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Analytic difference equations with small step size. Application to bifurcation delay

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Abstract: General results are presented on difference equations of the form $y(x+\varepsilon)=f(x,y(x))$, where $x\in\mathbb{C}, y\in\mathbb{C}^n$ and ε is a small parameter. These results are essentially existence of complex analytic solutions and exponentially small difference of two solutions. They are applied to the problem of delayed bifurcation at a point of period doubling for a real discrete dynamical system.

Keywords: difference equation, discrete dynamical system, bifurcation delay, singular perturbation, complex asymptotic.

AMS Classification: 39A, 34E15, 58F14.

1 Introduction

Difference equations with small step size are equations of the form

$$T_\varepsilon y=f(x,y)\ , \qquad x\in D\subset\mathbb{C},\ y\in\mathbb{C}^n \tag{1}$$

where f is an analytic function in some domain of $\mathbb{C}\times\mathbb{C}^n$, D is a domain (*i.e.* connected open subset) of $\mathbb{C}$, $\varepsilon>0$ or ε in a small sector S, and T_ε is one of the following operators $\sigma_\varepsilon,\Delta_\varepsilon,\delta_\varepsilon$ defined by

$$\begin{aligned}\sigma_\varepsilon y(x) &= y(x+\varepsilon)\\ \Delta_\varepsilon y(x) &= \tfrac{1}{\varepsilon}(y(x+\varepsilon)-y(x))\\ \delta_\varepsilon y(x) &= \tfrac{1}{\varepsilon}\left(y\left(x+\tfrac{\varepsilon}{2}\right)-y\left(x-\tfrac{\varepsilon}{2}\right)\right)\end{aligned}$$

The main questions are:

- Are there solutions $y=y(x,\varepsilon)$ with a nice behavior as $\varepsilon\to 0$? e.g. in the case $T_\varepsilon=\Delta_\varepsilon$ or δ_ε, are there solutions that converge to a solution of the corresponding ODE $y'=f(x,y)$ as $\varepsilon\to 0$?
- Are there analytic solutions w.r.t. $x\in D$ and to $\varepsilon\in S$?

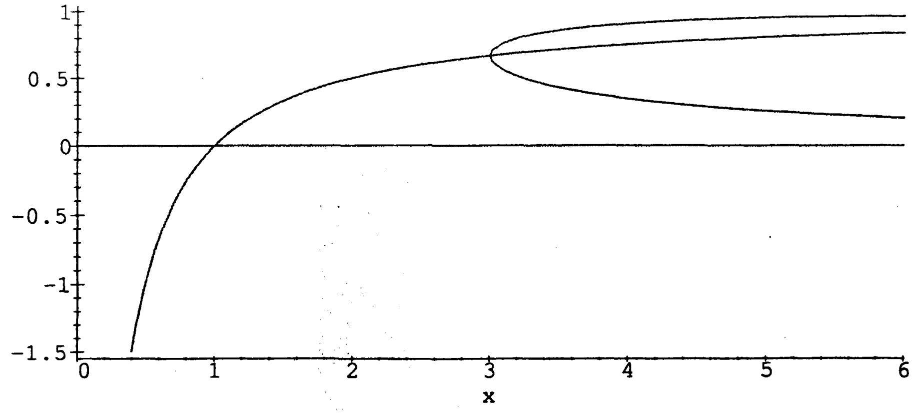


Figure 1: Curves of fixed points and 2-periodic points of the family $f_x : y \mapsto xy(1-y)$.

- Of course, perfect uniqueness is hopeless: Already the equation $\Delta_\varepsilon y = f(x)$ has all ε-periodic functions as solutions. However, to which extend can two solutions be close to each other?

To our knowledge, very few work exists in the literature about this topic. We mention all the references we know during the last 15 years.

The contents of the article is as follows. Section 2 presents the problem of bifurcation delay for slow-fast discrete dynamical systems and states the main results recently obtained in this domain [10]. As an introduction to the general theory, Section 3 deals with the *sum of a function*, which is a solution of equation $\Delta_\varepsilon y(x) = f(x)$. General results about the main questions are stated in Section 4.

2 Bifurcation delay

We consider here a real discrete system defined recursively by

$$(2) \qquad y_0 \text{ given }, \quad y_{n+1} = f_x(y_n) = f(x, y_n)$$

where, for this section, x is a real parameter, as is the variable y, and where $f : \mathbb{R}^2 \to \mathbb{R}$ is sufficiently smooth. In this article we only consider the example of the unimodal quadratic family

$$f_x : y \mapsto xy(1-y) .$$

Static bifurcation. Usually these mappings are considered as mappings of the interval $[0,1]$ into itself, i.e. for x in $[0,4]$ only. Here we will consider them as dynamical systems for all $y \in \mathbb{R}$ and $x > 0$. There are two curves of fixed points: $y = 0$, stable for $|x| < 1$, and $y = g_0(x) := 1 - \frac{1}{x}$, stable for $1 < x < 3$, see figure 1. Indeed the stability of the curve of fixed points $y = g_0(x)$ is governed by the function

$$a : x \mapsto \frac{\partial f}{\partial y}(x, g_0(x)) .$$

In our example, hence in the whole article, we have $a(x) = 2 - x$. We are mainly interested in the period doubling bifurcation, which appears on the curve $y = g_0(x)$ for $x > 3$. This curve is named the *slow curve* in the sequel.

Dynamic bifurcation. How did we draw figure 2? We replaced the parameter x by a variable x_n that changes slowly with each iteration: We introduced a small parameter $\varepsilon > 0$ and calculated

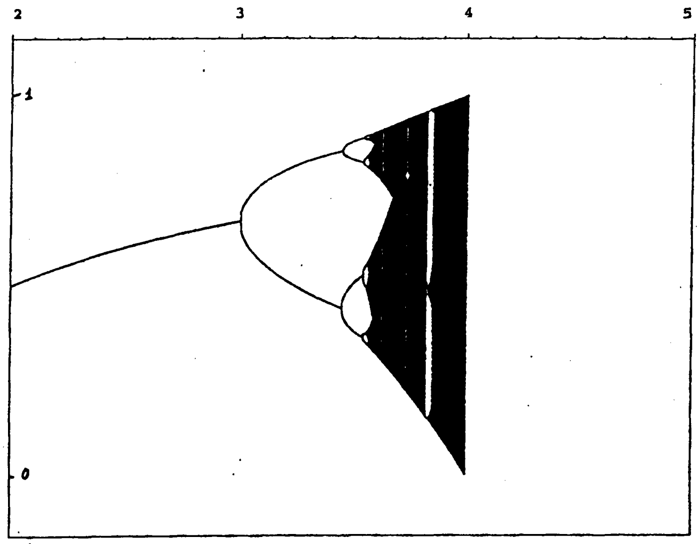


Figure 2: Bifurcation diagram of the quadratic family, $2 < x < 5,\ 0 < y < 1$.

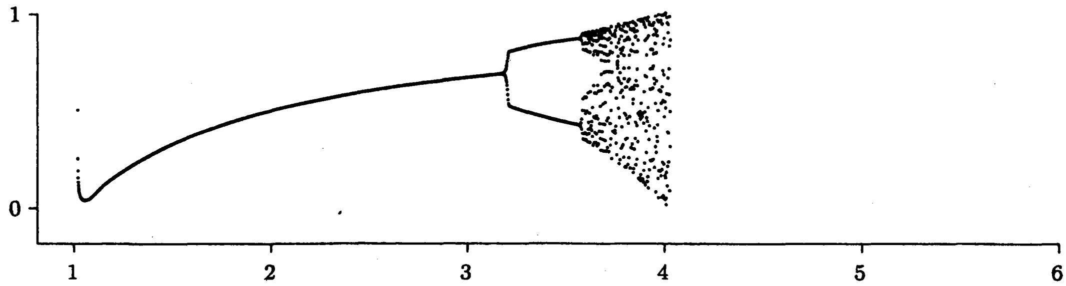
the orbit of the following discrete *slow-fast* system

$$(3) \qquad \begin{cases} x_{n+1} &= x_n + \varepsilon \\ y_{n+1} &= f(x_n, y_n) = x_n y_n (1 - y_n) \end{cases}$$

with initial condition $x_0 = 2, y_0 = 1/2$. In the sequel, we will call *orbit* a family $(x_n(\varepsilon), y_n(\varepsilon))_{n\in\mathbf{N}}$ of solutions of (3) depending upon ε. Here on figure 2, $\varepsilon = 10^{-6}$ and the accuracy is 8 digits.

It is quite natural to slowly move the parameter when one draws a bifurcation diagram. Indeed the description of this type of bifurcation in classical works often contains "dynamic" terms, see [10] and the references therein. Moreover in applications to physics, in particular non-linear optics, due to the very long time necessary for a system to reach a state of equilibrium, this move of parameter during the experiment is necessary. It is therefore interesting to understand the asymptotic behavior of some orbit when the small parameter tends to 0, and in particular to find out whether the *dynamic* bifurcation reflects the static bifurcation.

In figure 3, we have chosen $\varepsilon = 10^{-3}$ and as initial point $x_0 = 1,\ y_0 = \frac{1}{2}$. The figures differ only in the numerical accuracy used to calculate the orbits.



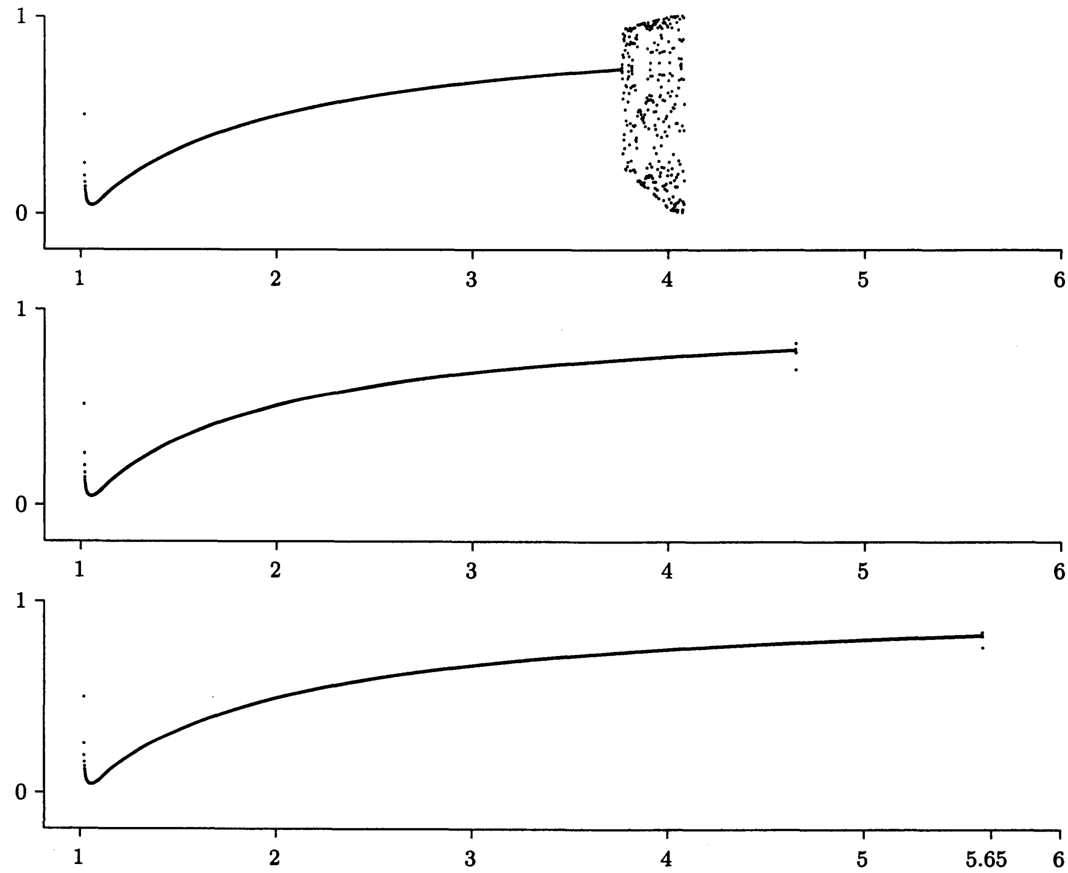


Figure 3: Successively 8, 100, 400 and 1000 digits

The first observation is that with sufficiently high precision the dynamic bifurcation corresponding to system (3) is completely different from the static bifurcation of (2). In particular, the orbits follow the curve of repulsive fixed points instead of switch to the curves of 2-periodic points. This is the so-called *bifurcation delay.* The second observation is an exponential sensitivity of this phenomenon. This will be explained below.

Difference equation. We will see later that the behavior of the orbits of (3) is closely related to the behavior of the solutions of the associated difference equation

$$\varphi(x+\varepsilon,\varepsilon) = x\varphi(x,\varepsilon)(1-\varphi(x,\varepsilon)) \ . \tag{4}$$

The smoothness of these solutions (w.r.t. x or ε) has no influence on the dynamics of the discrete solutions of (3). Their closeness to the slow curve, however, is important. This notion of closeness will be defined in definition 3 below.

The general results presented in section 4 lead to the following result.

Theorem 2.1. *Let x^* be determined by $x^* > 2$ and $\int_1^{x^*} \ln|2-x|dx = 0$. Put*

$$g_0 :]1, x^*[\to \mathbb{R}, x \mapsto 1 - \frac{1}{x} \ .$$

Then for every $\delta > 0$, there exist $\varepsilon_0 > 0$ and an analytic function $\varphi :]1+\delta, x^* - \delta[\times]0, \varepsilon_0[\to \mathbb{R}$, $(x,\varepsilon) \mapsto \varphi(x,\varepsilon)$, solution of (4) such that φ tends to g_0 as ε tend to 0, uniformly for x on $]1+\delta, x^* - \delta[$.

Numerically one finds $x^* \approx 5,65$. At the end of the present article we explain in few words how the general results of Section 4 are used to prove theorem 2.1.. For a complete proof, the reader is referred to [10]. Using the preliminary results of the following section, this theorem implies the following consequence for the dynamics of the orbits.

Corollary 2.2. *Let (x_0, y_0) an initial condition where $x_0 \in [1,3[$ and $y_0 \in]0,1[$ and let $((x_n, y_n))_{n\in\mathbb{N}^*}$ the sequence defined by*

$$(5) \qquad \begin{cases} x_{n+1} &= x_n + \varepsilon \\ y_{n+1} &= x_n y_n (1 - y_n) \,. \end{cases}$$

If $x_0 \in [1,2]$ then the orbit of (5) starting at (x_0, y_0) follows the slow curve $y = g_0(x) = 1 - \frac{1}{x}$ from $x = x_0$ up to $x = x_s \in]3, +\infty]$ satisfying $x_s \geq l(x_0)$, where the function $l : [1,3[\to]3, x^*]$ is defined by $\int_x^{l(x)} \ln|2-\xi|d\xi = 0$.

If $x_0 \in]2,3[$ and if $y_0 \neq g_0(x_0)$, then the exit abscissa $x = x_s$ is equal to $l(x_0)$.

Definitions. 1. We say that some orbit (i.e. solution) $((x_n(\varepsilon), y_n(\varepsilon)))_{n\in\mathbb{N}}$ of (3) *follows the slow curve $y = g_0(x)$ from the entry abscissa x_e to the exit abscissa x_s* if, for every $\delta > 0$ and $\rho > 0$, there exists ε_0 such that:

$$\forall n \in \mathbb{N},\ \forall \varepsilon \in]0, \varepsilon_0[, \quad (x_e + \delta \leq x_n(\varepsilon) \leq x_s - \delta \ \Rightarrow\ |y_n(\varepsilon) - g_0(x_n(\varepsilon))| < \rho)$$

and if the interval $[x_e, x_s]$ is maximal with this property.

Using the terminology of singular perturbation, this amounts more or less to saying that the orbit $((x_n(\varepsilon), y_n(\varepsilon)))_{n\in\mathbb{N}}$ has *boundary layers* near $x = x_e$ and $x = x_s$.

If the initial condition (x_0, y_0) satisfies $1 < x_0 < 3$ and $0 < y_0 < 1$ then, because of the attractiveness of the slow curve $y = g_0(x)$ for $1 < x < 3$, we obtain $x_e = x_0$ and $x_s \geq 3$. In other words, the orbit follows the slow curve at least on its attractive part.

2. We say that some orbit $((x_n(\varepsilon), y_n(\varepsilon)))_{n\in\mathbb{N}}$ *exhibits a bifurcation delay* if $x_s > 3$; we say that system (3) *exhibits a bifurcation delay* if every orbit with initial point (x_0, y_0), satisfying $1 < x_0 < 3$ and $0 < y_0 < 1$ exhibits a bifurcation delay.

In the references [3, 4, 5] these orbits are called "discrete canards". We avoid the word "canard" which might indicate a certain volatility. In the present context the phenomenon of bifurcation delay is robust in some sense.

3. We call *invariant curve* of (3) the graph of some solution (depending upon ε) of the associated difference equation (4). We say that the invariant curve $y = \varphi_\varepsilon(x)$ is *close* to the slow curve $y = g_0(x)$ on some compact interval I if $\lim_{\varepsilon\to 0} \varphi_\varepsilon(x) = g_0(x)$ uniformly on I. We say that the invariant curve is close to the slow curve on some open interval I if it is close on every compact sub-interval.

It is easy to construct invariant curves (which are not necessarily close to the slow curve): Define φ_ε arbitrarily on some interval of length ε, e.g. $\varphi_\varepsilon = g_0$ on $[x_0, x_0 + \varepsilon[$, and then use (4) to define φ_ε on the intervals $[x_0 + n\varepsilon, x_0 + (n+1)\varepsilon[$. In this way, it is even possible to construct invariant curves of class $\mathcal{C}^\infty$. On the other hand, the existence of *analytic* invariant curves is not clear, but also unnecessary for studying the discrete dynamics. The existence of any invariant curve, even not measurable, is sufficient, provided it is *close* to the slow curve on some appropriate interval. Ironically, the invariant curves we construct are analytic with respect to x.

To simplify notation, we will not indicate the ε-dependence of an orbit of (3). Thus the notation $((x_n, y_n))_{n\in\mathbb{N}}$ replaces the notation $((x_n(\varepsilon), y_n(\varepsilon)))_{n\in\mathbb{N}}$ used previously. We will indicate, however, the ε-dependence of the invariant curves.

Preliminary results.

1. There are orbits exhibiting bifurcation delay.
2. If some orbit $((x_n, y_n))_{n\in\mathbb{N}}$ exhibits bifurcation delay and if $((x_n, \tilde{y}_n))_{n\in\mathbb{N}}$ is an orbit (having the same first coordinates x_n) with $0 < \tilde{y}_0 < 1$ then $((x_n, \tilde{y}_n))_{n\in\mathbb{N}}$ also exhibits bifurcation delay.
   Moreover, if n is such that x_n is "properly" between x_e and x_s then the two points (x_n, y_n) and $(x_n, \tilde{y}_n)$ are exponentially close: $\forall\delta > 0\ \exists k, M > 0\ \forall n \in \mathbb{N}\ \forall\varepsilon > 0,\ (x_e + \delta \le x_n \le x_s - \delta \Rightarrow |y_n - \tilde{y}_n| \le M\exp(-k/\varepsilon))$.
3. System (3) exhibits bifurcation delay if and only if there exists an invariant curve close to the slow curve on some open interval containing 3.
   More precisely, if there is such an invariant curve then, for fixed (i.e. ε-independent) $x_0 \in]1, 3[$ and $y_0 \in]0, 1[$ with $y_0 \neq g_0(x_0)$, the orbit with initial point (x_0, y_0) follows the slow curve on the interval $]x_e, x_s[$, where $x_e = x_0$ and where $x_s > 3$ is determined by the *entry-exit relation* $\displaystyle\int_{x_e}^{x_s} \ln|2 - x|dx = 0$.
   Furthermore, for n such that x_n is properly between x_e and x_s, the point (x_n, y_n) is exponentially close to the invariant curve. In the same manner, two invariant curves close to the slow curve on some interval $[a, b]$ are exponentially close to each other on $]a, b[$.

Ideas of the proofs.

1. Put $m_n := (x_n, y_n)$ and consider the mapping

$$F_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x + \varepsilon, xy(1 - y)).$$

We construct an invariant curve by iterating the segment $[m_0, m_1]$ with F_ε. If m_n is not close and not too far from the slow curve, then m_{n+1} is on the other side of the slow curve (due to $a(x) = 2 - x < 0$, the orbits "oscillate" around the slow curve). Hence by intermediate value theorem, our invariant curve crosses the slow curve at some point. We then use induction to show that there is an orbit in some given neighborhood of the slow curve.
2. Using the following change of variables, the exponential closeness of two orbits is expressed differently: $Z_n = \varepsilon\ln|y_n - \tilde{y}_n|$. This yields an equation of the form

$$Z_{n+1} = Z_n + \varepsilon\ln|2 - x_n| + \varepsilon P(x_n, y_n, \tilde{y}_n, \varepsilon)$$

where P is negligible compared to $\ln|2 - x_n|$ if y_n and $\tilde{y}_n$ are close to $g_0(x_n)$. It follows that $Z_n - Z_0$ is close to $\displaystyle\int_{x_0}^{x_n} \ln|2 - x|dx$. This also shows the exponential closeness of y_n and $\tilde{y}_n$.

3. If (3) has bifurcation delay, then the invariant curve constructed as in the proof of item 1. is composed with orbits that exhibit bifurcation delay, hence is close to the slow curve on an open interval containing 3.
The rest of the proof of 3. is analogous to the proof of 2., using $Z_n = \varepsilon\ln|y_n - \varphi_\varepsilon(x_n)|$ (where

$y = \varphi_\varepsilon(x)$ parameterizes the invariant curve). Without giving any details, we mention that these results remain valid in a wide class of analogous discrete systems, but if the function a vanishes between x_e and x_s, then the approximation $\int_{x_0}^{x_n} \ln|2-x|dx$ of Z_n is not always valid and the entry-exit relation might be different. This explains why we had to consider separately $x_\varepsilon \leq 2$ and $x_\varepsilon > 2$ in the corollary 2.2. Note also that all remains valid for complex x and y. □

In addition to these preliminary results, let us mention the following local result [4, 1]:

If the function f is real analytic in a neighborhood of C then (3) exhibits bifurcation delay.

Two different methods of proof have been used independently. Both rely on the construction of some *quasi-invariant* curve, i.e. satisfying equation (4) except for exponentially small error terms. The first method [4] is an adaptation of a technique due to A.I. Neishtadt and consists of a sequence of changes of variable. The second method, due to Mrs Claude Baesens [1, 2], is a Gevrey analysis of the formal solution. We insist on the fact that both methods of proof can only give a local result. We present in section 4 the latest results on the subject [10], which are more global. Before the elaboration of these new technics, the only previous global results concerned analytical systems that are linear non homogeneous with respect to y [5].

3 The sum of a function

As an introduction to the topic, we study in this section the simplest difference equation $\Delta_\varepsilon y = f(x)$. In other words, we are concerned with the following problem.

Given a complex analytic function f in a domain D, is there a family $(y_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ of functions analytic on D such that:
i) *For all ε in $]0,\varepsilon_0]$ and all x in D such that $x+\varepsilon$ is in D,*

$$y_\varepsilon(x+\varepsilon) = y_\varepsilon(x) + \varepsilon f(x) \tag{6}$$

ii) *y_ε is bounded uniformly with respect to ε on every compact subset of D.*

In that case we say that f has a sum on D. The answer will essentially depend upon the domain D. We refer to [6] for more details.

Example. The equation

$$\Delta_\varepsilon y(x) = -\frac{1}{x^2} \tag{7}$$

has a formal solution in the complex plane:

$$y_1(x,\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon}{(x+n\varepsilon)^2}$$

which converges except for $x = -n\varepsilon$, $n \in \mathbb{N}$ and defines an analytic function with poles at $0,-\varepsilon,-2\varepsilon,\ldots$ It is easy to show that $\lim_{\varepsilon\to 0} y_1(x,\varepsilon) = \frac{1}{x}$ uniformly in any compact region of $\mathbb{C}\setminus\mathbb{R}^-$.

One can also consider the solution

$$y_2(x) = \sum_{k=0}^{\infty} \frac{-\varepsilon}{(x+(n+1)\varepsilon)^2}$$

which has poles at ε,2ε,.... and tends to $\frac{1}{x}$ on $\mathbb{C}\setminus\mathbb{R}^+$. However, no solution of (7) exists on $\mathbb{C}\setminus\{0\}$. This is somewhat surprising : although the primitive $\frac{1}{x}$ has only a singularity at 0 and is single valued, a sum of $-\frac{1}{x^2}$ must have a "cut" in one of the directions given by ε.

This example shows that a general result is hopeless without condition on the domain D. A necessary condition seems to be that D has to be *horizontally convex* in the following sense:

Definition 3.1. *A domain D is called horizontally convex if for all $x, y \in D$ with $\operatorname{Im} x = \operatorname{Im} y$, the whole segment $[x, y]$ is included in D.*

It is proven in [6] that this condition is also sufficient:

Theorem 3.2. *If f is analytic on a horizontally convex domain D then there is a sum of f on D.*

Idea of proof. To simplify we suppose D bounded and f bounded on D-. Let $x^+, x^- \in \mathrm{Cl}(D)$ with imaginary parts maximal, resp. minimal. For $x \in D$, consider a path γ_x^- from x^- to $x - \frac{\varepsilon}{2}$ with Im increasing and a path γ_x^+ from $x - \frac{\varepsilon}{2}$ to x^- with Im increasing, too. Put

$$y(x) := \int_{\gamma_x^- \cup \gamma_x^+} \frac{f}{1 - e_x^{-1}} = \int_{\gamma_x^-} f + \int_{\gamma_x^-} \frac{f}{1 - e_x} + \int_{\gamma_x^+} \frac{f}{1 - e_x^{-1}} \tag{8}$$

with $e_x : \xi \mapsto \exp\left(\frac{2\pi i}{\varepsilon}(\xi - x)\right)$. By Cauchy's formula we get

$$y(x + \varepsilon) - y(x) = 2\pi i f(x) \operatorname{Res}(1 - e_x^{-1}; x) = \varepsilon f(x) \ .$$

For any compact subset K of D there is a $c > 0$ such that for all $x \in K$, γ_x^- and γ_x^+ can be chosen "c-ascending", i.e.

$$y, z \in \gamma_x^\pm \ \Rightarrow \ |\operatorname{Im}(y - z)| \geq c|y - z| \ . \tag{9}$$

Thus $\left|\frac{1}{1-e_x(\xi)}\right|$, resp. $\left|\frac{1}{1-e_x^{-1}(\xi)}\right|$, are smaller than $\frac{1}{c}\exp\left(-\frac{\pi}{\varepsilon}|\operatorname{Im}(x - \xi)|\right) \leq \frac{1}{c}\exp\left(-\frac{\pi c}{\varepsilon}|x - \xi|\right)$ and we find that $g(x) - \int_{\gamma_x^-} f = O(\varepsilon/c^2)$. □

The followings statements answer to the question "what does a sum look like?" The first one is more or less the Euler-Mac Laurin's formula; here B_{2n} is the n-th Bernoulli number. The second one is clearly related to ε-periodic functions. It expresses that such a function bounded on some strip is exponentially flat inside the strip [7].

Proposition 3.3. *If f is analytic in a domain D and if y is a sum of f in D, then for any $N \in \mathbb{N}$ and for any x in D :*

$$y(x) - y(x_0) = \int_{x_0}^{x} f - \frac{\varepsilon}{2}\left(f(x) - f(x_0)\right) + \sum_{n=1}^{N} \frac{B_{2n}}{2n!}\left(f^{(2n-1)}(x) - f^{(2n-1)}(x_0)\right)\varepsilon^{2n} + O\left(\varepsilon^{2N+2}\right).$$

Proposition 3.4. *If y_1 and y_2 are two sums of f on D with $y_1(x_0) = y_2(x_0)$ for some $x_0 \in D$, then for any $x \in D$ and any $\delta > 0$ we have*

$$y_1(x) - y_2(x) = O\left(\exp\left(\tfrac{-2\pi}{\varepsilon}\left(r(x) - \delta\right)\right)\right)$$

with either

$$r(x) := \min\left\{\sup_{\xi\in D}(\operatorname{Im}(\xi - x), \sup_{\xi\in D}(\operatorname{Im}(\xi - x_0), \sup_{\xi\in D}(\operatorname{Im}(x - \xi), \sup_{\xi\in D}(\operatorname{Im}(x_0 - \xi)\right\}$$

if this minimum is finite, or $r(x)$ is any arbitrary real number if this minimum is $+\infty$.

Another expression for r is $r(x) = \min\{d(x), d(x_0)\}$ where $d(x)$ is the distance from x to the boundary of the smallest horizontal strip containing D.

Idea of proof. Take $x_1 \in D$ with $d(x_1)$ maximal. The function $w := y_1 - y_2$ is ε-periodic and defined and bounded in any strip $S := \{x \in \mathbb{C} \ ; \ |\text{Im}\,(x - x_1)| < (r(x_1) - \delta)/2\}$ for ε small enough. Hence the function $W : u \mapsto w\left(x_1 + \frac{\varepsilon}{2\pi i}\ln u\right)$ is analytic (single valued) and bounded (uniformly w.r.t ε) on the annular region $\{u \in \mathbb{C} \ ; \ \rho < |u| < R\}$ with $\rho = \exp\left(-\frac{\pi}{\varepsilon}(r(x_1) - \delta)\right) = 1/R$. Cauchy's formula then allows to show that $W(u) - W(1) = O\left(\max\{\rho/|u|, |u|/R\}\right)$. □

4 Analytic solutions of difference equations

The general results below are proved in details in [8, 9, 10].

Discretization of ODEs. We start with equations of the form $\Delta_\varepsilon y = f(\varepsilon, x, y)$. Therefore we consider an analytic function $f : \mathcal{D}_f \to \mathbb{C}^l$, $\mathcal{D}_f$ a domain contained in $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^l$, and a horizontally convex domain $D \subset \mathbb{C}$. We suppose that the differential equation

$$y' = f(0, x, y) \tag{10}$$

has an analytic solution $\tilde{y} : D \to \mathbb{C}^l$; in particular we assume $(0, x, \tilde{y}(x)) \in \mathcal{D}_f$ for any $x \in D$. As initial conditions, fix some $x_0 \in D$ and some function $y_0 :]0, \varepsilon_0] \to \mathbb{C}^l$ be given such that $y_0(\varepsilon) = \tilde{y}(x_0) + \mathcal{O}(|\varepsilon|)$.

Theorem 4.1. *With the above notation and assumptions, for every compact subset $K \subset D$ containing x_0, there exist $\eta > 0$ and a family $(y_\varepsilon)_{0<\varepsilon\leq\eta}$ of functions analytic on some domain containing K such that y_ε converges to $\tilde{y}$ uniformly on K as $\varepsilon \to 0$ and such that y_ε is a solution of*

$$y(x + \varepsilon) = y(x) + \varepsilon f(\varepsilon, x, y(x)) \tag{11}$$

with initial condition $y_\varepsilon(x_0) = y_0(\varepsilon)$.

Idea of proof. We first construct a right inverse of Δ_ε with integrals analogous to those in the proof of theorem 3.2. This right inverse is constructed on a "c-ascending" sub-domain Ω of D , i.e. a domain with two points x^+ and x^- in its closure with maximal and minimal imaginary part, and whose boundary consists of two paths from x^- to x^+ satisfying (9). If x is far enough from $x^\pm$ then the paths $\gamma_x^\pm$ can be chosen c'-ascending for some $c' < c$, too. However, as $x \to x^-$, say, the path γ_x^- has to pass near x, where $\frac{1}{1-e_x}$ has a simple pole. Therefore the function y given by (8) has a logarithmic singularity at x^+ and x^-. We overcome this difficulty by averaging the integral of (8) as follows.

Consider $\Omega_\varepsilon = \Omega + \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] \subset D$ for ε small enough and put

$$V_\varepsilon f(x) := 4 \int_{-1/8}^{1/8} dt \int_{x^- + \varepsilon t}^{x^+ + \varepsilon t} \frac{f}{1 - e_x}$$

where the integration path from $x^- + \varepsilon t$ to $x^+ + \varepsilon t$ passes between $x - \varepsilon$ and x, and Im increasing on it. Using the fact that the integral of ln converges at 0, we then show that V_ε is indeed a bounded right inverse of Δ_ε.

The next step is to linearize equation (11) around $\tilde{y}$, to solve the initial value problem $z' = \frac{\partial f}{\partial y}(0, x, \tilde{y}(x))$, $z(x_0) = h(x_0)$ by the variation of constant formula and to express its solution as the image of h by a bounded operator. The last step is an application of the fixed point theorem on some suitable Banach space. We refer to [9, 8] for the details. □

Together with this existence result, we have a result of exponential closeness between two solutions, analogous to proposition 3.4.:

Theorem 4.2. *With the above assumptions and notation, we introduce*

$$r(x) = \min(\mathrm{Im}\,(x^+ - x_0), \mathrm{Im}\,(x_0 - x^-), \mathrm{Im}\,(x^+ - x), \mathrm{Im}\,(x - x^-))\ ,$$

where x^+ *and* x^- *are points of* $\mathrm{Cl}(D)$ *with maximal and minimal imaginary part, or* $r(x)$ *arbitrary if this min is* $+\infty$.

Suppose that $y_{\varepsilon,1}$ *and* $y_{\varepsilon,2}$ *are two families of solutions of (11) that are analytic in* D*, that converge to* $\tilde{y}$ *uniformly on* D *as* $\varepsilon \to 0$ *and suppose that*

$$y_{\varepsilon,1}(x_0) - y_{\varepsilon,2}(x_0) = \mathcal{O}\left(\exp\left(-\tfrac{2\pi}{\varepsilon}(r(x_0) - \delta)\right)\right)$$

for every $\delta > 0$. *Then for every* $\delta > 0$ *we have*

$$y_{\varepsilon,1}(x) - y_{\varepsilon,2}(x) = \mathcal{O}\left(\exp\left(-\tfrac{2\pi}{\varepsilon}(r(x) - \delta)\right)\right)$$

on every compact subset K *of* D.

It is also possible to prove some analyticity result w.r.t. ε in sectors. This yields Gevrey estimates of the formal solution. As an application we can derive explicit errors bounds for Euler's scheme. See [8] for details.

The case of equations of the form $\delta_\varepsilon y = f(\varepsilon, x, y)$, i.e. with (11) replaced by

$$y\left(x + \tfrac{\varepsilon}{2}\right) = y\left(x - \tfrac{\varepsilon}{2}\right) + \varepsilon f(\varepsilon, x, y(x)) \tag{12}$$

leads to exactly the same results. We used these results in [9] to analyze in details the so-called ghost solutions of the discretized logistic equation

$$y_0 = 0,\ y_1 = \varepsilon, \qquad y_{n+1} = y_{n-1} + 2\varepsilon(1 - y_n^2)\ . \tag{13}$$

In particular, we estimated the length of the levels of these ghost solutions.

Slow-fast difference equations. Let us now return to our problem of bifurcation delay. We consider the following difference equation in the complex domain

$$y_\varepsilon(x + \varepsilon) = f(x, y_\varepsilon(x)) \tag{14}$$

where:

- The variable x varies in a bounded horizontally convex domain $D \subset \mathbb{C}$,
- The function $f : D \times \mathbb{C} \to \mathbb{C}$ is holomorphic,
- The letter ε denotes as usual a small positive parameter.

We suppose there exists an analytic function $g_0 : D \to \mathbb{C}$ verifying

$$f(x, g_0(x)) = g_0(x) \tag{15}$$

for all $x \in D$. We define $a(x) = \frac{\partial f}{\partial y}(x, g_0(x))$, and we suppose that, for $x \in D$, the values $a(x)$ are contained in some simply connected domain of $\mathbb{C} \setminus \{0\}$.

As before, we denote by x^- and x^+ the "peeks" of D, i.e. the points of $\mathrm{Cl}(D)$ such that $\forall x \in D,\ \mathrm{Im}\, x^- < \mathrm{Im}\, x < \mathrm{Im}\, x^+$.

Finally we denote by R_0 and R_1 the so-called *relief functions*, defined on D by

$$R_0 : x \mapsto \mathrm{Re}\left(\int_{x_0}^{x} \mathrm{Log}\, a(\xi)d\xi\right)$$

$$R_1 : x \mapsto R_0(x) - \mathrm{Re}\,(2\pi i(x-x_0)) = R_0(x) + 2\pi \mathrm{Im}\,(x-x_0)$$

where x_0 is some arbitrary point of D. These functions are natural generalizations to the complex variable of the entry-exit relation in Section 2. Already for singularly perturbed ODEs such a relief function appears. Here in the discrete case the situation is complicated by the fact that two reliefs are necessary. For the study of a pitchfork bifurcation, which leads to actual (i.e. volatile) canards, up to three reliefs are necessary, see the work of Miss A. El Rabih [11].

An *ascending path* is a path along which the imaginary part increases.

Theorem 4.3. *Suppose that for every $x \in D$ there exist two ascending paths γ_x^- from x^- to x and γ_x^+ from x to x^+ such that R_0 is decreasing on γ_x^- and R_1 is increasing on γ_x^+.*

Then for any compact subset K of D there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0]$ there is an analytic solution $y_\varepsilon : K \to \mathbb{C}$ of (14) tending to g_0 as $\varepsilon \to 0$ uniformly on K.

The second general result needed for a proof of theorem 2.1. concerns the exponential closeness of solutions of (14). Here, we only present this result in a situation symmetric with respect to the real axis; this is sufficient in our example.

Theorem 4.4. *Suppose that the functions f and g_0 have real values on the real axis.*

Suppose furthermore that y_1 and y_2 are two solutions of (14) defined in D, such that $y_j(x,\varepsilon) = g_0(x) + \mathcal{O}(\varepsilon)$ uniformly on D, $j = 1, 2$.

Then we have $y_1(x) - y_2(x) = \mathcal{O}\,(\exp(-r/\varepsilon))$ uniformly on D, with

$$r := \min(R_0(x^-) - R_0(x), R_1(x^+) - R_1(x))\ .$$

The *idea of proof of theorem 2.1.* is to construct (using theorem 4.3.) two solutions of (4) close to $1 - \frac{1}{x}$ on two domains not containing the point 2, then to show using theorem 4.4. that these solutions are exponentially close and finally to deduce the existence of a solution of (14) close to $1 - \frac{1}{x}$ on some domain containing the line segment $]1+\delta, x^* - \delta[$ of the theorem. Observe that theorem 4.3. only allows to prove the existence of invariant curves of (4) close to the slow curve on intervals *not* containing 2 and that theorem 4.4. yields the additional necessary ingredients. We refer to [10] for details.

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