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ON A SLOW-FAST SYSTEM IN R^6 WITH DUCKS

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ABSTRACT. The singular perturbation problem in $R^n (n > 3)$ includes a possibility having a constrained surface with a 3-dimensional dfferentiable manifold. We will take up the system in R^6 having such a constrained surface. Although it is difficult to analyze these systems in general, we will show some sufficient conditions to make it possible. Furthermore, we will reduce the system to the problem in R^3 and show the existence of the duck solutions using Benoit's criterion.

1. INTRODUCTION

S.A.Campbell, one of authors of [3], investigated first the coupled FitzHugh-Nagumo equations as a bifurcation problem. In the system, we have already proved the existence of the winding duck solutions in R^4 ([4]). As the associated slow-fast system (or singular pertubation problem) has a 2-dimensional slow manifold (constrained surface), it is able to reduce it to the slow-fast one in R^3. In this paper, we take up the system in R^6 with a 3-dimensional slow manifold. A typical example of this system is a 3-paralleled FitzHugh-Nagumo equations. There exists a 2-dimensional undefined region in the corresponding time scaled reduced sytem. This region makes it turn to have a projected slow-fast subsystem in R^3.

2. PRELIMINARIES

Let consider a constrained system(2.1):

$$
\begin{aligned}
&dx/dt = f(x, y, z, u),\\
&dy/dt = g(x, y, z, u),\\
&h(x, y, z, u) = 0,
\end{aligned}
\tag{2.1}
$$

where u is a parameter (any fixed) and f,g,h are defined in $R^3 \times R^1$. Furthermore, let consider the singular perturbation problem of the system (2.1):

$$
\begin{aligned}
&dx/dt = f(x, y, z, u),\\
&dy/dt = g(x, y, z, u),\\
&\epsilon dz/dt = h(x, y, z, u),
\end{aligned}
\tag{2.2}
$$

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where ϵ is infinitesimally small.

We assume that the system (2.1) satisfies the following conditions $(A1)-(A5)$:

$(A1)$ f and g are of class C^1 and h is of class C^2.

$(A2)$ The set $S=\{(x,y,z)\in R^3|h(x,y,z,u)=0\}$ is a 2-dimensional differentiable manifold and the set S intersects the set $T=\{(x,y,z)\in R^3|\partial h(x,y,z,u)/\partial z=0\}$ transversely so that the pli set $PL=\{(x,y,z)\in S\cap T\}$ is a 1-dimensional differentiable manifold.

$(A3)$ Either the value of f or that of g is nonzero at any point $p\in PL$.

Let $(x(t,u),y(t,u),z(t,u))$ be a solution of (2.1). By differentiating $h(x,y,z,u)$ with respect to the time t, the following equation holds:

$$(2.3)\quad h_x(x,y,z,u)f(x,y,z,u)+h_y(x,y,z,u)g(x,y,z,u)+h_z(x,y,z,u)dz/dt=0,$$

where $h_i(x,y,z,u)=\partial h(x,y,z,u)/\partial i$, $i=x,y,z$. The above system (2.1) becomes the following system:

$$(2.4)\quad \begin{aligned} dx/dt &= f(x,y,z,u),\\ dy/dt &= g(x,y,z,u),\\ dz/dt &= -\{h_x(x,y,z,u)f(x,y,z,u)+\\ &\quad h_y(x,y,z,u)g(x,y,z,u)\}/h_z(x,y,z,u), \end{aligned}$$

where $(x,y,z)\in S\setminus PL$. The system (2.1) coincides with the system (2.4) at any point $p\in S\setminus PL$. In order to study the system (2.4), let consider the following system:

$$(2.5)\quad \begin{aligned} dx/dt &= -h_z(x,y,z,u)f(x,y,z,u),\\ dy/dt &= -h_z(x,y,z,u)g(x,y,z,u),\\ dz/dt &= h_x(x,y,z,u)f(x,y,z,u)+h_y(x,y,z,u)g(x,y,z,u). \end{aligned}$$

As the system(2.5) is well defined at any point of R^3, it is well defined indeed at any point of PL. The solutions of (2.4) coincide with those of (2.1) on $S\setminus PL$ except the velocity when they start from the same initial points.

$(A4)$ For any $(x,y,z)\in S$, either of the following holds;

$$(2.6)\quad h_y(x,y,z,u)\neq 0,\, h_x(x,y,z,u)\neq 0,$$

that is, the surface S can be expressed as $y=\varphi(x,z,u)$ or $x=\psi(y,z,u)$ in the neighborhood of PL. Let $y=\varphi(x,z,u)$ exist, then the projected system, which restricts the system (2.5) is obtained:

$$(2.7)\quad \begin{aligned} dx/dt &= -h_z(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u),\\ dz/dt &= h_x(x,\varphi(x,z,u),z,u)f(x,\varphi(x,z,u),z,u)+\\ &\quad h_y(x,\varphi(x,z,u),z,u)g(x,\varphi(x,\varphi(x,z,u),z,u). \end{aligned}$$

$(A5)$ All the singular points of (2.7) are nondegenerate, that is, the matrix induced from the linearized system of (2.7) at a singular point has two nonzero eigenvalues. Note that all the points contained in $PS=\{(x,y,z)\in PL|dz/dt=0\}$, which is called *pseudo singular points* are the singular points of (2.5).

3. SLOW-FAST SYSTEM IN R^6

Let us consider the following slow-fast system:

$$
\begin{aligned}
\epsilon dx_1/dt &= h_1(x,y,u),\\
\epsilon dx_2/dt &= h_2(x,y,u),\\
\epsilon dx_3/dt &= h_3(x,y,u),\\
dy_1/dt &= f_1(x,y,u),\\
dy_2/dt &= f_2(x,y,u),\\
dy_3/dt &= f_3(x,y,u),
\end{aligned}
\tag{3.1}
$$

where $x^t = (x_1, x_2, x_3)$, $y^t = (y_1, y_2, y_3)$, are variables, $u \in R$ is a parameter and ϵ is infinitesimal in the sense of non-standard analysis of Nelson. We put $h^t = (h_1, h_2, h_3)$, $f^t = (f_1, f_2, f_3)$, then assume that $rank[Jh] = 3$ with respect to y, that is, there exists h_y^{-1}. Then, y is uniquely described like as $y = \psi(x,u)$. On the constrained surface, when ϵ tends to zero,

$$h(x,y,u) = 0, \tag{3.2}$$

differentiating it by t,

$$h_x dx/dt + h_y dy/dt = 0. \tag{3.3}$$

By the above assumption, we can obtain

$$
\begin{aligned}
dy/dt &= -h_y^{-1} h_x dx/dt\\
&= f(x,y,u).
\end{aligned}
\tag{3.4}
$$

In the system (3.4), we can reduce it to the time scaled reduced system:

$$dx/dt = -det(h_y^{-1}h_x)(h_y^{-1}h_x)^{-1} f(x, \psi(x,u), u). \tag{3.5}$$

The singular point of the system (3.5) is called a generalized pseudo- singular point (GPS). Here, we assume that the generalized pli set GPL:

$$GPL = \{(x,u) | det(h_y^{-1}h_x) = 0\} \tag{3.6}$$

gives a 2-dimensional differential manifold. Note that this is well defined in the original system (3.1).

Let us consider the following slow-fast system induced from the system (3.5):

$$
\begin{aligned}
\dot{x_1} &= -[det(h_y^{-1}h_x)(h_y^{-1}h_x)^{-1} f(x, \psi(x,u), u)]_1 = k_1(x,u),\\
\dot{x_2} &= -[det(h_y^{-1}h_x)(h_y^{-1}h_x)^{-1} f(x, \psi(x,u), u)]_2 = k_2(x,u),\\
\epsilon \dot{x_3} &= det(h_y^{-1}h_x) = k_0(x,u),
\end{aligned}
\tag{3.7}
$$

where $dx_i/dt = \dot{x_i}$ and $[*]_i = k_i(x,u)$ denotes the i th component of the vector $[*]$. Here, we assume that $|\dot{x_1} - \dot{x_2}|$, $|\dot{x_2} - \dot{x_3}|$, and $|\dot{x_3} - \dot{x_1}|$ are limited.

The above system is a projected subsystem into R^3 as an approximation of the original system (3.1). We suppose further that this system satisfies the assumptions $(A1)-(A5)$ in the section2 and the intersection of the set PS_3 of the pseudo-singular points in the system (3.7) and the set GPS is not empty. Especially, the assumption $(A2)$ makes an important role in this framework, that is, $S = \{(x,u) | k_0(x,u) = 0\}$ intersects $T_3 = \{(x,u) | \partial k_0(x,u)/\partial x_3 = 0\}$ transversely.

Theorem1 (Benoit[1],[2]). Let λ_1, λ_2 be the eigenvalues of the linearized system of the time scaled reduced system. If the PS has a saddle ($\lambda_1 > 0, \lambda_2 < 0$ or $\lambda_1 < 0, \lambda_2 > 0$) or a node point ($\lambda_1 > 0, \lambda_2 > 0$ or $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_1/\lambda_2 \notin Q$), there exist duck solutions in the slow-fast system.

Let us consider the projected subsystems:

$$\begin{aligned} &dx_i/dt = -[det(h_y^{-1}h_x)(h_y^{-1}h_x)^{-1}f(x,\psi(x,u),u)]_i = k_i(x,u), (i \neq j), \\ &\epsilon dx_j/dt = k_0(x,u). \end{aligned} \tag{3.8}$$

Here, in the above systems, we assume that there exists the number j such that the set S intersects the set T_j transversely and the intersection of the set GPS and the set PS_j is not empty.

Theorem2. In the j th system with $(A1)-(A5)$, if PS_j has a saddle or a node point satifying the above, there exist duck solutions in the original system(3.1).

4.A 3-PARALLELED FITZHUGH-NAGUMO EQUATIONS

Let h, f in the section 3 be

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} y_1 - x_1^3/3 + x_2 - x_3 \\ y_2 - x_2^3/3 + x_1 \\ y_3 - x_3^3/3 - x_2 + 2x_3 \end{pmatrix}, \tag{4.1}$$

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} -(x_1 + by_1 - a_1) \\ -(x_2 + by_2 - a_2) \\ -(x_3 + by_3 - a_3) \end{pmatrix}, \tag{4.2}$$

then we can obtain the following system like as the system (3.4)

$$\begin{pmatrix} x_1^2 & -1 & 1 \\ -1 & x_2^2 & 0 \\ 0 & 1 & x_3^2 - 2 \end{pmatrix} \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} bx_1^3/3 + x_1 - bx_2 + bx_3 - a_1 \\ -bx_1 + bx_2^3/3 + x_2 - a_2 \\ bx_2 + bx_3^3/3 + (1-2b)x_3 - a_3 \end{pmatrix}, \tag{4.3}$$

and then the set GPL:

$$GPL = \{(x,u) | det(h_y^{-1}h_x) = (x_3^2 - 2)(x_1^2 x_2^2 - 1) - 1 = 0\} \tag{4.4}$$

has 2-dimensional differentiable manifold satisfying the assumption $(A4)$. When putting the matrix in the equation (4.3) into A simply, the time scaled reduced system:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = (A_{ji}) \begin{pmatrix} bx_1^3/3 + x_1 - bx_2 + bx_3 - a_1 \\ -bx_1 + bx_2^3/3 + x_2 - a_2 \\ bx_2 + bx_3^3/3 + (1-2b)x_3 - a_3 \end{pmatrix}, \tag{4.5}$$

where (A_{ij}) is the cofactor matrix of A, gives a pseudo singular point $(0,0,a_1)$ $(3 < a_1 < 4, a_2 = 0, a_3 = 1)$ of GPS on GPL, when $b = -1$.

On the constrained surface: $detA = 0$, which has a 2-dimensional differentiable manifold, let us consider a slow-fast subsystem projected in R^3: putting $b = -1$,

$$
\begin{aligned}
&\dot{x_1} = x_2^2(x_3^2 - 2)(-x_1^3/3 + x_1 + x_2 - x_3 - a_1) \\
&+ (x_3^2 - 1)(x_1 - x_2^3/3 + x_2) + x_2^2(x_2 + x_3^3/3 - 3x_3 + 1), \\
&\epsilon\dot{x_2} = (x_3^2 - 2)(x_1^2x_2^2 - 1) - 1, \\
&\dot{x_3} = x_1^3/3 - x_1 - x_2 + x_3 + a_1 - x_1^2(x_1 - x_2^3/3 + x_2) \\
&+ (x_1^2x_2^2 - 1)(-x_2 - x_3^3/3 + 3x_3 - 1).
\end{aligned} \tag{4.6}
$$

Note that we use the variable x_2 instead of the variable x_3 in the system(3.7), that is, $j = 2$. The set T_2 is denoted as

$$T_2 = \{(x, -1) | \partial k_0/\partial x_2 = 2(x_3^2 - 2)x_1^2x_2 = 0\}. \tag{4.7}$$

On the set PL, the sign of $\partial k_0/\partial x_2$ changes when the variable x_2 increases from $-$ to $+$. Therefore, the set S intersects the set T_2 transversely. In case we choose the set T_1, it is possible to get the same result as the case choosing the set T_2. However, in case we do the set T_3, it is impossible to get the projected subsystem, because the set T_3:

$$T_3 = \{(x, -1) | \partial k_0/\partial x_3 = 2(x_1^2x_2^2 - 1)x_3 = 0\}, \tag{4.8}$$

does not intersect the set S transversely. In fact, as the third component of the pseudo-singular point is positive ($3 < a_1 < 4$), the sign of $\partial k_0/\partial x_3$ does not change on the set PL. The time scaled reduced system of the equation(3.5) is

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} F(x,u) \\ G(x,u) \\ H(x,u) \end{pmatrix}, \tag{4.9}$$

where

$$
\begin{aligned}
&F(x,u) = x_2^2(x_3^2 - 2)(-x_1^3/3 + x_1 + x_2 - x_3 - a_1) \\
&+ (x_3^2 - 1)(x_1 - x_2^3/3 + x_2) + x_2^2(x_2 + x_3^3/3 - 3x_3 + 1), \\
&G(x,u) = (x_3^2 - 2)(-x_1^3/3 + x_1 + x_2 - x_3 - a_1) \\
&+ x_1^2(x_3^2 - 2)(x_1 - x_2^3/3 + x_2) + x_2 + x_3^3/3 - 3x_3 + 1, \\
&H(x,u) = x_1^3/3 - x_1 - x_2 + x_3 + a_1 - x_1^2(x_1 - x_2^3/3 + x_2) \\
&+ (x_1^2x_2^2 - 1)(-x_2 - x_3^3/3 + 3x_3 - 1).
\end{aligned} \tag{4.10}
$$

Then first derivatives of the equations (4.10) are

$$
\begin{aligned}
&\partial F/\partial x_1 = x_2^2(x_3^2-2)(-x_1^2+1)+x_3^2-1),\\
&\partial F/\partial x_2 = (x_3^2-2)(-2x_1^3x_2/3+2x_1x_2+3x_2^2-2x_1x_2-2a_1x_2)\\
&+(x_3^2-1)(-x_2^2+1)+3x_2^2+2x_3^3x_2^{\prime}3-6x_3x_2+2x_2,\\
&\partial F/\partial x_3 = 2x_2^2x_3(-x_1^3/3+x_1+x_2-x_3-a_1)-x_2^2(x_3^2-2)\\
&+2x_3(x_1-x_2^3/3+x_2)+x_2^2(x_3^2-3),\\
&\partial G/\partial x_1 = (x_3^2-2)(-x_1^2+1)\\
&+(x_3^2-2)(3x_1^2-2x_2^3x_1/3+2x_2x_1),\\
&\partial G/\partial x_2 = (x_3^2-2)-x_1^2(x_3^2-2)(x_2^2-1)+1,\\
&\partial G/\partial x_3 = 2x_3(-x_1^3/3+x_1+x_2-x_3-a_1)-(x_3^2-2)\\
&+2x_1^2x_3(x_1-x_2^3/3+x_2)+x_3^2-3,\\
&\partial H/\partial x_1 = x_1^2-1-2x_1(x_1-x_2^3/3+x_2)-x_1^2\\
&+2x_1x_2^2(-x_2-x_3^3/3+3x_3-1),\\
&\partial H/\partial x_2 = -1-x_1^2(-x_2^2+1)+2x_2x_1^2(-x_2-x_3^3/3+3x_3-1)\\
&-(x_1^2x_2^2-1),\\
&\partial H/\partial x_3 = 1+(x_1^2x_2^2-1)(-x_3^2+x_3).
\end{aligned}
\tag{4.11}
$$

The corresponding Jacobian matrix J_p at the pseudo singular point $p=(0,0,a_1)\in PS\cap GPS$ is

$$
J_p=\begin{pmatrix} a_1^2-1 & a_1^2-1 & 0\\ a_1^2-2 & a_1^2-1 & -1\\ -1 & 0 & a_1^2-2\end{pmatrix}. \tag{4.12}
$$

For some constant $a_1(3<a_1<4)$, the above eigenvalues $\lambda_1,\lambda_2,\lambda_3$ are positive and $\lambda_1/\lambda_2\notin Q$,$\lambda_2/\lambda_3\notin Q$. Even if the parameter b changes, it is established almost everywhere. Therefore, in the time scaled reduced system projected into R^2, the eigenvalues ensures that the pseudo singular points are node at around $b=-1$.

Theorem3. In the system (3.1) with the equations (4.1), (4.2), putting the parameter $b=-1$, it has duck solutions for some constant $3<a_1<4$, $a_2=0$ and $a_3=1$.

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