Path Model for a Level-Zero Extremal Weight Module over a Quantum Affine Algebra

佐垣 大輔 (Daisuke SAGAKI)  内藤 聡 (Satoshi NAITO)
筑波大学 数学系                       筑波大学 数学系
Institute of Mathematics, University of Tsukuba
sagaki@math.tsukuba.ac.jp               naito@math.tsukuba.ac.jp

0 Introduction.

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra over $\mathbb{Q}$ with the Cartan subalgebra $\mathfrak{h}$ and the Weyl group $W$. We fix an integral weight lattice $P \subset \mathfrak{h}^* := \text{Hom}_\mathbb{Q}(\mathfrak{h}, \mathbb{Q})$ that contains all simple roots of $\mathfrak{g}$. Let $\lambda \in P$ be an integral weight. In [L1] and [L2], Littelmann introduced the notion of Lakshmibai–Seshadri paths of shape $\lambda$, which are piecewise linear, continuous maps $\pi : [0, 1] \to P$ parametrized by pairs of a sequence of elements of $W\lambda$ and a sequence of rational numbers satisfying a certain condition, called the chain condition. Denote by $\mathcal{B}(\lambda)$ the set of Lakshmibai–Seshadri paths of shape $\lambda$. Littelmann proved that $\mathcal{B}(\lambda)$ has a normal crystal structure in the sense of [Kas3], and that if $\lambda$ is a dominant integral weight, then the formal sum $\sum_{\pi \in \mathcal{B}(\lambda)} e(\pi(1))$ is equal to the character $\text{ch} L(\lambda)$ of the integrable highest weight $\mathfrak{g}$-module $L(\lambda)$ of highest weight $\lambda$. Then he conjectured that $\mathcal{B}(\lambda)$ for dominant $\lambda \in P$ would be isomorphic to the crystal base of the integrable highest weight module of highest weight $\lambda$ as crystals. This conjecture was affirmatively proved independently by Kashiwara [Kas4] and Joseph [J].

In [Kas2] and [Kas5], Kashiwara introduced an extremal weight module $V(\lambda)$ of extremal weight $\lambda \in P$ over the quantized universal enveloping algebra $U_q(\mathfrak{g})$ over $\mathbb{Q}(q)$, and showed that it has a crystal base $\mathcal{B}(\lambda)$. The extremal weight module is a natural generalization of an integrable highest (lowest) weight module. In fact, we know from [Kas2, §8] that if $\lambda \in P$ is dominant (resp. anti-dominant), then the extremal weight module $V(\lambda)$ is isomorphic to the integrable highest (resp. lowest) weight module of highest (resp. lowest) weight $\lambda$, and the crystal base $\mathcal{B}(\lambda)$ of $V(\lambda)$ is isomorphic to the crystal base of the integrable highest (resp. lowest) weight module as a crystal.
Now, we assume that \( g \) is of affine type. Let \( I \) be the index set of the simple roots of \( g \), and fix a special vertex \( 0 \in I \) as in [Kas5, §5.2]. In this paper, as an extension of the isomorphism theorem due to Kashiwara and Joseph, we prove that if \( \lambda \) is a level-zero fundamental weight \( \varpi_i \in P \) for \( i \in I_0 := I \setminus \{0\} \) (see [Kas5, §5.2]; note that \( \varpi_i \) is not dominant), then the connected component \( \mathcal{B}_0(\varpi_i) \) of \( \mathcal{B}(\varpi_i) \) containing \( \pi_{\varpi_i}(t) := t\varpi_i \) is isomorphic to the crystal base \( \mathcal{B}(\varpi_i) \) of the extremal weight module \( V(\varpi_i) \) as crystals. Namely, we prove the following:

**Theorem 1.** Assume that \( g \) is of affine type. There exists a unique isomorphism \( \Phi_{\varpi_i} : \mathcal{B}(\varpi_i) \cong \mathcal{B}_0(\varpi_i) \) of crystals such that \( \Phi_{\varpi_i}(u_{\varpi_i}) = \pi_{\varpi_i} \), where \( u_{\varpi_i} \in \mathcal{B}(\varpi_i) \) is the unique extremal weight element of weight \( \varpi_i \).

Let \( g_S \) be the Levi subalgebra corresponding to a proper subset \( S \) of the index set \( I \), and let \( U_q(g_S) \subset U_q(g) \) be the quantized universal enveloping algebra of \( g_S \). By restriction, we can regard the crystals \( \mathcal{B}(\varpi_i) \) and \( \mathcal{B}_0(\varpi_i) \) for \( U_q(g) \) as crystals for \( U_q(g_S) \). We show the following branching rule for \( \mathcal{B}(\varpi_i) \) and \( \mathcal{B}_0(\varpi_i) \) as crystals for \( U_q(g_S) \):

**Theorem 2.** As crystals for \( U_q(g_S) \), \( \mathcal{B}(\varpi_i) \) and \( \mathcal{B}_0(\varpi_i) \) decompose as follows:

\[
\mathcal{B}(\varpi_i) \cong \bigoplus_{\pi \in \mathcal{B}(\varpi_i), \pi \text{ is } g_S\text{-dominant}} \mathcal{B}_S(\pi(1)), \quad \mathcal{B}_0(\varpi_i) \cong \bigoplus_{\pi \in \mathcal{B}_0(\varpi_i), \pi \text{ is } g_S\text{-dominant}} \mathcal{B}_S(\pi(1)).
\]

where \( \mathcal{B}_S(\lambda) \) is the set of Lakshmibai-Seshadri paths of shape \( \lambda \) for \( U_q(g_S) \), and \( \pi \in \mathcal{B}(\varpi_i) \) is said to be \( g_S \)-dominant if \( (\pi(t))(\alpha_i^\vee) \geq 0 \) for all \( t \in [0,1] \) and \( i \in S \).

We also show that the extremal weight module \( V(\varpi_i) \) of extremal weight \( \varpi_i \) is completely reducible as a \( U_q(g_S) \)-module. Then, as an application of Theorems 1 and 2 above, we obtain the following branching rule for \( V(\varpi_i) \):

**Theorem 3.** The extremal weight module \( V(\varpi_i) \) of extremal weight \( \varpi_i \) is completely reducible as a \( U_q(g_S) \)-module, and the decomposition of \( V(\varpi_i) \) as a \( U_q(g_S) \)-module is given by:

\[
V(\varpi_i) \cong \bigoplus_{\pi \in \mathcal{B}_0(\varpi_i), \pi \text{ is } g_S\text{-dominant}} V_S(\pi(1)),
\]

where \( V_S(\lambda) \) is the integrable highest weight \( U_q(g_S) \)-module of highest weight \( \lambda \).

Assume that \( \varpi_i \) is minuscule, i.e., \( \varpi_i(\alpha^\vee) \in \{\pm 1, 0\} \) for every dual real root \( \alpha^\vee \) of \( g \). Then we can check that \( \mathcal{B}(\varpi_i) \) is connected, and hence \( \mathcal{B}(\varpi_i) = \mathcal{B}_0(\varpi_i) \).
In this case, we get the following decomposition rule of Littelmann type for the concatenation $\mathcal{B}(\lambda) \ast \mathcal{B}(\varpi_i)$. Here we note that unlike Theorems 2 and 3, this theorem does not necessarily imply the decomposition rule for tensor products of corresponding $U_q(\mathfrak{g})$-modules.

**Theorem 4.** Let $\lambda$ be a dominant integral weight which is not a multiple of the null root $\delta$ of $\mathfrak{g}$, and assume that $\varpi_i$ is minuscule. Then, the concatenation $\mathcal{B}(\lambda) \ast \mathcal{B}(\varpi_i)$ decomposes as follows:

$$
\mathcal{B}(\lambda) \ast \mathcal{B}(\varpi_i) \cong \bigoplus_{\pi \in \mathcal{B}(\varpi_i), \pi: \lambda\text{-dominant}} \mathcal{B}(\lambda + \pi(1)),
$$

where $\pi \in \mathcal{B}(\varpi_i)$ is said to be $\lambda$-dominant if $(\lambda + \pi(t))(\alpha_i^\vee) \geq 0$ for all $t \in [0,1]$ and $i \in I$.

**Remark.** The reader should compare Theorems 1 and 4 with the corresponding results [G, Theorems 1.5 and 1.6] of Greenstein for bounded modules.

**Acknowledgments.** We are grateful to Professors Jonathan Beck and Hiraku Nakajima for informing us their results in [BN], and permitting us to use them.

1 Preliminaries and Notation.

1.1 Quantized universal enveloping algebras. Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix, and $\mathfrak{g} := \mathfrak{g}(A)$ the Kac–Moody algebra over $Q$ associated to the generalized Cartan matrix $A$. Denote by $\mathfrak{h}$ the Cartan subalgebra, by $\Pi := \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ and $\Pi^\vee := \{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}$ the set of simple roots and simple coroots, and by $W = \langle r_i \mid i \in I \rangle$ the Weyl group. We take (and fix) an integral weight lattice $P \subset \mathfrak{h}^*$ such that $\alpha_i \in P$ for all $i \in I$.

Denote by $U_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $Q(q)$ of rational functions in $q$, and by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the negative (resp. positive) part of $U_q(\mathfrak{g})$. We denote by $\widetilde{U}_q(\mathfrak{g}) = \bigoplus_{\lambda \in P} U_q(\mathfrak{g})a_\lambda$ the modified quantized universal enveloping algebra of $\mathfrak{g}$, where $a_\lambda$ is a formal element of weight $\lambda$ (cf. [Kas2, §1.2]).

1.2 Affine Lie algebras. Assume that $\mathfrak{g}$ is of affine type. Let

$$
\delta = \sum_{i \in I} a_i \alpha_i \in \mathfrak{h}^* \quad \text{and} \quad c = \sum_{i \in I} a_i^\vee \alpha_i^\vee \in \mathfrak{h}
$$

(1.2.1)
be the null root and the canonical central element of $g$. We denote by $(\cdot, \cdot)$ the bilinear form on $\mathfrak{h}^*$, which is normalized by: $a_i^\vee = \frac{[\alpha_i, \alpha_i]}{2} a_i$ for all $i \in I$. Set $\mathfrak{h}_0^* := \bigoplus_{i \in I} \mathbb{Q} \alpha_i \subset \mathfrak{h}^*$, and let $\operatorname{cl} : \mathfrak{h}_0^* \to \mathfrak{h}_0^*/\mathbb{Q} \delta$ the canonical map from $\mathfrak{h}_0^*$ onto the quotient space $\mathfrak{h}_0^*/\mathbb{Q} \delta$. We have a bilinear form (also denoted by $(\cdot, \cdot)$) on $\mathfrak{h}_0^*/\mathbb{Q} \delta$ induced from the bilinear form $(\cdot, \cdot)$, which is positive-definite.

We take (and fix) a special vertex $0 \in I$ as in [Kas5, §5.2], and set $I_0 := I \setminus \{0\}$. For $i \in I_0$, let $\varpi_i$ be a unique element in $\bigoplus_{i \in I_0} \mathbb{Q} \alpha_i$ such that $\varpi_i(\alpha_j^\vee) = \delta_{i,j}$ for all $j \in I_0$. Notice that $\Lambda_i := \varpi_i + a_i^\vee \Lambda_0$ is an $i$-th fundamental weight for $g$, where $\Lambda_0$ is a 0-th fundamental weight for $g$. So, we may assume that all the $\varpi_i$'s are contained in the integral weight lattice $P$.

1.3 Crystal bases. Let $B(\infty)$ be the crystal base of the negative part $U_q^{-}(g)$ with $u_{\infty}$ the highest weight element. Denote by $e_i$ and $f_i$ the raising and lowering Kashiwara operator on $B(\infty)$, respectively, and define $\varepsilon_i : B(\infty) \to \mathbb{Z}$ and $\varphi_i : B(\infty) \to \mathbb{Z}$ by

$$\varepsilon_i(b) := \max\{n \geq 0 \mid e_i^n b \neq 0\}, \quad \varphi_i(b) := \varepsilon_i(b) + (\mathrm{wt}(b))(\alpha_i^\vee). \quad (1.3.1)$$

Denote by $*: B(\infty) \to B(\infty)$ the $*$-operation on $B(\infty)$ (cf. [Kas1, Theorem 2.1.1] and [Kas3, §8.3]). We put $e_i^\ast := * \circ e_i \circ *$ and $f_i^\ast := * \circ f_i \circ *$ for each $i \in I$.

**Theorem 1.3.1** (cf. [Kas1, Theorem 2.2.1]). For each $i \in I$, there exists an embedding $\Psi_i^- : B(\infty) \hookrightarrow B(\infty) \otimes B_i$ of crystals that maps $u_{\infty}$ to $u_{\infty} \otimes b_i(0)$, where $B_i := \{b_i(n) \mid n \in \mathbb{Z}\}$ is a crystal in [Kas1, Example 1.2.6]. In addition, if $b = (f_i^\ast)^k b_0$ for some $k \in \mathbb{Z}_{\geq 0}$ and $b_0 \in B(\infty)$ such that $e_i^\ast b_0 = 0$, then $\Psi_i^-(b) = b_0 \otimes b_i(-k)$.

We denote by $B(-\infty)$ the crystal base of the positive part $U_q^{+}(g)$ with $u_{-\infty}$ the lowest weight vector, and by $e_i$ and $f_i$ the raising and lowering Kashiwara operator on $B(-\infty)$, respectively. We set

$$\varepsilon_i(b) := \varphi_i(b) - (\mathrm{wt}(b))(\alpha_i^\vee), \quad \varphi_i(b) := \max\{n \geq 0 \mid f_i^n b \neq 0\}. \quad (1.3.2)$$

We also have the $*$-operation $*: B(-\infty) \to B(-\infty)$ on $B(-\infty)$. We can easily show that there exists an embedding $\Psi_i^+ : B(-\infty) \hookrightarrow B_i \otimes B(-\infty)$ of crystals with properties similar to $\Psi_i^-$ in Theorem 1.3.1.

Let $B(\tilde{U}_q(g)) = \bigsqcup_{\lambda \in P} B(U_q(g) a_\lambda)$ be the crystal base of the modified quantized universal enveloping algebra $\tilde{U}_q(g)$ with $u_\lambda$ the element of $B(U_q(g) a_\lambda)$ corresponding to $a_\lambda \in U_q(g) a_\lambda$ (cf. [Kas2, Theorem 2.1.2]). We denote by $e_i$ and $f_i$ the raising
and lowering Kashiwara operator on $B(\tilde{U}_q(\mathfrak{g}))$, and define $\varepsilon_i : B(\tilde{U}_q(\mathfrak{g})) \to \mathbb{Z}$ and $\varphi_i : B(\tilde{U}_q(\mathfrak{g})) \to \mathbb{Z}$ by

$$
\varepsilon_i(b) := \max\{n \geq 0 \mid e_i^n b \neq 0\}, \quad \varphi_i(b) := \max\{n \geq 0 \mid f_i^n b \neq 0\}.
$$

(1.3.3)

We know the following theorem from [Kas2, Theorem 3.1.1].

**Theorem 1.3.2.** There exists an isomorphism $\Xi_\lambda : B(U_q(\mathfrak{g})a_\lambda) \xrightarrow{\sim} B(\infty) \otimes T_\lambda \otimes B(-\infty)$ of crystals such that $\Xi_\lambda(u_\lambda) = u_\infty \otimes t_\lambda \otimes u_{-\infty}$, where $T_\lambda := \{t_\lambda\}$ is a crystal consisting of a single element $t_\lambda$ of weight $\lambda$ (cf. [Kas3, Example 7.3]).

We also denote by $* : B(\tilde{U}_q(\mathfrak{g})) \to B(\tilde{U}_q(\mathfrak{g}))$ the $*$-operation on $B(\tilde{U}_q(\mathfrak{g}))$ (cf. [Kas2, Theorem 4.3.2]). We know the following theorem from [Kas2, Corollary 4.3.3].

**Theorem 1.3.3.** Let $b \in B(U_q(\mathfrak{g})a_\lambda)$, and assume that $\Xi_\lambda(b) = b_1 \otimes t_\lambda \otimes b_2$ with $b_1 \in B(\infty)$ and $b_2 \in B(-\infty)$. Then, $b^*$ is contained in $B(U_q(\mathfrak{g})a_{\lambda'})$, where $\lambda' := -\lambda - \text{wt}(b_1) - \text{wt}(b_2)$, and $\Xi_{\lambda'}(b^*) = b_1^* \otimes t_{\lambda'} \otimes b_2^*$.

**1.4 The crystal base of an extremal weight module.** Since $B(\tilde{U}_q(\mathfrak{g}))$ is a normal crystal, we can define an action of the Weyl group $W$ on $B(\tilde{U}_q(\mathfrak{g}))$ (see [Kas2, §7.1]); for $i \in I$, we define an action of the simple reflection $r_i$ by

$$
r_i b := \begin{cases} e_i^n b & \text{if } n := (\text{wt}(b))(\alpha_i^\vee) \geq 0 \\
e_i^{-n} b & \text{if } n := (\text{wt}(b))(\alpha_i^\vee) \leq 0
\end{cases}
$$

for $b \in B(\tilde{U}_q(\mathfrak{g}))$. (1.4.1)

An element $b \in B(\tilde{U}_q(\mathfrak{g}))$ is said to be extremal if the elements $\{wb\}_{w \in W} \subset B(\tilde{U}_q(\mathfrak{g}))$ satisfy the following condition for all $i \in I$:

$$
\text{if } (\text{wt}(wb))(\alpha_i^\vee) \geq 0, \text{ then } e_i(wb) = 0,
$$

and

$$
\text{if } (\text{wt}(wb))(\alpha_i^\vee) \leq 0, \text{ then } f_i(wb) = 0.
$$

(1.4.2)

For $\lambda \in P$, we define a subcrystal $B(\lambda)$ of $B(U_q(\mathfrak{g})a_\lambda)$ by

$$
B(\lambda) := \{b \in B(U_q(\mathfrak{g})a_\lambda) \mid b^* \text{ is extremal}\}.
$$

(1.4.3)

Remark that $u_\lambda \in B(U_q(\mathfrak{g})a_\lambda)$ is contained in $B(\lambda)$. We know from [Kas2, Proposition 8.2.2] and [Kas5, §3.1] that $B(\lambda)$ is the crystal base of the extremal weight module $V(\lambda)$ of extremal weight $\lambda$ over $U_q(\mathfrak{g})$. 

2 Some Tools for Crystal Bases.

2.1 Multiple maps. We know the following theorem.

Theorem 2.1.1 ([Kas4, Theorem 3.2]). Let $m \in \mathbb{Z}_{>0}$. There exists a unique injective map $S_{m,\infty} : B(\infty) \hookrightarrow B(\infty)$ such that for each $b \in B(\infty)$ and $i \in I$, we have

$$\begin{align*}
\mathrm{wt}(S_{m,\infty}(b)) &= m \mathrm{wt}(b), \\
\epsilon_i(S_{m,\infty}(b)) &= m \epsilon_i(b), \\
\varphi_i(S_{m,\infty}(b)) &= m \varphi_i(b), \\
S_{m,\infty}(u_\infty) &= u_\infty, \\
S_{m,\infty}(e_i b) &= e_i^m S_{m,\infty}(b), \\
S_{m,\infty}(f_i b) &= f_i^m S_{m,\infty}(b). 
\end{align*}$$

(2.1.1)

Proposition 2.1.2. We set $S_{m,\infty}^* := \ast \circ S_{m,\infty} \circ \ast$. Then we have $S_{m,\infty}^* = S_{m,\infty}$ on $B(\infty)$. Namely, the $\ast$-operation commutes with the map $S_{m,\infty} : B(\infty) \hookrightarrow B(\infty)$.

The proposition above can be shown in a way similar to [NS2, Theorem 2.3.1]. Before giving a proof of the proposition, we show the following lemma.

Lemma 2.1.3. The following diagram is commutative:

$$
\begin{array}{ccc}
B(\infty) & \xrightarrow{\Psi_j^-} & B(\infty) \otimes B_j \\
S_{m,\infty}^* \downarrow & & \downarrow S_{m,\infty}^* \otimes S_{m,j} \\
B(\infty) & \xrightarrow{\Psi_j^-} & B(\infty) \otimes B_j.
\end{array}
$$

(2.1.3)

Here $S_{m,j} : B_j \rightarrow B_j$ is a map defined by $S_{m,j}(b_j(n)) := b_j(mn)$.

Proof. For $b \in B(\infty)$, there exists $b_0 \in B(\infty)$ such that $b = (f_j^*)^k b_0$ for some $k \in \mathbb{Z}_{>0}$ and $e_j^* b_0 = 0$. Then, by Theorem 1.3.1, we have $\Psi_j^-(b) = b_0 \otimes b_j(-k)$, and hence

$$
(S_{m,\infty}^* \otimes S_{m,j})(\Psi_j^-(b)) = S_{m,\infty}^*(b_0) \otimes b_j(-mk).
$$

On the other hand, we see that $S_{m,\infty}^*(b) = (f_j^*)^m k S_{m,\infty}^*(b_0)$. If $e_j^* S_{m,\infty}^*(b_0) \neq 0$, then we have $\epsilon_j(S_{m,\infty}(b_0)) \geq 1$. Since $\epsilon_j(S_{m,\infty}(b)) = m \epsilon_j(b) \in m \mathbb{Z}$ for all $b \in B(\infty)$, we deduce that $\epsilon_j(S_{m,\infty}(b_0)) \geq m$, and hence $(e_j^*)^m S_{m,\infty}^*(b_0) \neq 0$. However, since $e_j^* b_0 = 0$, we get $(e_j^*)^m S_{m,\infty}^*(b_0) = S_{m,\infty}^*(e_j^* b_0) = 0$, which is a contradiction. Therefore, we conclude that $e_j^* S_{m,\infty}^*(b_0) = 0$. It follows from Theorem 1.3.1 that

$$
\Psi_j^-(S_{m,\infty}(b)) = \Psi_j^-((f_j^*)^m S_{m,\infty}^*(b_0)) = S_{m,\infty}^*(b_0) \otimes b_j(-mk).
$$

Hence we have $(S_{m,\infty}^* \otimes S_{m,j})(\Psi_j^-(b)) = \Psi_j^-(S_{m,\infty}^*(b))$. This completes the proof of the lemma. \qed
Proof of Proposition 2.1.2. We will prove that \( S_{\infty}^{*}(b) = S_{m,\infty}(b) \) for \( b \in B(\infty)_{-\xi} \) by induction on the height \( \text{ht}(\xi) \) of \( \xi \) (note that \( -\text{wt}(b) \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_{i} \) for all \( b \in B(\infty) \)). If \( \text{ht}(\xi) = 0 \), then \( b \) is the highest weight element \( u_{\infty} \in B(\infty) \), and hence the assertion is obvious.

Assume that \( \text{ht}(\xi) \geq 1 \). Then, there exists some \( i \in I \) such that \( b_{1} := e_{i}b \neq 0 \). If \( e_{j}^{*}b_{1} = 0 \) for all \( j \in I \), then \( b_{1} = u_{\infty} \), and hence \( b = f_{i}u_{\infty} \). Because \( f_{i}^{k}u_{\infty} \) is a unique element of weight \(-k\alpha_{i} \) for each \( k \in \mathbb{Z}_{\geq 0} \), and \( \text{wt}(b^{*}) = \text{wt}(b) \) for all \( b \in B(\infty) \), we deduce that \( b^{*} = b \), and hence

\[
S_{m,\infty}^{*}(b) = (S_{m,\infty}(b))^{*} = (f_{i}^{m}u_{\infty})^{*} = f_{i}^{m}u_{\infty} = S_{m,\infty}(b).
\]

So, we may assume that there exists \( j \in I \) such that \( e_{j}^{*}b_{1} \neq 0 \). Let \( b_{2} \in B(\infty) \) be such that \( e_{j}^{*}b_{2} = 0 \) and \( b_{1} = (f_{j}^{*})^{k}b_{2} \) for some \( k \in \mathbb{Z}_{\geq 1} \). Namely, \( b = f_{i}(f_{j}^{*})^{k}b_{2} \) for some \( k \geq 1 \) and \( b_{2} \in B(\infty) \) such that \( e_{j}^{*}b_{2} = 0 \).

Case 1: \( i \neq j \). We show that \( \Psi_{j}^{-}(S_{m,\infty}^{*}(b)) = \Psi_{j}^{-}(S_{m,\infty}(b)) \) (recall that \( \Psi_{j}^{-} : B(\infty) \rightarrow B(\infty) \otimes B_{j} \) is an embedding of crystals). We have

\[
\Psi_{j}^{-}(b) = \Psi_{j}^{-}(f_{i}(f_{j}^{*})^{k}b_{2}) = f_{i}\Psi_{j}^{-}((f_{j}^{*})^{k}b_{2}) = f_{i}(b_{2} \otimes b_{j}(-k)) = f_{i}b_{2} \otimes b_{j}(-k).
\]

Here the last equality immediately follows from the definition of the tensor product of crystals (see, for example, [Kas3, §7.3]) and the condition that \( i \neq j \). Therefore, we obtain

\[
\Psi_{j}^{-}(S_{m,\infty}^{*}(b)) = (S_{m,\infty} \otimes S_{m,j})(\Psi_{j}^{-}(b)) \quad \text{by Lemma 2.1.3}
\]

\[
= S_{m,\infty}(f_{1}b_{2}) \otimes b_{j}(-mk)
\]

\[
= S_{m,\infty}(f_{1}b_{2}) \otimes b_{j}(-mk) \quad \text{by the inductive assumption}
\]

\[
= f_{i}^{m}S_{m,\infty}(b_{2}) \otimes b_{j}(-mk).
\]

On the other hand,

\[
S_{m,\infty}(b) = S_{m,\infty}(f_{i}(f_{j}^{*})^{k}b_{2}) = f_{i}^{m}S_{m,\infty}((f_{j}^{*})^{k}b_{2})
\]

\[
= f_{i}^{m}(f_{j}^{*})^{mk}S_{m,\infty}(b_{2}) \quad \text{by the inductive assumption}
\]

As in the proof of Lemma 2.1.3, we deduce that \( e_{j}^{*}S_{m,\infty}(b_{2}) = 0 \), and hence

\[
eq e_{j}^{*}S_{m,\infty}(b_{2}) = e_{j}^{*}S_{m,\infty}(b_{2}) = 0 \quad \text{by the inductive assumption. Therefore,}
\]

\[
\Psi_{j}^{-}(S_{m,\infty}(b)) = \Psi_{j}^{-}(f_{i}^{m}(f_{j}^{*})^{mk}S_{m,\infty}(b_{2})) = f_{i}^{m}\Psi_{j}^-((f_{j}^{*})^{mk}S_{m,\infty}(b_{2}))
\]

\[
= f_{i}^{m}(S_{m,\infty}(b_{2}) \otimes b_{j}(-mk)) = (f_{i}^{m}S_{m,\infty}(b_{2})) \otimes b_{j}(-mk).
\]
Here the last equality immediately follows again from the definition of the tensor product of crystals and the condition that $i \neq j$. Thus, we get that $\Psi_{j}^{-}(S_{m,\infty}(b)) = \Psi_{j}^{-}(S_{m,\infty}(b))$, and hence $S_{m,\infty}(b) = S_{m,\infty}(b)$.

**Case 2 : $i = j$.** As in Case 1, we have $\Psi_{j}^{-}(b) = f_{i}(b_{2} \otimes b_{i}(-k))$. We deduce from the definition of the tensor product of crystals that

$$\Psi_{i}^{-}(b) = f_{i}(b_{2} \otimes b_{i}(-k)) = \begin{cases} f_{i}b_{2} \otimes b_{i}(-k) & \text{if } \varphi_{i}(b_{2}) > k, \\ b_{2} \otimes b_{i}(-k-1) & \text{if } \varphi_{i}(b_{2}) \leq k. \end{cases}$$

Hence, as in Case 1, we get

$$\Psi_{i}^{-}(S_{m,\infty}(b)) = \begin{cases} f_{i}^{m}S_{m,\infty}(b_{2}) \otimes b_{i}(-mk) & \text{if } \varphi_{i}(b_{2}) > k, \\ S_{m,\infty}(b_{2}) \otimes b_{i}(-mk-m) & \text{if } \varphi_{i}(b_{2}) \leq k. \end{cases}$$

On the other hand, in exactly the same way as in Case 1, we can show that $\Psi_{i}^{-}(S_{m,\infty}(b)) = f_{i}^{m}(S_{m,\infty}(b_{2}) \otimes b_{i}(-mk))$. Because $\varphi_{i}(S_{m,\infty}(b_{2})) = m\varphi_{i}(b_{2})$ by (2.1.1), we deduce from the definition of the tensor product of crystals that

$$f_{i}^{m}(S_{m,\infty}(b_{2}) \otimes b_{i}(-mk)) = \begin{cases} f_{i}^{m}S_{m,\infty}(b_{2}) \otimes b_{i}(-mk) & \text{if } \varphi_{i}(b_{2}) > k, \\ S_{m,\infty}(b_{2}) \otimes b_{i}(-mk-m) & \text{if } \varphi_{i}(b_{2}) \leq k. \end{cases}$$

Therefore, we obtain that $\Psi_{i}^{-}(S_{m,\infty}(b)) = \Psi_{i}^{-}(S_{m,\infty}(b))$, and hence $S_{m,\infty}(b) = S_{m,\infty}(b)$. Thus, we have proved the proposition. □

**Remark 2.1.4.** A similar result holds for the crystal base $B(-\infty)$. Namely, for each $m \in \mathbb{Z}_{>0}$, there exists a unique injective map $S_{m,-\infty} : B(-\infty) \hookrightarrow B(-\infty)$ with properties similar to $S_{m,\infty}$ in Theorem 2.1.1, and it commutes with the $*$-operation on $B(-\infty)$.

For $m \in \mathbb{Z}_{>0}$, we define an injective map $\tilde{S}_{m,\lambda} : B(U_{q}(\mathfrak{g})a_{\lambda}) \hookrightarrow B(U_{q}(\mathfrak{g})a_{m\lambda})$ as in the following commutative diagram (cf. Theorem 1.3.2):

$$
\begin{array}{ccc}
B(U_{q}(\mathfrak{g})a_{\lambda}) & \xrightarrow{\Xi_{\lambda}} & B(\infty) \otimes T_{\lambda} \otimes B(-\infty) \\
\tilde{S}_{m,\lambda} & \sim & \downarrow S_{m,\infty} \otimes \tau_{m,\lambda} \otimes S_{m,-\infty} \\
B(U_{q}(\mathfrak{g})a_{m\lambda}) & \xleftarrow{\Xi_{m\lambda}^{-1}} & B(\infty) \otimes T_{m\lambda} \otimes B(-\infty),
\end{array}
$$

where $\tau_{m,\lambda} : T_{\lambda} \rightarrow T_{m\lambda}$ is defined by $\tau_{m,\lambda}(t_{\lambda}) := t_{m\lambda}$. We define $\tilde{S}_{m} : \tilde{U}_{q}(\mathfrak{g}) \hookrightarrow \tilde{U}_{q}(\mathfrak{g})$ as the direct sum of all the $\tilde{S}_{m,\lambda}$'s.
Proposition 2.1.5. The maps $\tilde{S}_{m,\lambda} : B(U_q(\mathfrak{g})a_\lambda) \hookrightarrow B(U_q(\mathfrak{g})a_{m\lambda})$ and $\tilde{S}_m : B(\tilde{U}_q(\mathfrak{g})) \hookrightarrow B(\tilde{U}_q(\mathfrak{g}))$ have properties similar to $S_{m,\infty}$ in Theorem 2.1.1. In addition, the map $\tilde{S}_m$ commutes with the $*$-operation on $B(\tilde{U}_q(\mathfrak{g}))$.

Proof. The first assertion immediately follows from Theorem 2.1.1, Remark 2.1.4, and the definition of the tensor product of crystals (see also [Kas5, Appendix B]). Let us prove the second assertion. We set $\tilde{S}_m^* := \ast \circ \tilde{S}_m \circ \ast$. It suffices to show the following:

Claim. Let $\lambda \in P$, and $b \in B(U_q(\mathfrak{g})a_\lambda)$. Then, we have that $\tilde{S}_m^*(b) \in B(U_q(\mathfrak{g})a_{m\lambda})$, and that $\Xi_{m\lambda}(\tilde{S}_m^*(b)) = \Xi_{m\lambda}(\tilde{S}_m(b))$.

Assume that $\Xi_{\lambda}(b) = b_1 \otimes t_\lambda \otimes b_2$ with $b_1 \in B(\infty)$ and $b_2 \in B(-\infty)$. Then we see by the definition of $\tilde{S}_m$ that

$$\Xi_{m\lambda}(\tilde{S}_m(b)) = (S_{m,\infty} \otimes \tau_{m,\lambda} \otimes S_{m,-\infty})(\Xi_{\lambda}(b)) = S_{m,\infty}(b_1) \otimes t_{m\lambda} \otimes S_{m,-\infty}(b_2).$$

On the other hand, we know from Theorem 1.3.3 that $b^* \in B(U_q(\mathfrak{g})a_{\lambda'})$ and $\Xi_{\lambda'}(b^*) = b_1^* \otimes t_{\lambda'} \otimes b_2^*$, where $\lambda' := -\lambda - \text{wt}(b_1) - \text{wt}(b_2)$. Hence we have

$$\Xi_{m\lambda}(\tilde{S}_m(b^*)) = (S_{m,\infty} \otimes \tau_{m,\lambda} \otimes S_{m,-\infty})(\Xi_{\lambda'}(b^*)) = S_{m,\infty}(b_1^*) \otimes t_{m\lambda} \otimes S_{m,-\infty}(b_2^*).$$

We deduce again from Theorem 1.3.3 that $\tilde{S}_m^*(b) = (\tilde{S}_m(b^*))^* \in B(U_q(\mathfrak{g})a_{m\lambda})$, and that

$$\Xi_{m\lambda}(\tilde{S}_m^*(b)) = S_{m,\infty}^*(b_1) \otimes t_{m\lambda} \otimes S_{m,-\infty}^*(b_2)$$

$$= S_{m,\infty}(b_1) \otimes t_{m\lambda} \otimes S_{m,-\infty}(b_2)$$

by Proposition 2.1.2 and Remark 2.1.4.

Thus, we obtain $\Xi_{m\lambda}(\tilde{S}_m^*(b)) = \Xi_{m\lambda}(\tilde{S}_m(b))$, as desired. \hfill \square

Theorem 2.1.6. Let $m \in \mathbb{Z}_{>0}$. There exists an injective map $S_{m,\lambda} : B(\lambda) \hookrightarrow B(m\lambda)$ such that $S_{m,\lambda}(u_\lambda) = u_{m\lambda}$ and such that for each $b \in B(\infty)$ and $i \in I$, we have

$$\text{wt}(S_{m,\lambda}(b)) = m \text{wt}(b), \quad \epsilon_i(S_{m,\lambda}(b)) = m \epsilon_i(b), \quad \varphi_i(S_{m,\lambda}(b)) = m \varphi_i(b), \quad (2.1.5)$$

$$S_{m,\lambda}(e_i b) = e_i^m S_{m,\lambda}(b), \quad S_{m,\lambda}(f_i b) = f_i^m S_{m,\lambda}(b). \quad (2.1.6)$$

Proof. Set $S_{m,\lambda} := \tilde{S}_m|_{B(\lambda)}$. Then, it is obvious from Proposition 2.1.5 that $S_{m,\lambda}(B(\lambda)) \subset B(U_q(\mathfrak{g})a_{m\lambda})$. Hence we need only show that $(S_{m,\lambda}(b))^*$ is extremal for every $b \in B(\lambda)$. We can easily check that the action of the Weyl group $W$ commutes with $S_{m,\lambda}$. So, it follows from Proposition 2.1.5 that

$$w((S_{m,\lambda}(b))^*) = wS_{m,\lambda}(b^*) = S_{m,\lambda}(wb^*)$$

for all $b \in B(\lambda)$ and $w \in W$. 

Assume that $\text{wt}(b^*) = \mu$. Then we see that $\text{wt}((S_{m,\lambda}(b))^*) = m\mu$. Suppose that $(w(m\mu))(\alpha_i^\vee) \geq 0$ and $e_i((S_{m,\lambda}(b))^*) \neq 0$. As in the proof of Lemma 2.1.3, we deduce that $e_i^m(w((S_{m,\lambda}(b))^*)) \neq 0$. Hence we have

$$S_{m,\lambda}(e_i(wb^*)) = e_i^mS_{m,\lambda}(wb^*) = e_i^m(wS_{m,\lambda}(b^*)) = e_i^m(w((S_{m,\lambda}(b))^*)) \neq 0.$$  

However, since $(w(\mu))(\alpha_i^\vee) \geq 0$ and $b^*$ is extremal, we have $e_i(wb^*) = 0$, and hence $S_{m,\lambda}(e_i(wb^*)) = 0$, which is a contradiction. Therefore, we obtain that $e_i(w((S_{m,\lambda}(b))^*)) = 0$. Similarly, we can prove that if $(w(m\mu))(\alpha_i^\vee) \leq 0$, then $f_i(w((S_{m,\lambda}(b))^*)) = 0$. This completes the proof of the theorem.

2.2 Embedding into tensor products. In this subsection, we assume that $\mathfrak{g}$ is an affine Lie algebra (for the notation, see §1.2). We know the following theorem from [B, §2], [N, §3] in the symmetric case, and from [BN, §4] in the nonsymmetric case.

**Theorem 2.2.1.** We have an embedding $G_{m,\varpi} : B_0(m\varpi_i) \hookrightarrow B(\varpi_i)^{\otimes m}$ of crystals that maps $u_{m\varpi_i}$ to $u_{\varpi_i}^{\otimes m}$.

**Remark 2.2.2.** In [BN], they take a vertex $0 \in I$ such that $a_0 = 1$ (see [BN, §2.1]). So, in the case of $A_2^{(2)}$, the choice of the vertex 0 is different from that in [Kas5, §5.2], and hence from ours. However, this does not cause a serious problem. For details, see the comment after [BN, Theorem 2.15].

Since $B(\varpi_i)$ is connected (see [Kas5, Theorem 5.5]), we see that $S_{m,\varpi_i}(B(\varpi_i)) \subset B_0(m\varpi_i)$. Hence we can define $\sigma_{m,\varpi_i} : B(\varpi_i) \hookrightarrow B(\varpi_i)^{\otimes m}$ by $\sigma_{m,\varpi_i} := G_{m,\varpi_i} \circ S_{m,\varpi_i}$ for each $m \in \mathbb{Z}_{>0}$. Remark that $\sigma_{m,\varpi_i}$ has the following properties:

$$\text{wt}(\sigma_{m,\varpi_i}(b)) = m \text{wt}(b), \quad \epsilon_j(\sigma_{m,\varpi_i}(b)) = m\epsilon_j(b), \quad \varphi_j(\sigma_{m,\varpi_i}(b)) = m\varphi_j(b), \quad (2.2.1)$$

$$\sigma_{m,\varpi_i}(u_{\varpi_i}) = u_{\varpi_i}^{\otimes m}, \quad \sigma_{m,\varpi_i}(e_j b) = e_j^m \sigma_{m,\varpi_i}(b), \quad \sigma_{m,\varpi_i}(f_j b) = f_j^m \sigma_{m,\varpi_i}(b). \quad (2.2.2)$$

**Lemma 2.2.3.** Let $m, n \in \mathbb{Z}_{>0}$. Then we have $\sigma_{m,\varpi_i} = \sigma_{n,\varpi_i}^{\otimes m} \circ \sigma_{m,\varpi_i}$.

**Proof.** Since $B(\varpi_i)$ is connected, every $b \in B(\varpi_i)$ is of the form

$$b = x_{j_1}x_{j_2} \cdots x_{j_k}u_{\varpi_i}$$

for some $j_1, j_2, \ldots, j_k \in I$, where $x_j$ is either $e_j$ or $f_j$. We will show by induction on $k$ that $\sigma_{m,\varpi_i}(b) = \sigma_{n,\varpi_i}^{\otimes m} \circ \sigma_{m,\varpi_i}(b)$ for all $b \in B(\varpi_i)$. If $k = 0$, then the assertion is obvious, since $b = u_{\varpi_i}$. Assume that $k \geq 1$. We set $b' := x_{j_2} \cdots x_{j_k}u_{\varpi_i}$, and $\sigma_{m,\varpi_i}(b') := u_1 \otimes u_2 \otimes \cdots \otimes u_m \in B(\varpi_i)^{\otimes m}$. Assume that

$$\sigma_{m,\varpi_i}(b) = x_{j_1}^m \sigma_{m,\varpi_i}(b') = x_{j_1}^k u_1 \otimes x_{j_1}^{k_2} u_2 \otimes \cdots \otimes x_{j_1}^{k_m} u_m$$
for some $k_1, k_2, \ldots, k_m \in \mathbb{Z}_{\geq 0}$. Then we have

$$\sigma_{n, \varpi_{j}}^{m} \circ \sigma_{m, \varpi_{j}}(b) = x_{j_{1}}^{nk_{1}} \sigma_{n, \varpi_{j}}(u_{1}) \otimes x_{j_{1}}^{nk_{2}} \sigma_{n, \varpi_{j}}(u_{2}) \otimes \cdots \otimes x_{j_{1}}^{nk_{m}} \sigma_{n, \varpi_{j}}(u_{m}).$$

Here we remark (cf. [Kasl, Lemma 1.3.6]) that for all $u_{1} \otimes u_{2} \otimes \cdots \otimes u_{m} \in B(\varpi_{i})^{\otimes m}$,

$$x_{j}(u_{1} \otimes u_{2} \otimes \cdots \otimes u_{m}) = u_{1} \otimes u_{2} \otimes \cdots \otimes x_{j}u_{l} \otimes \cdots \otimes u_{m}$$

if and only if

$$x_{j}^{n}(\sigma_{n, \varpi_{j}}(u_{1}) \otimes \sigma_{n, \varpi_{j}}(u_{2}) \otimes \cdots \otimes \sigma_{n, \varpi_{j}}(u_{m})) = \sigma_{n, \varpi_{j}}(u_{1}) \otimes \sigma_{n, \varpi_{j}}(u_{2}) \otimes \cdots \otimes x_{j}^{n} \sigma_{n, \varpi_{j}}(u_{l}) \otimes \cdots \otimes \sigma_{n, \varpi_{j}}(u_{m}).$$

So we obtain

$$\sigma_{n, \varpi_{j}}^{m} \circ \sigma_{m, \varpi_{j}}(b) = x_{j_{1}}^{mn} \sigma_{n, \varpi_{j}}(u_{1}) \otimes \sigma_{n, \varpi_{j}}(u_{2}) \otimes \cdots \otimes \sigma_{n, \varpi_{j}}(u_{m})$$

$$= x_{j_{1}}^{mn} (\sigma_{n, \varpi_{j}}^{m} \circ \sigma_{m, \varpi_{j}}(b')).$$

We see that $\sigma_{n, \varpi_{j}}^{m} \circ \sigma_{m, \varpi_{j}}(b') = \sigma_{mn, \varpi_{j}}(b')$ by the inductive assumption, and that $\sigma_{mn, \varpi_{j}}(b) = x_{j_{1}}^{mn} \sigma_{mn, \varpi_{j}}(b')$. Therefore, we obtain $\sigma_{n, \varpi_{j}}^{m} \circ \sigma_{m, \varpi_{j}}(b) = \sigma_{mn, \varpi_{j}}(b)$. \( \square \)

For each $w \in W$, we set $u_{w} := vu_{\varpi}$. By [Kas5, Proposition 5.8], we see that $u_{w}$ is well-defined. We can easily show the following lemma.

**Lemma 2.2.4.** For each $m \in \mathbb{Z}_{>0}$ and $w \in W$, we have $\sigma_{m, \varpi_{j}}(u_{w}) = (u_{w})^{\otimes m}$.  

**Proposition 2.2.5.** Let $b \in B(\varpi_{i})$. Assume that $b = x_{j_{1}}x_{j_{2}} \cdots x_{j_{k}}u_{\varpi}$, where $x_{j}$ is either $e_{j}$ or $f_{j}$, and set $b_{l} := x_{j_{l}}x_{j_{l+1}} \cdots x_{j_{k}}u_{\varpi}$ for $l = 1, 2, \ldots, k + 1$ (here $b_{k+1} := u_{\varpi}$). Then there exists sufficiently large $m \in \mathbb{Z}$ such that for every $l = 1, 2, \ldots, k + 1$,

$$\sigma_{m, \varpi_{j}}(b_{l}) = u_{w_{l,1}} \otimes u_{w_{l,2}} \otimes \cdots \otimes u_{w_{l,m}} \cdot (2.2.3)$$

for some $w_{l,1}, w_{l,2}, \ldots, w_{l,m} \in W$.

**Proof.** We show the assertion by induction on $k$. If $k = 0$, then the assertion is obvious. Assume that $k \geq 1$. By the inductive assumption, there exists $m \in \mathbb{Z}_{>0}$ such that $\sigma_{m, \varpi_{j}}(b_{l})$ is of the desired form for every $l = 2, \ldots, k + 1$. Assume that

$$\sigma_{m, \varpi_{j}}(b_{1}) = \sigma_{m, \varpi_{j}}(x_{j_{1}}b_{2}) = x_{j_{1}}^{m} \sigma_{m, \varpi_{j}}(b_{2})$$

$$= x_{j_{1}}^{c_{1}}u_{w_{2,1}} \otimes x_{j_{1}}^{c_{2}}u_{w_{2,2}} \otimes \cdots \otimes x_{j_{1}}^{c_{m}}u_{w_{2,m}}.$$
for some $c_1, c_2, \ldots, c_m \in \mathbb{Z}_{\geq 0}$. We can easily check by Lemma 2.2.4 and [Kas1, Lemma 1.3.6] that if $n_p \in \mathbb{Z}_{>0}$ satisfies the condition that $(w_{2,p} \omega_i)(\alpha_j^\vee) | n_p c_p$, then
\[
\sigma_{n_p, \omega_i}(\alpha_j^\vee u_{w_{2,p} \omega_i}) = u_{w_1 \omega_i} \otimes u_{w_2 \omega_i} \otimes \cdots \otimes u_{w_n \omega_i}
\] for some $w_1, w_2, \ldots, w_n \in W$. Therefore, by Lemma 2.2.4, we see that there exists $N \gg 0$ (for example, put $N = \prod_{p=1}^m n_p$) such that
\[
(\sigma_{N, \omega_i})^{\otimes m} \circ \sigma_{m, \omega_i}(b_1) = u_{w_{1,1} \omega_i} \otimes u_{w_{1,2} \omega_i} \otimes \cdots \otimes u_{w_{1,N_m} \omega_i}
\] for some $w_{1,1}, w_{1,2}, \ldots, w_{1,N_m} \in W$. Furthermore, we deduce from Lemma 2.2.4 that $(\sigma_{N, \omega_i})^{\otimes m} \circ \sigma_{m, \omega_i}(b_l)$ is of the desired form for every $l = 2, \ldots, k + 1$. It follows from Lemma 2.2.3 that $(\sigma_{N, \omega_i})^{\otimes m} \circ \sigma_{m, \omega_i} = \sigma_{N_m, \omega_i}$. Thus we have proved the proposition. \hfill $\Box$

3 Preliminary Results.

3.1 Some tools for path models. A path is, by definition, a piecewise linear, continuous map $\pi : [0, 1] \rightarrow \mathbb{Q} \otimes \mathbb{Z} P$ such that $\pi(0) = 0$. We regard two paths $\pi$ and $\pi'$ as equivalent if there exist piecewise linear, nondecreasing, surjective, continuous maps $\psi, \psi' : [0, 1] \rightarrow [0, 1]$ (reparametrization) such that $\pi \circ \psi = \pi' \circ \psi$. We denote by $\mathcal{P}$ the set of paths (modulo reparametrization) such that $\pi(1) \in P$, and by $e_i$ and $f_i$ the raising and lowering root operator (see [L2, §1]). By using root operators, we can endow $\mathcal{P}$ with a normal crystal structure (see [L2, §1 and §2]); we set $\text{wt}(\pi) := \pi(1)$, and define $\epsilon_i : \mathcal{P} \rightarrow \mathbb{Z}$ and $\varphi_i : \mathcal{P} \rightarrow \mathbb{Z}$ by
\[
\epsilon_i(\pi) := \max\{n \geq 0 \mid e_i^n \pi \neq 0\}, \quad \varphi_i(\pi) := \max\{n \geq 0 \mid f_i^n \pi \neq 0\}. \quad (3.1.1)
\]

Let $\lambda \in P$ be an (arbitrary) integral weight. We denote by $\mathcal{B}(\lambda) \subset \mathcal{P}$ the set of Lakshmibai–Seshadri paths of shape $\lambda$ (see [L2, §4]), and set $\pi_{\lambda}(t) := t \lambda \in \mathcal{B}(\lambda)$. Denote by $\mathcal{B}_0(\lambda)$ the connected component of $\mathcal{B}(\lambda)$ containing $\pi_{\lambda}$. We obtain the following lemma by [L2, Lemma 2.4].

Lemma 3.1.1. For $\pi \in \mathcal{P}$, we define $S_m : \mathcal{P} \hookrightarrow \mathcal{P}$ by $S_m(\pi) := m\pi$, where $(m\pi)(t) := m\pi(t)$ for $t \in [0, 1]$. Then we have $S_m(\mathcal{B}_0(\lambda)) = \mathcal{B}_0(m\lambda)$. In addition, the map $S_m$ has properties similar to $S_{m,\infty}$ in Theorem 2.1.1.

For paths $\pi_1, \pi_2 \in \mathcal{P}$, we define a concatenation $\pi_1 \ast \pi_2 \in \mathcal{P}$ as in [L2, §1]. Because $\pi_{\lambda} \ast \pi_{\lambda} \ast \cdots \ast \pi_{\lambda}$ ($m$-times) is just $\pi_{m\lambda}$ modulo reparametrization, we obtain the following lemma.
Lemma 3.1.2. We have a canonical embedding $G_{m,\lambda} : B_{0}(m\lambda) \hookrightarrow B(\lambda)^{\ast m}$ of crystals that maps $\pi_{m\lambda}$ to $\pi_{\lambda}^{\ast m}$, where $B(\lambda)^{\ast m} := \{ \pi_{1} \ast \pi_{2} \ast \cdots \ast \pi_{m} | \pi_{i} \in B(\lambda) \}$, and $\pi_{\lambda}^{\ast m} := \pi_{\lambda} \ast \pi_{\lambda} \ast \cdots \ast \pi_{\lambda} \in B(\lambda)^{\ast m}$.

By combining Lemmas 3.1.1 and 3.1.2, we get an embedding $\sigma_{m,\lambda} : B_{0}(\lambda) \hookrightarrow B(\lambda)^{\ast m}$ defined by $\sigma_{m,\lambda} := G_{m,\lambda} \circ S_{m}$. It can easily be seen that this map has properties similar to (2.2.1) and (2.2.2).

Since $B(\lambda)$ is a normal crystal, we can define an action of the Weyl group $W$ on $B(\lambda)$ (cf. (1.4.1); see also [L2, Theorem 8.1]). We set $\pi_{w\lambda} := w\pi_{\lambda}$ for $w \in W$. Note that $(w\pi_{\lambda})(t) = t(w\lambda)$ for each $w \in W$. Using [L2, Lemma 2.7], we can prove the following proposition in a way similar to Proposition 2.2.5.

Proposition 3.1.3. Let $\pi \in B_{0}(\lambda)$. Assume that $\pi = x_{j_{1}}x_{j_{2}}\cdots x_{j_{k}}\pi_{\lambda}$, where $x_{j}$ is either $e_{j}$ or $f_{j}$, and set $\pi_{l} := x_{j_{l}}x_{j_{l+1}}\cdots x_{j_{k}}\pi_{\lambda}$ for $l = 1, 2, \ldots, k + 1$ (here $\pi_{k+1} := \pi_{\lambda}$). Then, there exists sufficiently large $m \in \mathbb{Z}$ such that for every $l = 1, 2, \ldots, k + 1$,

$$
\sigma_{m,\lambda}(\pi_{l}) = \pi_{w_{l,1}\lambda} \ast \pi_{w_{l,2}\lambda} \ast \cdots \ast \pi_{w_{l,m}\lambda}
$$

(3.1.2)

for some $w_{l,1}, w_{l,2}, \ldots, w_{l,m} \in W$.

3.2 Preliminary lemmas. In this subsection, $g$ is assumed to be of affine type (for the notation, see §1.2). By using [L2, Lemma 2.1 c)], we can easily show the following lemma.

Lemma 3.2.1. Let $i \in I_{0}$. For each $w \in W$ and $j \in I$, we have $\text{wt}(\pi_{w\varpi_{i}}) = \text{wt}(u_{w\varpi_{i}})$, $\epsilon_{j}(\pi_{w\varpi_{i}}) = \epsilon_{j}(u_{w\varpi_{i}})$, and $\varphi_{j}(\pi_{w\varpi_{i}}) = \varphi_{j}(u_{w\varpi_{i}})$.

It follows from [Kas1, Lemma 1.3.6], [L2, Lemma 2.7], and Lemma 3.2.1 that

$$
x_{j}^{k}(u_{w_{1}\varpi_{i}} \otimes u_{w_{2}\varpi_{i}} \otimes \cdots \otimes u_{w_{m}\varpi_{i}}) = x_{j}^{k_{1}}u_{w_{1}\varpi_{i}} \otimes x_{j}^{k_{2}}u_{w_{2}\varpi_{i}} \otimes \cdots \otimes x_{j}^{k_{m}}u_{w_{m}\varpi_{i}}
$$

for some $k_{1}, k_{2}, \ldots, k_{m} \in \mathbb{Z}_{\geq 0}$ if and only if

$$
x_{j}^{k}(\pi_{w_{1}\varpi_{i}} \ast \pi_{w_{2}\varpi_{i}} \ast \cdots \ast \pi_{w_{m}\varpi_{i}}) = x_{j}^{k_{1}}\pi_{w_{1}\varpi_{i}} \ast x_{j}^{k_{2}}\pi_{w_{2}\varpi_{i}} \ast \cdots \ast x_{j}^{k_{m}}\pi_{w_{m}\varpi_{i}}
$$

for every $k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{>0}$ and $w_{1}, w_{2}, \ldots, w_{m} \in W$. So, we obtain the following lemma.

Lemma 3.2.2. (1) Let $b = x_{j_{1}}x_{j_{2}}\cdots x_{j_{k}}u_{\varpi_{i}} \in B(\varpi_{i})$. Take $m \in \mathbb{Z}_{>0}$ such that the assertion of Proposition 2.2.5 holds, and assume that $\sigma_{m,\varpi_{i}}(b) = u_{w_{1}\varpi_{i}} \otimes$
Then we have \( \pi := x_j x_{j_2} \cdots x_{j_k} \pi_{\omega_i} \neq 0 \), and \( \sigma_{m, \omega_i}(\pi) = \pi_{w_1 \omega_i} \star \pi_{w_2 \omega_i} \star \cdots \star \pi_{w_m \omega_i}. \)

(2) The converse of (1) holds. Namely, let \( \pi = x_j x_{j_2} \cdots x_{j_k} \pi_{\omega_i} \in B(\omega_i) \). Take \( m \in \mathbb{Z}_{>0} \) such that the assertion of Proposition 3.1.3 holds, and assume that \( \sigma_{m, \omega_i}(\pi) = \pi_{w_1 \omega_i} \star \pi_{w_2 \omega_i} \star \cdots \star \pi_{w_m \omega_i}. \) Then we have \( b := x_j x_{j_2} \cdots x_{j_k} \pi_{\omega_i} \neq 0 \), and \( \sigma_{m, \omega_i}(b) = u_{w_1 \omega_i} \otimes u_{w_2 \omega_i} \otimes \cdots \otimes u_{w_m \omega_i}. \)

4 Main Results.

4.1 Isomorphism theorem. From now on, we assume that \( g \) is an affine Lie algebra. We can carry out the proof of our isomorphism theorem, following the general line of that for [Kas5, Theorem 4.1].

**Theorem 4.1.1.** There exists a unique isomorphism \( \Phi_{\omega_i} : B(\omega_i) \sim B_0(\omega_i) \) of crystals such that \( \Phi_{\omega_i}(u_{\omega_i}) = \pi_{\omega_i}. \)

**Proof.** It suffices to prove that for \( j_1, j_2, \ldots, j_p \in I \) and \( k_1, k_2, \ldots, k_q \in I \),

(1) \( x_j x_{j_2} \cdots x_{j_p} u_{\omega_i} = x_{k_1} x_{k_2} \cdots x_{k_q} u_{\omega_i} \) \( \Leftrightarrow \) \( x_j x_{j_2} \cdots x_{j_p} \pi_{\omega_i} = x_{k_1} x_{k_2} \cdots x_{k_q} \pi_{\omega_i} \),

(2) \( x_j x_{j_2} \cdots x_{j_p} u_{\omega_i} = 0 \) \( \Leftrightarrow \) \( x_j x_{j_2} \cdots x_{j_p} \pi_{\omega_i} = 0. \)

Part (2) has already been proved in Lemma 3.2.2. Let us show the direction \( (\Rightarrow) \) of part (1). Take \( m \in \mathbb{Z}_{>0} \) such that the assertion of Proposition 2.2.5 holds for both \( b_1 := x_{j_1} x_{j_2} \cdots x_{j_p} u_{\omega_i} \) and \( b_2 := x_{k_1} x_{k_2} \cdots x_{k_q} u_{\omega_i}: \)

\[
\sigma_{m, \omega_i}(b_1) = u_{w_1 \omega_i} \otimes u_{w_2 \omega_i} \otimes \cdots \otimes u_{w_m \omega_i};
\]

\[
\sigma_{m, \omega_i}(b_2) = u_{w_1' \omega_i} \otimes u_{w_2' \omega_i} \otimes \cdots \otimes u_{w_m' \omega_i}.
\]

Since \( b_1 = b_2 \), we get \( u_{w_1 \omega_i} = u_{w_1' \omega_i} \), and hence \( w_l \omega_i = w_l' \omega_i \) for all \( l = 1, 2, \ldots, m \). By Lemma 3.2.2 (1), we see that

\[
\sigma_{m, \omega_i}(\pi_1) = \pi_{w_1 \omega_i} \star \pi_{w_2 \omega_i} \star \cdots \star \pi_{w_m \omega_i};
\]

\[
\sigma_{m, \omega_i}(\pi_2) = \pi_{w_1' \omega_i} \star \pi_{w_2' \omega_i} \star \cdots \star \pi_{w_m' \omega_i},
\]

where \( \pi_1 := x_{j_1} x_{j_2} \cdots x_{j_p} \pi_{\omega_i} \) and \( \pi_2 := x_{k_1} x_{k_2} \cdots x_{k_q} \pi_{\omega_i}. \) Since \( w_l \omega_i = w_l' \omega_i \) and \( \pi_{w_l \omega_i} = \pi_{w_l' \omega_i} \) for all \( w \in W \), we get \( \sigma_{m, \omega_i}(\pi_1) = \sigma_{m, \omega_i}(\pi_2). \) Since \( \sigma_{m, \omega_i} \) is injective, we conclude that \( \pi_1 = \pi_2. \)

We show the reverse direction \( (\Leftarrow) \) of part (1). Take \( m \in \mathbb{Z}_{>0} \) such that the assertion of Proposition 3.1.3 holds for both \( \pi_1 := x_{j_1} x_{j_2} \cdots x_{j_p} \pi_{\omega_i} \) and \( \pi_2 :=
\[ x_{k_1} x_{k_2} \cdots x_{k_q} \pi_{\omega_i} \]

\[ \sigma_{m, \omega_i}(\pi_1) = \pi_{w_1} \omega_i * \pi_{w_2} \omega_i * \cdots * \pi_{w_m} \omega_i ; \]

\[ \sigma_{m, \omega_i}(\pi_2) = \pi_{w'_1} \omega_i * \pi_{w'_2} \omega_i * \cdots * \pi_{w'_m} \omega_i. \]

Since \( \pi_1 = \pi_2 \), and hence \( \sigma_{m, \omega_i}(\pi_1) = \sigma_{m, \omega_i}(\pi_2) \) in \( P \), the two paths \( \pi_{w_1} \omega_i * \pi_{w_2} \omega_i * \cdots * \pi_{w_m} \omega_i \) and \( \pi_{w'_1} \omega_i * \pi_{w'_2} \omega_i * \cdots * \pi_{w'_m} \omega_i \) are identical modulo reparametrization.

Hence we can deduce that \( w_l \omega_i = w'_l \omega_i \) for all \( l = 1, 2, \ldots, m \) from the fact that if \( a \omega_j \in W \omega_i \) for some \( a \in \mathbb{Q}_{\geq 0} \) and \( j, i \in I_0 \), then \( i = j \) and \( a = 1 \). By Lemma 3.2.2 (2), we have

\[ \sigma_{m, \omega_i}(b_1) = u_{w_1} \omega_i \otimes u_{w_2} \omega_i \otimes \cdots \otimes u_{w_m} \omega_i, \]

\[ \sigma_{m, \omega_i}(b_2) = u_{w'_1} \omega_i \otimes u_{w'_2} \omega_i \otimes \cdots \otimes u_{w'_m} \omega_i. \]

Since \( w_l \omega_i = w'_l \omega_i \) for all \( l = 1, 2, \ldots, m \), it follows from [Kas5, Proposition 5.8 (i)] that \( u_{w_l} \omega_i = u_{w'_l} \omega_i \) for all \( l = 1, 2, \ldots, m \). Therefore we have \( \sigma_{m, \omega_i}(b_1) = \sigma_{m, \omega_i}(b_2) \). Since \( \sigma_{m, \omega_i} \) is injective, we conclude that \( b_1 = b_2 \). \( \square \)

Remark 4.1.2. In general, an isomorphism of crystals between \( B(\lambda) \) and \( B_0(\lambda) \) does not exist, even if \( B(\lambda) \) is connected. For example, let \( g \) be of type \( A_2^{(1)} \), and \( \lambda = \omega_1 + \omega_2 \) (we know from [Kas5, Proposition 5.4] that \( B(\lambda) \) is connected). If \( B(\lambda) \cong B_0(\lambda) \) as crystals, then we would have \( wu_\lambda = w'u_\lambda \) in \( B(\lambda) \) for every \( w, w' \in W \) with \( w_\lambda = w_\lambda \), but we have an example of \( w, w' \in W \) such that \( wu_\lambda \neq w'u_\lambda \) in \( B(\lambda) \) and \( w_\lambda = w_\lambda \) (see [Kas5, Remark 5.10]).

Remark 4.1.3. In [G], Greenstein proved that if \( g \) is of type \( A_2^{(1)} \), then the connected component \( B_0(m \omega_i + n \delta) \) is a path model for a certain bounded module \( L(\ell, m, n) \). He also showed a decomposition rule for tensor products, which seems to be closely related to Theorem 4.3.3 below.

4.2 Branching rule for \( V(\omega_i) \).

Lemma 4.2.1. For every \( \pi \in B(\omega_i) \), we have \( (\pi(1), \pi(1)) \leq (\omega_i, \omega_i) \).

Proof. Let \( \pi = (\nu_1, \nu_2, \ldots, \nu_s; a_0, a_1, \ldots, a_s) \) with \( \nu_j \in W \omega_i \) and \( a_j \in [0, 1] \) be a Lakshmibai-Seshadri path of shape \( \omega_i \) (cf. [L2, §4]). By the definition of a Lakshmibai-Seshadri path, we see that \( \pi(1) = \sum_{j=1}^{s}(a_j - a_{j-1})\nu_j \). Hence we have

\[ (\pi(1), \pi(1)) = \sum_{j=1}^{s}(a_j - a_{j-1})^2(\nu_j, \nu_j) + 2 \sum_{1 \leq k < l \leq s} (a_k - a_{k-1})(a_l - a_{l-1})(\nu_k, \nu_l) \]

\[ = \sum_{j=1}^{s}(a_j - a_{j-1})^2(\omega_i, \omega_i) + 2 \sum_{1 \leq k < l \leq s} (a_k - a_{k-1})(a_l - a_{l-1})(\omega_i, \omega_i). \]
for some $w_{kl} \in W$. By [Kac, Proposition 6.3], we deduce that $w_{kl} \omega_{i} = \omega_{i} - \beta_{kl} + n_{kl} \delta$ for some $\beta_{kl} \in \sum_{i \in I_{0}} \mathbb{Z}_{\geq 0} \alpha_{i}$ and $n_{kl} \in \mathbb{Z}$. Therefore, we have (note that $\omega_{i}$ is of level 0)

$$(\pi(1), \pi(1)) = \sum_{j=1}^{s} (a_{j} - a_{j-1})^{2} (\omega_{i}, \omega_{i}) + 2 \sum_{1 \leq k < l \leq s} (a_{k} - a_{k-1})(a_{l} - a_{l-1})(\omega_{i}, \omega_{i} - \beta_{kl} + n_{kl} \delta)$$

Since $(\omega_{i}, \beta_{kl}) \geq 0$ for all $1 \leq k < l \leq s$, we deduce that $(\pi(1), \pi(1)) \leq (\omega_{i}, \omega_{i})$, as desired. 

Let $S$ be a proper subset of $I$, i.e., $S \subset I$. Let $\mathfrak{g}_{S}$ be the Levi subalgebra of $\mathfrak{g}$ corresponding to $S$, and $U_{q}(\mathfrak{g}_{S}) \subset U_{q}(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}_{S}$. Note that a crystal for $U_{q}(\mathfrak{g})$ can be regarded as a crystal for $U_{q}(\mathfrak{g}_{S})$ by restriction.

**Theorem 4.2.2.** As crystals for $\mathfrak{g}_{S}$, $B(\omega_{i})$ and $B_{0}(\omega_{i})$ decompose as follows:

$$B(\omega_{i}) \cong \bigcup_{\pi \in B(\omega_{i}), \pi: \mathfrak{g}_{S}-dominant} B_{S}(\pi(1)), \quad B_{0}(\omega_{i}) \cong \bigcup_{\pi \in B_{0}(\omega_{i}), \pi: \mathfrak{g}_{S}-dominant} B_{S}(\pi(1)), \quad (4.2.1)$$

where $B_{S}(\lambda)$ is the set of Lakshmibai-Seshadri paths of shape $\lambda$ for $U_{q}(\mathfrak{g}_{S})$, and a path $\pi$ is said to be $\mathfrak{g}_{S}$-dominant if $(\pi(t))(\alpha_{i}^{\vee}) \geq 0$ for all $t \in [0, 1]$ and $i \in S$.

**Proof.** We will show only the first equality in (4.2.1), since the second one can be shown in the same way. As in [Kas1, §9.3], we deduce, using Lemma 4.2.1, that each connected component of $B(\omega_{i})$ (as a crystal for $U_{q}(\mathfrak{g}_{S})$) contains an extremal weight element $\pi'$ with respect to $W_{S} := \langle r_{j} \mid j \in S \rangle$. Because $\mathfrak{g}_{S}$ is a finite-dimensional reductive Lie algebra, there exists $w \in W_{S}$ such that $((w \pi')(1))(\alpha_{i}^{\vee}) \geq 0$ for all $j \in S$. Put $\pi := w \pi'$ for this $w \in W_{S}$. Since $\pi$ is also extremal, we have that $e_{j} \pi = 0$ for all $j \in S$. Because $\pi$ is a Lakshmibai-Seshadri path of shape
we deduce from [L2, Lemmas 2.2 b) and 4.5 d)] that \((\pi(t))(\alpha_j^\wedge) \geq 0\) for all \(t \in [0,1]\) and \(j \in S\), i.e., \(\pi\) is \(g_S\)-dominant. We see from [L2, Theorem 7.1] that the connected component containing \(\pi\) as a crystal for \(U_q(g_S)\) is isomorphic to \(B_S(\pi(1))\), thereby completing the proof of the theorem.

\textbf{Theorem 4.2.3.} (1) \textit{The extremal weight module} \(V(\varpi_i)\) \textit{of extremal weight} \(\varpi_i\) \textit{is completely reducible as a} \(U_q(g_S)\)-\textit{module}.

(2) \textit{The decomposition of} \(V(\varpi_i)\) \textit{as a} \(U_q(g_S)\)-\textit{module is given by}:

\[ V(\varpi_i) \cong \bigoplus_{\pi \in B_0(\varpi_i)} V_S(\pi(1)), \tag{4.2.2} \]

\textit{where} \(V_S(\lambda)\) \textit{is the integrable highest weight} \(U_q(g_S)\)-\textit{module of highest weight} \(\lambda\).

\textbf{Proof.} (1) First we prove that \(U := U_q(g_S)u\) is finite-dimensional for each weight vector \(u \in V(\varpi_i)\). To prove this, it suffices to show that the weight system \(\text{Wt}(U)\) of \(U\) is a finite set, since each weight space of \(V(\varpi_i)\) is finite-dimensional (see [Kas5, Proposition 5.16 (iii)]). Remark that if \(\mu, \nu \in P\) are weights of \(U\), then \(\mu, \nu \in \mathfrak{h}_0^\ast\), and \(\mu - \nu \in Q_S := \sum_{i \in S} \mathbb{Z}\alpha_i\). Hence the canonical map \(\text{cl} : \mathfrak{h}_0^\ast \to \mathfrak{h}_0^\ast/\mathbb{Q}\delta\) is injective on \(\text{Wt}(U)\), since \(k\delta \notin Q_S\) for any \(k \in \mathbb{Z}\setminus\{0\}\). Since \(\text{Wt}(U)\) is contained in the weight system \(\text{Wt}(V(\varpi_i))\) of \(V(\varpi_i)\), it follows from Theorem 4.1.1 and Lemma 4.2.1 that

\[ \text{cl}(\text{Wt}(U)) \subset \text{cl}(\text{Wt}(V(\varpi_i))) = \text{cl}(\{\pi(1) \mid \pi \in B_0(\varpi_i)\}) \quad \text{by Theorem 4.1.1} \]

\[ \subset \{\mu' \in \mathfrak{h}_0^\ast/\mathbb{Q}\delta \mid (\mu', \mu') \leq (\text{cl}(\varpi_i), \text{cl}(\varpi_i))\} \quad \text{by Lemma 4.2.1}. \]

Because the bilinear form \((\cdot, \cdot)\) on \(\mathfrak{h}_0^\ast/\mathbb{Q}\delta\) is positive-definite, the set \(\text{cl}(\text{Wt}(U))\) is discrete and contained in a compact set with respect to the usual metric topology on \(\mathbb{R}\otimes_{\mathbb{Q}} (\mathfrak{h}_0^\ast/\mathbb{Q}\delta)\) defined by \((\cdot, \cdot)\). Therefore, we see that \(\text{cl}(\text{Wt}(U))\) is a finite set, and hence so is \(\text{Wt}(U)\). Thus, we conclude that \(U = U_q(g_S)u\) is finite-dimensional.

Since \(q\) is assumed to generic, the finite-dimensional \(U_q(g_S)\)-module \(U_q(g_S)u\) is completely reducible for each weight vector \(u \in V(\varpi_i)\). Because \(V(\varpi_i)\) is a sum of all such modules \(U_q(g_S)u\), we deduce that \(V(\varpi_i)\) is also completely reducible.

(2) Because each weight space of \(V(\varpi_i)\) is finite-dimensional, we can define the formal character \(\text{ch}V(\varpi_i)\) of \(V(\varpi_i)\). By Theorem 4.2.2, we have

\[ \text{ch}V(\varpi_i) = \sum_{\pi \in B_0(\varpi_i)} \text{ch}V_S(\pi(1)). \]
Therefore, in order to prove part (2), we need only show that this is the unique way of writing $\text{ch} \, V(\varpi_i)$ as a sum of the characters of integrable highest weight $U_q(\mathfrak{g}_S)$-modules. Assume that

$$\text{ch} \, V(\varpi_i) = \sum_{\lambda \in P} c_{\lambda} \text{ch} \, V_S(\lambda) \quad \text{and} \quad \text{ch} \, V(\varpi_i) = \sum_{\lambda \in P} c'_{\lambda} \text{ch} \, V_S(\lambda)$$

with $c_{\lambda}, c'_{\lambda} \in \mathbb{Z}$ for $\lambda \in P$. Then we have $\sum_{\lambda \in P} (c_{\lambda} - c'_{\lambda}) \text{ch} \, V_S(\lambda) = 0$. Suppose that there exists $\lambda \in P$ such that $c_{\lambda} - c'_{\lambda} \neq 0$, and set $X := \{ \lambda \in P \mid c_{\lambda} - c'_{\lambda} \neq 0 \} (\neq \emptyset)$. Note that $X$ is contained in the weight system $\text{Wt}(V(\varpi_i))$ of $V(\varpi_i)$. As in the proof of part (1), we deduce that

$$\text{cl} \, (\text{Wt}(V(\varpi_i))) \subset \{ \mu' \in \mathfrak{h}_0^*/Q_\delta \mid (\mu', \mu') \leq (\text{cl}(\varpi_i), \text{cl}(\varpi_i)) \},$$

and hence $\text{Wt}(V(\varpi_i))$ modulo $\mathbb{Z}\delta$ is a finite set.

Now, we define a partial order $\geq_s$ on $P$ as follows:

$$\mu \geq_s \nu \quad \text{for} \quad \mu, \nu \in P \iff \mu - \nu \in (Q_S)_+ := \sum_{i \in S} \mathbb{Z}_{\geq 0} \alpha_i.$$

Let us show that the set $X$ has a maximal element with respect to this order $\geq_s$. Let $\mu \in X$. Then $\text{Wt}(V(\varpi_i)) \cap (\mu + Q_S)$ is a finite set. Indeed, if this is not a finite set, then there exist elements $\nu, \nu'$ of it such that $\nu - \nu' = k\delta$ with $k \in \mathbb{Z} \setminus \{0\}$, since $\text{Wt}(V(\varpi_i))$ modulo $\mathbb{Z}\delta$ is a finite set. However, since $\nu - \nu' \in Q_S$ and $k\delta \notin Q_S$ for any $k \in \mathbb{Z} \setminus \{0\}$, this is a contradiction. Therefore, we see that $X \cap (\mu + (Q_S)_+)$ is also a finite set, and hence that $X$ has a maximal element of the form $\mu + \beta$ for some $\beta \in (Q_S)_+$.

Let $\nu \in X$ be a maximal element with respect to this order $\geq_s$. We can easily see that the coefficient of $e(\nu)$ in $\sum_{\lambda \in P} (c_{\lambda} - c'_{\lambda}) \text{ch} \, V_S(\lambda)$ is equal to $c_\nu - c'_\nu$. Since $\nu \in X$, we have $c_\nu - c'_\nu \neq 0$, which contradicts $\sum_{\lambda} (c_{\lambda} - c'_{\lambda}) \text{ch} \, V_S(\lambda) = 0$. This completes the proof of the theorem.

4.3 Decomposition rule for tensor products. In this subsection, we assume that $\varpi_i$ is minuscule, i.e., $\varpi_i(\alpha^\vee) \in \{ \pm 1, 0 \}$ for every dual real root $\alpha^\vee$ of $g$.

Remark 4.3.1. The following is the list of minuscule weights (cf. [H, p. 174]). We use the numbering of vertices of the Dynkin diagrams in [Kac, Ch. 4]:
Remark 4.3.2. If $\varpi_i$ is minuscule, then, for any $\mu, \nu \in W\varpi_i$ and rational number $0 < \alpha < 1$, there does not exist an $a$-chain for $(\mu, \nu)$. Hence it follows from the definition of Lakshmi-Seshadri paths that $\mathcal{B}(\varpi_i) = \{\pi_{w\varpi_i} | w \in W\}$. Since $w\pi_{\varpi_i} = \pi_{w\varpi_i}$, we see that $\mathcal{B}(\varpi_i)$ is connected, and hence $\mathcal{B}(\varpi_i) = \mathcal{B}_0(\varpi_i)$.

Theorem 4.3.3. Let $\lambda$ be a dominant integral weight which is not a multiple of the null root $\delta$ of $\mathfrak{g}$. Then, the concatenation $\mathcal{B}(\lambda) * \mathcal{B}(\varpi_i)$ decomposes as follows:

$$
\mathcal{B}(\lambda) * \mathcal{B}(\varpi_i) \cong \bigcup_{\pi \in \mathcal{B}(\varpi_i)} \mathcal{B}(\lambda + \pi(1)),
$$

where $\pi \in \mathcal{B}(\varpi_i)$ is said to be $\lambda$-dominant if $(\lambda + \pi(t))(\alpha_i^\vee) \geq 0$ for all $t \in [0, 1]$ and $i \in I$.

Proof. We will prove that each connected component contains a (unique) path of the form $\pi \lambda \ast \pi$ for a $\lambda$-dominant path $\pi \in \mathcal{B}(\varpi_i)$. Then the assertion of the theorem follows from [L2, Theorem 7.1].

Let $\pi_1 \ast \pi_2 \in \mathcal{B}(\lambda) * \mathcal{B}(\varpi_i)$. It can easily be seen that $e_{i_1} e_{i_2} \cdots e_{i_k} (\pi_1 \ast \pi_2) = \pi_\lambda \ast \pi'_2$ for some $i_1, i_2, \ldots, i_k \in I$, where $\pi'_2 \in \mathcal{B}(\varpi_i)$ (cf. [G, §5.6]). Set $S := \{i \in I \mid \lambda(\alpha_i^\vee) = 0\}$ (note that $S \subseteq I$, since $\lambda$ is not a multiple of $\delta$), and let $\mathcal{B}$ be the set of paths of the form $e_{j_1} e_{j_2} \cdots e_{j_l} (\pi_\lambda \ast \pi'_j)$ for $j_1, j_2, \ldots, j_l \in S$. Remark that if $e_{j_1} e_{j_2} \cdots e_{j_l} (\pi_\lambda \ast \pi'_j) \neq 0$, then $e_{j_1} e_{j_2} \cdots e_{j_l} (\pi_\lambda \ast \pi'_j) = \pi_\lambda \ast (e_{j_1} e_{j_2} \cdots e_{j_l} \pi'_j)$. As in the proof of part (2) of Theorem 4.2.3, we deduce that

$$
\{\pi(1) \mid \pi \in \mathcal{B}(\varpi_i)\} \cap (\pi'_2(1) + (Q_S)_+) = \text{Wt}(V(\varpi_i)) \cap (\pi'_2(1) + (Q_S)_+)
$$

is a finite set. Hence we have $\pi_\lambda \ast \pi''_2 \in \mathcal{B}$ for some $\pi''_2 \in \mathcal{B}(\varpi_i)$ such that $e_j (\pi_\lambda \ast \pi''_2) = 0$ for all $j \in S$. Because $\varpi_i$ is minuscule and $\pi''_2 = \pi_{w\varpi_i}$ for some $w \in W$ (cf. Remark 4.3.2), we see that $e_j (\pi_\lambda \ast \pi''_2) = 0$ for all $j \in I \setminus S$. Therefore, we conclude that $\pi''_2 \in \mathcal{B}(\varpi_i)$ is $\lambda$-dominant. Thus, we have completed the proof of the theorem.

\[\square\]
Remark 4.3.4. Unlike Theorems 4.2.2 and 4.2.3, this theorem does not necessarily imply the decomposition rule for tensor products of corresponding $U_q(\mathfrak{g})$-modules.

References


[G] J. Greenstein, Littelmann’s path crystal and combinatorics of certain integrable $\overline{\mathfrak{sl}}_{\ell+1}$ modules of level zero, math.QA/0206263.


