'$, for which $k > 3$ is sufficient. Hence, the second assertion holds. The last assertion is clear because U_i and V_i are the same in distribution. □

Lemma 4.2. Let $q \ge 1$.
(i) $\{(t/n^*)^{-q},\ c > 0\}$ is uniformly integrable if $k > 2q$.
(ii) $\{(t/n^*)^q,\ 0 < c \le c_0\}$ is uniformly integrable if $k > 2q$.

Proof. From the definition (4.1) of t, we have $(t/n^*)^{-q} \le (\overline{V}_t/\overline{U}_t)^q$. Thus, for $a > 1$, from the Hölder inequality with $u > 1$ and $u^{-1} + v^{-1} = 1$,

$$E(t/n^*)^{-aq} \le \{E(\overline{V}_t)^{aqu}\}^{1/u}\{E(\overline{U}_t)^{-aqv}\}^{1/v}.$$

Hence, from Lemma 4.1, $\{(t/n^*)^{-q},\ c > 0\}$ is uniformly integrable if $k > 2q$. So (i) holds. For (ii), observe that $(t-1)\overline{V}_{t-1}/\overline{U}_{t-1} < n^*$ on $\{t > k\}$, so that for some $c_0 > 0$,

$$\begin{aligned} t/n^* &\le \{(\overline{U}_{t-1}/\overline{V}_{t-1}) + (1/n^*)\}I_{\{t>k\}} + (k/n^*)I_{\{t=k\}} \\ &\le (\overline{U}_{t-1}/\overline{V}_{t-1})I_{\{t>k\}} + (k+1), \end{aligned}$$

where $I_{\{\cdot\}}$ denotes the indicator function. Therefore, by c_r-inequality (see Loève (1977), p. 157), for $0 < c \le c_0$,

$$(t/n^*)^q \le \left\{(\overline{U}_{t-1}/\overline{V}_{t-1})I_{\{t>k\}} + (k+1)\right\}^q \le M\left\{(\overline{U}_{t-1}/\overline{V}_{t-1})^q I_{\{t>k\}} + (k+1)^q\right\}.$$

For $a > 1$, from the Hölder inequality with $u > 1$ and $u^{-1} + v^{-1} = 1$,

$$\begin{aligned} E\left\{(\overline{U}_{t-1}/\overline{V}_{t-1})^q I_{\{t>k\}}\right\}^a &\le \left\{E(\overline{U}_{t-1})^{aqu} I_{\{t>k\}}\right\}^{1/u}\left\{E(\overline{V}_{t-1})^{-aqv} I_{\{t>k\}}\right\}^{1/v} \\ &\le \left[E\left\{\sup_{n\ge k}(\overline{U}_n)^{aqu}\right\}\right]^{1/u}\left[E\left\{\sup_{n\ge k}(\overline{V}_n)^{-aqv}\right\}\right]^{1/v}, \end{aligned}$$

which, together with (4.4) and (4.5), proves (ii). □

From Theorem 2 of Chow et al. (1979), we have the next lemma.

Lemma 4.3. For $q \geq 1$, if $\{(t/n^*)^q,\ 0 < c \leq c_0\}$ is uniformly integrable for some $c_0 > 0$, then $\{\,(n^{*-\frac{1}{2}}|D_t|)^q,\ 0 < c \leq c_0\,\}$ and $\{\,(n^{*-\frac{1}{2}}|Q_t|)^q,\ 0 < c \leq c_0\,\}$ are uniformly integrable.

Let $\boldsymbol{W} = (\zeta_1, \zeta_2)$ be distributed according to a bivariate normal distribution with mean vector $(0,0)$ and covariance matrix $\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. In the notation of Aras and Woodroofe (1993), letting

$$\boldsymbol{X}_i = (U_i - 1,\ V_i - 1), \quad \boldsymbol{S}_n = (D_n,\ Q_n) \quad \text{and} \quad \boldsymbol{c} = (-1,\ 1), \tag{4.6}$$

we have the following lemma.

Lemma 4.4. If $k > 6$, then the conditions (C1)–(C6) of Aras and Woodroofe (1993) are satisfied with $p = 3$.

Proof. Clearly, (C1) holds for $p = 3$. From Proposition 4 of Aras and Woodroofe (1993), (C4) is satisfied, (C5) holds for all $\alpha \geq 3/2$ and (C6) holds with $\xi = \zeta_1^2 - \zeta_1\zeta_2$. We shall show (C2) with $p = 3$. Let $0 < \varepsilon < \frac{1}{2}$. Since $Z_n - (n/\varepsilon) = n\{(\overline{V}_n/\overline{U}_n) - \varepsilon^{-1}\} \leq 0$ on $\{\overline{V}_n/\overline{U}_n \leq 1/\varepsilon\}$, we have for some $s > 3$,

$$\begin{aligned} E\left\{\left(Z_n - \tfrac{n}{\varepsilon}\right)^+\right\}^s &= E\left[\left(Z_n - \tfrac{n}{\varepsilon}\right)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon\}}\right] \leq n^s\, E\left[(\overline{V}_n/\overline{U}_n)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon\}}\right] \\ &= n^s\, E\left[(\overline{V}_n/\overline{U}_n)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon,\ \overline{U}_n < 1-\varepsilon\}}\right] + n^s\, E\left[(\overline{V}_n/\overline{U}_n)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon,\ \overline{U}_n \geq 1-\varepsilon\}}\right] \\ &= \quad J_1(n) \quad + \quad J_2(n), \quad \text{say.} \end{aligned}$$

By the independency of $\overline{U}_n$ and $\overline{V}_n$, (4.4) and (4.5), we have, for $u > 1$ and $u^{-1} + v^{-1} = 1$,

$$\begin{aligned} J_1(n) &\leq n^s\, E\left[(\overline{V}_n/\overline{U}_n)^s I_{\{\overline{U}_n < 1-\varepsilon\}}\right] \leq n^s\{E(\overline{V}_n)^s\}\{E(\overline{U}_n)^{-su}\}^{1/u}\{P(\overline{U}_n < 1-\varepsilon)\}^{1/v} \\ &\leq M\, n^s\{P(\overline{U}_n - 1 < -\varepsilon)\}^{1/v} \quad \text{if } k > 2su. \end{aligned}$$

Since by Tchebichev's inequality and the Marcinkiewicz-Zygmund inequality, for $n \geq 1$,

$$P(\overline{U}_n - 1 < -\varepsilon) \leq (\varepsilon n)^{-q} E|D_n|^q = O(n^{-q/2}) \quad \text{for } q \geq 2, \tag{4.7}$$

we obtain $J_1(n) \leq M n^{s - \frac{q}{2v}}$ for $n \geq k$. If $k > 6$, then we can choose $s > 3$, $q \geq 2$ and (u, v) such that $k > 2su$ and $s - q/(2v) \leq 0$, so that $J_1(n) \leq M$ for $n \geq k$. For $J_2(n)$, since $\{\overline{V}_n/\overline{U}_n > 1/\varepsilon,\ \overline{U}_n \geq 1 - \varepsilon\} \subset \{\overline{V}_n - 1 > \delta\}$ where $\delta = (1 - 2\varepsilon)/\varepsilon > 0$, we have, from (4.4), for $u > 1$ and $u^{-1} + v^{-1} = 1$,

$$\begin{aligned} J_2(n) &\leq (1-\varepsilon)^{-s} n^s\, E\left[(\overline{V}_n)^s I_{\{\overline{V}_n/\overline{U}_n > 1/\varepsilon,\ \overline{U}_n \geq 1-\varepsilon\}}\right] \\ &\leq M\, n^s\{E(\overline{V}_n)^{su}\}^{1/u}\{P(\overline{V}_n - 1 > \delta)\}^{1/v} \leq M\, n^s\{P(\overline{V}_n - 1 > \delta)\}^{1/v}. \end{aligned}$$

By (4.7), $P(\overline{V}_n - 1 > \delta) = O(n^{-q/2})$ for $q \geq 2$, so that $J_2(n) \leq M n^{s - q/(2v)}$ for $n \geq k$. Choosing q such that $s - q/(2v) \leq 0$ for some $s > 3$ and (u, v), we have $J_2(n) \leq M$ for

$n \geq k$. Therefore, $\{[(Z_n - \frac{n}{\varepsilon})^+]^3,\ n \geq k\}$ is uniformly integrable, that is, (C2) holds. Finally, we shall show (C3). From (4.2), Tchebichev's inequality, the independency of $\overline{U}_n$ and $\overline{V}_n$ and the Marcinkiewicz-Zygmund inequality, we have, for $0 < \varepsilon < 1$,

$$\begin{aligned} P\{\xi_n < -\varepsilon n\} &= P\left\{(\overline{U}_n - 1)(\overline{V}_n - 1) - \overline{V}_n(\overline{U}_n - 1)^2\eta_n^{-3} > \varepsilon\right\} \\ &\leq P\left\{(\overline{U}_n - 1)(\overline{V}_n - 1) > \varepsilon\right\} \leq \varepsilon^{-3}E(\overline{U}_n - 1)^3 E(\overline{V}_n - 1)^3 = O(n^{-3}), \end{aligned}$$

which implies $\sum_{n=1}^{\infty} n\,P\{\xi_n < -\varepsilon n\} < \infty$, so that (C3) holds. □

Let

$$H_c = Z_t - n^* = t - n^* - D_t + Q_t + \xi_t. \tag{4.8}$$

It follows from Propositions 2 and 3 of Aras and Woodroofe (1993) that as $c \to 0$,

$$\frac{t}{n^*} \xrightarrow{a.s.} 1 \quad \text{and} \quad \left(\frac{\boldsymbol{S}_t}{\sqrt{t}},\ \xi_t,\ H_c\right) \xrightarrow{d} (\boldsymbol{W},\ \xi,\ H) \quad \text{with} \quad \xi = \zeta_1^2 - \zeta_1\zeta_2 \tag{4.9}$$

where '$\xrightarrow{a.s.}$' and '$\xrightarrow{d}$' stand for almost sure convergence and convergence in distribution, respectively and H is a certain random variable with $\rho_0 = E(H)$ which is given in (4.11). From Proposition 7 of Aras and Woodroofe (1993),

$$\{|\xi_t - H_c|^2,\ 0 < c \leq c_0\} \text{ is uniformly integrable.} \tag{4.10}$$

Now we are in a position to prove Theorems 3.1–3.3.

Proof of Theorem 3.1. Using the notation (4.6),

$$t = \inf\{n \geq k :\ n + \langle \boldsymbol{c},\ \boldsymbol{S}_n\rangle + \xi_n \geq n^*\},$$

where $\langle\cdot,\cdot\rangle$ denotes inner product. Let

$$\tau = \inf\{n \geq 1 :\ n + \langle \boldsymbol{c},\ \boldsymbol{S}_n\rangle > 0\} \quad \text{and} \quad \rho_0 = \frac{E\{(\tau + \langle \boldsymbol{c},\ \boldsymbol{S}_\tau\rangle)^2\}}{2\,E(\tau + \langle \boldsymbol{c},\ \boldsymbol{S}_\tau\rangle)}. \tag{4.11}$$

It follows from Theorem 1 and Proposition 3 of Aras and Woodroofe (1993), Corollary 2.2 of Woodroofe (1982) and Lemma 4 that if $k = k_0 - 1 > 6$, then

$$E(N) = E(t+1) = n^* + \rho_0 - E(\xi) + 1 + o(1) = n^* + \rho_0 - 1 + o(1) \quad \text{as } c \to 0.$$

From Corollary 2.7 of Woodroofe (1982), $\rho_0 = \frac{5}{2} - \sum_{n=1}^{\infty} \frac{1}{n} E\{(n - D_n + Q_n)^-\}$, and so $0 \leq \rho_0 \leq \frac{5}{2}$. Thus, the first assertion holds. We shall prove (ii). Observe that

$$R(\hat{\theta}_N) - 4cn^* = \theta^2 E(\overline{U}_t/\overline{V}_t - 1)^2 + 2cE(t+1) - 4cn^*$$

and by Taylor's theorem,

$$\begin{aligned}(\overline{U}_t/\overline{V}_t-1)^2 &= \{\overline{U}_t-1-(\overline{V}_t-1)\}^2(\overline{V}_t)^{-2}\\ &= \{\overline{U}_t-1-(\overline{V}_t-1)\}^2\{1-2(\overline{V}_t-1)+3(\overline{V}_t-1)^2\varphi^{-4}\},\end{aligned}$$

where φ is a random variable lying between 1 and $\overline{V}_t$. Hence,

$$\begin{aligned}R(\hat{\theta}_N)-4cn^* &= \theta^2E\{\overline{U}_t-1-(\overline{V}_t-1)\}^2+2cE(t+1)-4cn^*\\ &\quad -2\theta^2E\left[\{\overline{U}_t-1-(\overline{V}_t-1)\}^2(\overline{V}_t-1)\right]\\ &\quad +3\theta^2E\left[\{\overline{U}_t-1-(\overline{V}_t-1)\}^2(\overline{V}_t-1)^2\varphi^{-4}\right]\\ &= J_1 \quad + \quad J_2 \quad + \quad J_3, \quad \text{say.}\end{aligned} \tag{4.12}$$

Since from (3.1), $J_1=2c[\,\frac{1}{4}(n^*)^2E\{\overline{U}_t-1-(\overline{V}_t-1)\}^2+E(t)+1-2n^*\,]$, we get from Corollary 1 of Theorem 2 of Aras and Woodroofe (1993) with $\boldsymbol{b}=(\frac{1}{2},-\frac{1}{2})$ and Lemma 4.4,

$$\begin{aligned}&J_1/(2c)\\ &= \tfrac{1}{2}E\{\xi(\zeta_1-\zeta_2)^2\}-2E(\xi)+4+8-\tfrac{1}{2}E\{U_1-1-(V_1-1)\}^3+1+o(1)\\ &= \tfrac{1}{2}E\{\zeta_1(\zeta_1-\zeta_2)^3\}+9+o(1)\\ &= 21+o(1),\end{aligned}$$

which implies

$$J_1=42c+o(c) \quad \text{as } c\to 0. \tag{4.13}$$

Observe from (3.1) that $J_3=\frac{3}{2}c\,E[\,(n^*)^2\{(\overline{U}_t-1)-(\overline{V}_t-1)\}^2(\overline{V}_t-1)^2\varphi^{-4}\,]$. We shall show the uniform integrability of $\{(n^*)^2\{(\overline{U}_t-1)-(\overline{V}_t-1)\}^2(\overline{V}_t-1)^2\varphi^{-4},\, c\le c_0\}$ for some $c_0>0$. Clearly,

$$\begin{aligned}&(n^*)^2\{(\overline{U}_t-1)-(\overline{V}_t-1)\}^2(\overline{V}_t-1)^2\varphi^{-4}\\ &= (n^*)^2(\overline{U}_t-1)^2(\overline{V}_t-1)^2\varphi^{-4}-2(n^*)^2(\overline{U}_t-1)(\overline{V}_t-1)^3\varphi^{-4}+(n^*)^2(\overline{V}_t-1)^4\varphi^{-4}\\ &= J_{31} \quad - \quad 2J_{32} \quad + \quad J_{33}, \quad \text{say.}\end{aligned}$$

From the Hölder inequality, for $a>1$,

$$\begin{aligned}E|J_{31}|^a &= E\left|(n^*/t)^4\{(n^*)^{-\frac{1}{2}}D_t\}^2\{(n^*)^{-\frac{1}{2}}Q_t\}^2\varphi^{-4}\right|^a\\ &\le \{E(n^*/t)^{12a}\}^{1/3}\{E|(n^*)^{-\frac{1}{2}}D_t|^{12a}\}^{1/6}\{E|(n^*)^{-\frac{1}{2}}Q_t|^{12a}\}^{1/6}\{E(\varphi^{-12a})\}^{1/3}\end{aligned}$$

and by the convexity, $E(\varphi^{-12a})\le 1+E(\overline{V}_t)^{-12a}$. Thus, from Lemmas 4.1–4.3, if $k=k_0-1>24$, then $\{|J_{31}|,\, c\le c_0\}$ is uniformly integrable. Similarly, we can show the uniform integrabilities of $\{|J_{32}|,\, c\le c_0\}$ and $\{|J_{33}|,\, c\le c_0\}$ provided $k>24$, so that we obtain the uniform integrability of $\{(n^*)^2\{(\overline{U}_t-1)-(\overline{V}_t-1)\}^2(\overline{V}_t-1)^2\varphi^{-4},\, c\le c_0\}$. From (4.9) and the fact that $\varphi\xrightarrow{a.s.}1$ as $c\to 0$,

$$(n^*)^2\{(\overline{U}_t - 1) - (\overline{V}_t - 1)\}^2(\overline{V}_t - 1)^2\varphi^{-4} \xrightarrow{d} (\zeta_1 - \zeta_2)^2\zeta_2^2 \quad \text{as } c \to 0,$$

which yields

$$J_3 = \tfrac{3}{2}cE\{(\zeta_1 - \zeta_2)^2\zeta_2^2\} + o(c) = 24c + o(c). \tag{4.14}$$

Finally, we shall calculate J_2. From (3.1),

$$\begin{aligned} J_2 &= -cE\left\{(n^*)^2t^{-3}(D_t - Q_t)^2Q_t\right\} \\ &= -cE\left\{(n^*)^{-1}\left((n^*/t)^3 - 1\right)(D_t - Q_t)^2Q_t + (n^*)^{-1}(D_t - Q_t)^2Q_t\right\} \\ &= -cE\{\ J_{21} \ + \ J_{22}\ \}, \quad \text{say.} \end{aligned} \tag{4.15}$$

Observe from (4.8) that

$$\begin{aligned} J_{21} &= (n^*)^{-1}\left((n^*/t)^3 - 1\right)(D_t - Q_t)^2Q_t \\ &= \frac{(n^*)^2 + n^*t + t^2}{n^*t^3}(n^* - t)(D_t - Q_t)^2Q_t \\ &= -\frac{(n^*)^2 + n^*t + t^2}{n^*t^3}(D_t - Q_t)^3Q_t + \frac{(n^*)^2 + n^*t + t^2}{n^*t^3}(D_t - Q_t)^2Q_t(\xi_t - H_c) \\ &= \ J_{211} \ + \ J_{212}, \quad \text{say.} \end{aligned}$$

For $a > 1$, by the Hölder inequality,

$$\begin{aligned} E|J_{211}|^a &= E\left|\frac{(n^*)^3 + (n^*)^2t + n^*t^2}{t^3}\,\frac{(D_t - Q_t)^3Q_t}{(n^*)^2}\right|^a \\ &\le \left\{E\left(\frac{(n^*)^3}{t^3} + \frac{(n^*)^2}{t^2} + \frac{n^*}{t}\right)^{7a/3}\right\}^{3/7}\left\{E\left|\frac{(D_t - Q_t)^3Q_t}{(n^*)^2}\right|^{7a/4}\right\}^{4/7}, \end{aligned}$$

so that from Lemmas 4.2 and 4.3, $\{|J_{211}|,\ 0 < c \le c_0\}$ is uniformly integrable provided $k > 14$. Similarly, for $a > 1$, $s > 1$, $s^{-1} + u^{-1} = 1$ and $v > 1$, $v^{-1} + w^{-1} = 1$,

$$\begin{aligned} E|J_{212}|^a &= E\left|\frac{(n^*)^2 + n^*t + t^2}{t^2}\,\frac{(D_t - Q_t)^2}{n^*}(\overline{V}_t - 1)(\xi_t - H_c)\right|^a \\ &\le \left\{E\left(\frac{(n^*)^2}{t^2} + \frac{n^*}{t} + 1\right)^{2as}\right\}^{\frac{1}{2s}}\left\{E\left|\frac{(D_t - Q_t)^2}{n^*}\right|^{2as}\right\}^{\frac{1}{2s}} \\ &\quad \times\left\{E|\overline{V}_t - 1|^{auv}\right\}^{\frac{1}{uv}}\{E|\xi_t - H_c|^{auw}\}^{\frac{1}{uw}}, \end{aligned}$$

whence, taking $(s, u) = (\frac{21}{10}, \frac{21}{11})$ and $(v, w) = (43, \frac{43}{42})$, from Lemmas 4.1–4.3 and (4.10), $\{|J_{212}|,\ 0 < c \le c_0\}$ is uniformly integrable provided $k > 16$. Since from (4.9), $J_{21} \xrightarrow{d} -3(\zeta_1 - \zeta_2)^3\zeta_2$ as $c \to 0$, we obtain

$$E(J_{21}) = -3E\{(\zeta_1 - \zeta_2)^3\zeta_2\} + o(1) = 72 + o(1). \tag{4.16}$$

For J_{22},

$$\begin{aligned} J_{22} &= (n^*)^{-1}D_t^2Q_t - 2(n^*)^{-1}D_tQ_t^2 + (n^*)^{-1}Q_t^3 \\ &= J_{221} - 2\,J_{222} + J_{223}, \quad \text{say.} \end{aligned} \tag{4.17}$$

Since $E(t^2) < \infty$ for all $c \in (0,\, c_0]$ by Proposition 2 of Aras and Woodroofe (1993), it follows from Theorem 9 of Chow et al. (1965), Lemma 4.2 and (4.9) that

$$E(J_{223}) = (n^*)^{-1}\left\{8E(t) + 6E(tQ_t)\right\} = 8 + 6E\left\{(t/n^*)Q_t\right\} + o(1) \quad \text{as } c \to 0,$$

where by Wald's lemma and (4.8),

$$E\left\{(t/n^*)Q_t\right\} = E\left\{\left(\frac{t}{n^*} - 1\right)Q_t\right\} = E\left\{\frac{D_t - Q_t - \xi_t + H_c}{n^*}Q_t\right\}.$$

From (4.10) and Lemmas 4.2 and 4.3, for $a > 1$ and some $c_0 > 0$ such that $n^* \geq 1$, if $k > 6$, then

$$\begin{aligned} &E\left|\frac{D_t - Q_t - \xi_t + H_c}{n^*}Q_t\right|^a \\ &\quad \leq M\left[E\left|\frac{(D_t - Q_t)Q_t}{n^*}\right|^a + \left\{E|\xi_t - H_c|^{\frac{3a}{2}}\right\}^{\frac{2}{3}}\left\{E\left|(n^*)^{-\frac{1}{3}}Q_t\right|^{3a}\right\}^{\frac{1}{3}}\right] \leq M, \end{aligned}$$

and from (4.9), $(n^*)^{-1}(D_t - Q_t - \xi_t + H_c)Q_t \xrightarrow{d} (\zeta_1 - \zeta_2)\zeta_2$ as $c \to 0$. Therefore,

$$E\left\{(t/n^*)Q_t\right\} = E\{(\zeta_1 - \zeta_2)\zeta_2\} + o(1) = -2 + o(1) \quad \text{as } c \to 0, \tag{4.18}$$

which yields

$$E(J_{223}) = 8 + 6\{-2 + o(1)\} + o(1) = -4 + o(1). \tag{4.19}$$

From (4.18), as $c \to 0$,

$$\begin{aligned} E(J_{221}) &= (n^*)^{-1}E\{(D_t^2 - 2t)Q_t\} + 2E\{(t/n^*)Q_t\} \\ &= (n^*)^{-1}E\{(D_t^2 - 2t)Q_t\} - 4 + o(1) \end{aligned} \tag{4.20}$$

and we have

$$E\{(D_t^2 - 2t)Q_t\} = \tfrac{1}{2}\left\{E(D_t^2 - 2t + Q_t)^2 - E(D_t^2 - 2t)^2 - E(Q_t^2)\right\}. \tag{4.21}$$

For $\boldsymbol{X}_i = (U_i - 1,\, V_i - 1)$, $i = 1, 2, \ldots$, let $\mathcal{F}_n = \sigma(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ for $n \geq 1$ be the σ-algebra generated by $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ with $\mathcal{F}_0 = \{\phi,\, \Omega\}$, and let $x_i = 2D_{i-1}(U_i - 1) + (U_i - 1)^2 - 2$ for $i \geq 1$ with $D_0 = 0$. By the same argument as (2.14) of Chow and Martinsek (1982), it follows from Lemma 4.2 (ii) and $E(t^2) < \infty$ that for fixed $c \in (0,\, c_0]$, as $n \to \infty$

$$\int_{\{t>n\}} |D_n^2 - 2n| dP = o(1).$$

Therefore, from Theorem 1 and Lemma 6 of Chow et al. (1965),

$$\begin{aligned} E(D_t^2 - 2t)^2 &= E\left(\sum_{i=1}^{t} x_i^2\right) = E\left\{\sum_{i=1}^{t} E\left(x_i^2 \,|\mathcal{F}_{i-1}\right)\right\} \\ &= E\left\{\sum_{i=1}^{t}(8D_{i-1}^2 + 32D_{i-1} + 56)\right\} = 8E\left(\sum_{i=1}^{t} D_{i-1}^2\right) + 32E(tD_t) + 56E(t), \end{aligned}$$

which is finite because from Theorems 2, 7 and Lemma 9 of Chow et al. (1965),

$$\begin{aligned} &E(tD_t) \leq \{E(t^2)\}^{\frac{1}{2}}\{E(D_t^2)\}^{\frac{1}{2}} < \infty \quad \text{and} \\ &\quad E\left(\sum_{i=1}^{t} D_{i-1}^2\right) \leq E\left(\sum_{i=1}^{t} D_i^2\right) \leq E(tD_t^2) \leq \{E(t^2)\}^{\frac{1}{2}}\{E(D_t^4)\}^{\frac{1}{2}} < \infty. \end{aligned}$$

Similarly, we get

$$\begin{aligned} E(D_t^2 - 2t + Q_t)^2 &= E\left\{\sum_{i=1}^{t}(x_i + V_i - 1)^2\right\} = E\left[\sum_{i=1}^{t} E\left\{(x_i + V_i - 1)^2 \,|\mathcal{F}_{i-1}\right\}\right] \\ &= E\left[\sum_{i=1}^{t} E\left\{x_i^2 + 2x_i(V_i - 1) + (V_i - 1)^2 \,|\mathcal{F}_{i-1}\right\}\right] = E(D_t^2 - 2t)^2 + 2E(t) < \infty, \end{aligned}$$

which, together with (4.21) and $E(Q_t^2) = 2E(t)$, yields

$$E\{(D_t^2 - 2t)Q_t\} = 0 \tag{4.22}$$

and hence, from (4.20), we obtain

$$E(J_{221}) = -4 + o(1) \quad \text{as } c \to 0. \tag{4.23}$$

By an argument similar to (4.18), if $k > 6$, then $E\{(t/n^*)D_t\} = E\{(\zeta_1 - \zeta_2)\zeta_1\} + o(1) = 2 + o(1)$ as $c \to 0$, so that

$$\begin{aligned} E(J_{222}) &= (n^*)^{-1}E\{D_t(Q_t^2 - 2t)\} + 2E\{(t/n^*)D_t\} \\ &= (n^*)^{-1}E\{D_t(Q_t^2 - 2t)\} + 4 + o(1). \end{aligned}$$

By the same argument as (4.22), we have $E\{D_t(Q_t^2 - 2t)\} = 0$, and so $E(J_{222}) = 4 + o(1)$, which, together with (4.17), (4.19) and (4.23), yields $E(J_{22}) = -4 - 8 - 4 + o(1) = -16 + o(1)$. Therefore, from (4.15) and (4.16),

$$J_2 = -c(72 - 16) + o(1) = -56c + o(c) \quad \text{as } c \to 0,$$

from which, together with (4.12)–(4.14), we get $R(\hat{\theta}_N) - 4cn^* = (42 - 56 + 24)c + o(c) = 10c + o(c)$. Thus, the proof is complete. □

Proof of Theorem 3.2. From (3.1) and Taylor's theorem,

$$\begin{aligned}\frac{E(\hat{\theta}_N) - \theta}{\sqrt{c/2}} &= n^* E\left\{\frac{\overline{U}_t - 1 - (\overline{V}_t - 1)}{\overline{V}_t}\right\} \\ &= n^* E\{\overline{U}_t - 1 - (\overline{V}_t - 1)\} - n^* E\left[\{\overline{U}_t - 1 - (\overline{V}_t - 1)\}(\overline{V}_t - 1)\varphi^{-2}\right] \\ &= \quad J_1 \quad - \quad J_2, \quad \text{say,} \qquad (4.24)\end{aligned}$$

where φ is a random variable lying between 1 and $\overline{V}_t$. By Wald's lemma, (4.8) and (4.9),

$$\begin{aligned} J_1 &= E\left\{\frac{n^* - t}{t}(D_t - Q_t)\right\} = E\left\{\frac{-(D_t - Q_t) + \xi_t - H_c}{t}(D_t - Q_t)\right\} \\ &= -E(\zeta_1 - \zeta_2)^2 + o(1) = -4 + o(1) \quad \text{as } c \to 0 \qquad (4.25)\end{aligned}$$

because for $a > 1$,

$$\begin{aligned} & E\left|\frac{-(D_t - Q_t) + \xi_t - H_c}{t}(D_t - Q_t)\right|^a \\ & \leq M\left[E\left(\frac{(D_t - Q_t)^2}{t}\right)^a + E\left|(\xi_t - H_c)(\overline{U}_t - \overline{V}_t)\right|^a\right] \\ & \leq M\left\{E\left(\frac{n^*}{t}\right)^{3a}\right\}^{\frac{1}{3}}\left\{E\left|\frac{D_t - Q_t}{(n^*)^{1/2}}\right|^{3a}\right\}^{\frac{2}{3}} + M\left\{E|\xi_t - H_c|^{\frac{3a}{2}}\right\}^{\frac{2}{3}}\left\{E|\overline{U}_t - \overline{V}_t|^{3a}\right\}^{\frac{1}{3}},\end{aligned}$$

which is bounded, that is, $\{\frac{n^* - t}{t}(D_t - Q_t),\ 0 < c \leq c_0\}$ is uniformly integrable for some $c_0 > 0$ such that $n^* \geq 1$, by Lemmas 4.1–4.3 and (4.10), provided $k > 6$. Since for $a > 1$,

$$\begin{aligned} & E|n^*\{\overline{U}_t - 1 - (\overline{V}_t - 1)\}(\overline{V}_t - 1)\varphi^{-2}|^a \\ & \leq \left\{E\left(\frac{n^*}{t}\right)^{6a}\right\}^{1/3}\left\{E\left(\frac{(D_t - Q_t)Q_t}{n^*}\right)^{3a}\right\}^{1/3}\left\{E(\varphi^{-6a})\right\}^{1/3}\end{aligned}$$

and $E(\varphi^{-6a}) \leq 1 + E(\overline{V}_t)^{-6a}$, it follows from Lemmas 4.1–4.3 that if $k = k_0 - 1 > 12$, then $\{n^*\{\overline{U}_t - 1 - (\overline{V}_t - 1)\}(\overline{V}_t - 1)\varphi^{-2},\ 0 < c \leq c_0\}$ is uniformly integrable. Thus, from (4.9) and the fact that $\varphi \xrightarrow{a.s.} 1$ as $c \to 0$,

$$J_2 = E\{(\zeta_1 - \zeta_2)\zeta_2\} + o(1) = -2 + o(1),$$

which, together with (4.24) and (4.25), yields $E(\hat{\theta}_N) - \theta = \sqrt{c/2}\{-4 + 2 + o(1)\} = -\sqrt{2c} + o(\sqrt{c})$. The theorem holds. □

Proof of Theorem 3.3. It follows from Theorems 3.1 (ii) and 3.2 that as $c \to 0$,

$$\begin{aligned} R(\hat{\theta}_N^*) &= R(\hat{\theta}_N) + 2\sqrt{2c}\,E(\hat{\theta}_N - \theta) + 2c = R(\hat{\theta}_N) + 2\sqrt{2c}\{-\sqrt{2c} + o(\sqrt{c})\} + 2c \\ &= R(\hat{\theta}_N) - 2c + o(c) = 8c + o(c), \end{aligned}$$

proving Theorem 3.3. □

References

Aras, G. and Woodroofe, M. (1993). Asymptotic expansions for the moments of a randomly stopped average. *Ann. Statist.* **21**, 503–519.

Chow, Y.S., Hsiung, C.A. and Lai, T.L. (1979). Extended renewal theory and moment convergence in Anscombe's theorem. *Ann. Probab.* **7**, 304–318.

Chow, Y.S. and Martinsek, A.T. (1982). Bounded regret of a sequential procedure for estimation of the mean. *Ann. Statist.* **10**, 909–914.

Chow, Y.S., Robbins, H. and Teicher, H. (1965). Moments of randomly stopped sums. *Ann. Math. Statist.* **36**, 789–799.

Ghosh, M. and Mukhopadhyay, N. (1980). Sequential point estimation of the difference of two normal means. *Ann. Statist.* **8**, 221–225.

Loève, M. (1977). *Probability Theory I*, 4th ed. Springer-Verlag, New York.

Mukhopadhyay, N. and Chattopadhyay, S. (1991). Sequential methodologies for comparing exponential mean survival times. *Sequential Anal.* **10**, 139–148.

Mukhopadhyay, N. and Purkayastha, S. (1994). On sequential estimation of the difference of means. *Statist. Decisions* **12**, 41–52.

Uno, C. (2003). Sequential estimation of the ratio of scale parameters in the exponential two-sample problem, *submitted.*

Woodroofe, M. (1982). *Nonlinear Renewal Theory in Sequential Analysis.* CBMS Monograph No.39, SIAM Philadelphia.