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成層回転乱流中の波動成分と渦成分

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Abstract

Rotating stratified turbulence is analyzed using the rapid distortion theory (RDT) in the Craya-Herring frame, so that the unsteady time development of the wave components and the vortex components becomes much clearer than the previous analysis which utilized the usual Eulerian frame. In this study we have explicitly calculated the energy partition among the wave, vortex and potential energy components which would be useful for clarifying the mechanisms controlled by the buoyancy and Coriolis forces. We have found, for example the equi-partition between the wave components of kinetic energy (E_W) and the potential energy (PE) in a long time as observed in previous DNS for non-rotating stratified turbulence. This holds irrespective of the initial energy partition $E_W(0)/PE(0)$ at least when the initial turbulence is isotropic.

1 RDT equations

We consider a homogeneous turbulent flow with vertical density stratification ($\mathrm{d}\overline{\rho}/\mathrm{d}x_3$) and system rotation around the vertical axis. The governing equations for an inviscid fluid in the rotating frame under Boussinesq approximations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + 2\boldsymbol{\Omega}\times\boldsymbol{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla}p - g\hat{x}_3\frac{\rho}{\rho_0}, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho + u_3\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}x_3} = \kappa\nabla^2\rho, \tag{2}$$

and

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{3}$$

where ρ is the density perturbation from $\overline{\rho}(x_3)$, $\boldsymbol{u}$ is the velocity fluctuations, $\boldsymbol{\Omega} = (0, 0, \Omega)$ denotes the angular velocity $\boldsymbol{\Omega}$ of the system rotation, g is the accerelation due to gravity, $\hat{x_3}$ is the unit vector in the vertical upward direction, ρ_0 is the representative density, and ν and κ are the viscosity and diffusion coefficient respectively.

We then substitute the Fourier decomposition of velocity and density perturbations given by

$$u_i = \sum_{\boldsymbol{k}} \hat{u}_i(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \quad (i = 1, 2, 3), \tag{4}$$

and

$$\frac{g}{\rho_0}\rho = \sum_{\boldsymbol{k}} \hat{\rho}(\boldsymbol{k}, t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{5}$$

Then, RDT equations for stratified rotating turbulence with stratification in the vertical (x_3) direction and rotation around the vertical axis are obtained as (Hanazaki 2002)

$$\frac{d\hat{u}_i}{dt} + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \epsilon_{j3l} f \hat{u}_l = \left(\frac{k_i k_3}{k^2} - \delta_{i3}\right) \hat{\rho}, \tag{6}$$

and

$$\frac{d\hat{\rho}}{dt} = N^2 \hat{u}_3, \tag{7}$$

where N is the Brunt-Väisälä frequency and f is the Coriolis parameter (twice the angular frequency of rotation). The wave number vector $\boldsymbol{k}(t) = (k_1, k_2, k_3)(= \boldsymbol{k}(0))$ does not change with time when there is no shear. We next rewrite the equations in Craya-Herring frame by rotating the usual Eulerian frame so that one of the new coordinate axes $\boldsymbol{e}^3(= \boldsymbol{k}/|\boldsymbol{k}|)$ agrees with the direction of the wave number vector $\boldsymbol{k}$. Due to the incompressibility condition, the velocity vector must be perpendicular to $\boldsymbol{e}^3$ and is rewritten as $\hat{\boldsymbol{u}}(t) = \hat{\phi}_1 \boldsymbol{e}^1 + \hat{\phi}_2 \boldsymbol{e}^2$ ($\hat{\phi}_3 = 0$). We use the spherical coordinates (k, θ, ϕ) defined by

$$k_1 = k \sin\theta \cos\phi,\ k_2 = k \sin\theta \sin\phi,\ k_3 = k\cos\theta, \tag{8}$$

where θ is the angle between the polar (x_3) axis and vector $\boldsymbol{k}$. Since the new coordinate, i.e. the Craya-Herring frame $(\boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3)$ is obtained by rotating the original coordinate (x_1, x_2, x_3) by $\pi/2 + \phi$ around the x_3 axis and then rotating by angle θ around the new x_1 (or $\boldsymbol{e}^1$) axis, the old components $\hat{u}_i$ and new components $\hat{\phi}_i$ have the relation

$$\begin{aligned}
\hat{u}_1 &= -\hat{\phi}_1 \sin\phi - \hat{\phi}_2 \cos\theta \cos\phi, \\
\hat{u}_2 &= \hat{\phi}_1 \cos\phi - \hat{\phi}_2 \cos\theta \sin\phi, \\
\hat{u}_3 &= \hat{\phi}_2 \sin\theta.
\end{aligned} \tag{9}$$

or equivalently

$$\hat{\phi}_1 = \hat{u}_2 \cos\phi - \hat{u}_1 \sin\phi = (k_1 \hat{u}_2 - k_2 \hat{u}_1)/k_H = -i\hat{\omega}_3/k_H, \tag{10}$$

$$\hat{\phi}_2 = -(\hat{u}_1 \cos\phi + \hat{u}_2 \sin\phi)\cos\theta + \hat{u}_3 \sin\theta = \frac{k}{k_H}\hat{u}_3 = \hat{u}_3/\sin\theta, \tag{11}$$

where k_H is the horizontal wave number defined by $k_H = (k_1^2 + k_2^2)^{1/2}$ and the incompressibility condition $k_1\hat{u}_1 + k_2\hat{u}_2 + k_3\hat{u}_3 = 0$ has been used in (11). These expressions show that $\hat{\phi}_1$ is related to the vertical vorticity while $\hat{\phi}_2$ is related to the vertical velocity. Then $\hat{\phi}_1$ is called 'vortex mode' and $\hat{\phi}_2$ is called 'wave mode'.

Using (10) and (11), equations (6) and (7) can be rewritten in the Craya-Herring frame as

$$\frac{d\hat{\phi}_1}{dt} = f\hat{\phi}_2 \cos\theta,\ \frac{d\hat{\phi}_2}{dt} = -f\hat{\phi}_1 \cos\theta - \hat{\rho}\sin\theta,\ \frac{d\hat{\rho}}{dt} = N^2 \hat{\phi}_2 \sin\theta. \tag{12}$$

Solving these set of equations, we obtain

$$\hat{\phi}_1(t) = \frac{1}{a^2}\hat{\phi}_{10}(N^2\sin^2\theta + f^2\cos^2\theta\cos at) + \frac{1}{a}\hat{\phi}_{20}f\cos\theta\sin at + \frac{1}{a^2}\hat{\rho}_0 f\sin\theta\cos\theta(\cos at - 1), \tag{13}$$

$$\hat{\phi}_2(t) = \hat{\phi}_{20}\cos at - \frac{1}{a}(\hat{\phi}_{10}f\cos\theta + \hat{\rho}_0\sin\theta)\sin at, \tag{14}$$

$$\hat{\rho}(t) = \frac{1}{a^2}\hat{\phi}_{10}N^2 f\sin\theta\cos\theta(\cos at - 1) + \frac{1}{a}\hat{\phi}_{20}N^2\sin\theta\sin at + \frac{1}{a^2}\hat{\rho}_0(N^2\sin^2\theta\cos at + f^2\cos^2\theta), \tag{15}$$

where subscript 0 denotes the initial values and a is the frequency of the internal gravity wave defined by

$$a^2 = \frac{N^2(k_1^2 + k_2^2) + f^2k_3^2}{k_1^2 + k_2^2 + k_3^2} = N^2\sin^2\theta + f^2\cos^2\theta. \tag{16}$$

Since the density is a scalar quantity and not a vector, $\hat{\rho}(t)$ is independent of the frame of reference and agrees with the expression in the usual Eulerian frame given by Hanazaki (2002).

In this study we assume that the initial density flux is zero, i.e. $\Phi_{\rho i}^{CH}(\boldsymbol{k}, 0) = (1/2)\overline{\hat{\rho}_0\hat{\phi}_{i0}^* + \hat{\rho}_0^*\hat{\phi}_{i0}} = 0$ $(i = 1, 2)$ in agreement with the usual DNS and experiments. Then, the three-dimensional spectra in the Craya-Herring frame become, for example,

$$\begin{aligned}\Phi_{\rho\rho}(\boldsymbol{k}, t) &= \overline{\hat{\rho}\hat{\rho}^*} \\ &= \frac{1}{a^4}\Phi_{11}^{CH}(\boldsymbol{k}, 0)N^4 f^2\sin^2\theta\cos^2\theta(\cos at - 1)^2 \\ &+ \frac{2}{a^3}\Phi_{12}^{CH}(\boldsymbol{k}, 0)N^4 f\sin^2\theta\cos\theta\sin at \\ &+ \frac{1}{a^2}\Phi_{22}^{CH}(\boldsymbol{k}, 0)N^4\sin^2\theta\sin^2 at \\ &+ \frac{1}{a^4}\Phi_{\rho\rho}(\boldsymbol{k}, 0)(N^2\sin^2\theta\cos at + f^2\cos^2\theta)^2,\end{aligned} \tag{17}$$

Using equation (12), we can derive general and fundamental relations among the three-dimensional spectra. Some of them are

$$\frac{d}{dt}\Phi_{11}^{CH} = 2f\cos\theta\Phi_{12}^{CH}, \tag{18}$$

$$\frac{d}{dt}\Phi_{22}^{CH} = -2f\cos\theta\Phi_{12}^{CH} - 2\sin\theta\Phi_{\rho 2}^{CH}, \tag{19}$$

$$\frac{d}{dt}\Phi_{\rho\rho} = 2N^2\sin\theta\Phi_{\rho 2}^{CH}. \tag{20}$$

Since $d\Phi_{\rho\rho}/dt = 2N^2\Phi_{\rho 3}$ (Hanazaki & Hunt 1996), (20) shows that $\Phi_{\rho 3} = \sin\theta\Phi_{\rho 2}^{CH}$. Then,

$$\overline{\rho u_3} = \int \Phi_{\rho 3} d\boldsymbol{k} = \int_0^\infty dk k^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \Phi_{\rho 3} \tag{21}$$

is different from $\overline{\rho\phi_2} = \int \Phi_{\rho 2}^{CH} d\boldsymbol{k}$ although they agree in the long-time limit as will be shown later.

Integrations of the three-dimensional spectra such as (17) in the whole spectral space ($\int d\boldsymbol{k} = \int k^2 dk \sin\theta d\theta d\phi$) give

$$\frac{d}{dt}E_V = \int f\cos\theta\Phi_{12}^{CH} d\boldsymbol{k}, \tag{22}$$

$$\frac{d}{dt}E_W = -\int (f\cos\theta\Phi_{12}^{CH} + \sin\theta\Phi_{\rho 2}) d\boldsymbol{k} = -\int f\cos\theta\Phi_{12}^{CH} d\boldsymbol{k} - \overline{\rho u_3}, \tag{23}$$

and

$$\frac{d}{dt}PE = \int \sin\theta\Phi_{\rho 2}^{CH} d\boldsymbol{k} = \overline{\rho u_3}, \tag{24}$$

where

$$E_V(t) = \frac{1}{2}\int \Phi_{11}^{CH}(\boldsymbol{k}, t) d\boldsymbol{k}, \tag{25}$$

$$E_W(t) = \frac{1}{2}\int \Phi_{22}^{CH}(\boldsymbol{k}, t) d\boldsymbol{k}, \tag{26}$$

$$PE(t) = \frac{1}{2N^2}\int \Phi_{\rho\rho}(\boldsymbol{k}, t) d\boldsymbol{k}, \tag{27}$$

are the vortex-mode kinetic energy, wave-mode kinetic energy and potential energy respectively, and $KE = E_V + E_W$ is the total kinetic energy.

The kinetic energies of vortex mode and wave mode (E_V and E_W) exchange energy via the integral $\int f \cos\theta \Phi_{12}^{CH} d\boldsymbol{k}$, which vanishes when there is no rotation ($f = 0$). Then, without rotation (e.g., pure stratified flow), the spectrum $\Phi_{11}^{CH}(t)(= \Phi_{11}^{CH}(0))$ is constant (cf. (18)) and vortex mode energy $E_V(t)(= E_V(0))$ is constant. Even with the rotation ($f \neq 0$), the energy exchange vanishes if $\Phi_{12}^{CH}(t) = 0$ at all times. This occurs if $N = 0$ (pure rotation) and if the turbulence is initially isotropic. This recovers the previous results (e.g., Cambon & Jacquin 1989, Hanazaki 2002) that initial isotropy is conserved for pure rotating turbulence in the most general form.

It is important here to note that the integral decays rapidly with time (e.g., $\propto t^{-3/2}$) even if $f, N \neq 0$. This rapid decay occurs since the dispersion relation of the inertial gravity wave gives that the most contributions to the integral should have come from near $\theta = \pi/2$, while the integral contains $\cos\theta(= 0)$ (cf. §2), as verified mathematicaly by the method of stationary phase (Hanazaki & Hunt 1996, Hanazaki 2002).

On the other hand, E_W and PE exchange energy via $\int \sin\theta \Phi_{\rho 2} d\boldsymbol{k}$ (or $\overline{\rho u_3}$), which does not exist if $N = 0$ (e.g., pure rotation) but decays slowly ($\propto t^{-1/2}$) since $\theta = \pi/2$ gives $\sin\theta = 1$ which is contained in $d\boldsymbol{k}$ and the integrand does not vanish for the most contributing wave number components.

These characteristics of the interaction integrals show that the rotation contributes to the interaction between the vortex mode and the wave mode only for a short time, while the stratification contributes to the periodic energy exchange between the wave mode and the potential energy for a longer time. In ohter words, the system rotation does not contribute to the energy exchange for a long time at least in the linear dynamics.

This fact has been already found in the solutions for the usual Eulerian frame (Hanazaki 2002) for both the initially isotropic and the axisymmetric purely horizontal turbulence. Above result shows that the characteristics is quite general and independent of the initial conditions such as the initial isotropy or anisotropy of turbulence.

We should remember that the internal gravity wave is a transverse wave and the wave-number vector is perpendicular to the velocity of the fluid, i.e. $\boldsymbol{k}\cdot\boldsymbol{u} = 0$) the energy exchange/oscillation is maintained by the components of $\theta \sim \pi/2$, i.e. the vertical motion of the fluid which has angular frequency N, noting that the dispersion relation $a^2 = N^2 \sin^2\theta + f^2 \cos^2\theta$ gives $a = N$ at $\theta = \pi/2$.

Steady non-decaying components are also contributed mainly from $\theta \sim \pi/2$ since they also are multiplied by $\sin\theta$ in the integration in spherical coordinates. This result suggests that the energy containing cone is defined by $\theta \sim \pi/2$ which corresponds to a horizontal wave-numver vector with vertical fluid motion rather than $\theta \sim 0$ and π with horizontal motion??

The sum of (22) and (23) gives

$$\frac{d}{dt}KE = \frac{d}{dt}(E_V + E_W) = -\overline{\rho u_3} = -\frac{d}{dt}PE. \tag{28}$$

This shows the conservation of total energy $(E_V + E_W) + PE = KE + PE$ in the inviscid fluid. At the same time the energy exchange between the kinetic energy and the potential energy is maintained by the vertical density flux $\overline{\rho u_3}$, which decays slowly with time.

Sum of (23) and (24) gives

$$\frac{d}{dt}(E_W + PE) = -\int f\cos\theta\Phi_{12}^{CH}d\boldsymbol{k} = -\frac{d}{dt}E_V. \tag{29}$$

This again shows the conservation of total energy $E_V + E_W + PE$ in the inviscid fluid. At the same time the energy exchange between the vortex mode energy E_V and the wave-like energy components (PE and E_W) occur only with the effect of rotation f, which decays rapidly with time.

2 Initially isotropic turbulence

If we further assume that the turbulence is initially isotropic, the initial three-dimensional spectra are given by

$$\Phi_{ij}(\boldsymbol{k}, 0) = \frac{E(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \tag{30}$$

and

$$\Phi_{\rho\rho}(\boldsymbol{k}, 0) = \frac{S(k)}{4\pi k^2}2N^2, \tag{31}$$

where $E(k)$ and $S(k)$ are the initial radial kinetic and potential energy spectra and the initial kinetic energy KE_0 and potential energy PE_0 are given by

$$KE_0 = \int_0^\infty E(k)dk, \tag{32}$$

and

$$PE_0 = \frac{1}{2N^2}\int \Phi_{\rho\rho}d\boldsymbol{k} = \int_0^\infty S(k)dk. \tag{33}$$

Using the relation (6), the isotropic condition for the kinetic energy spectra can be rewritten in the Craya-Herring frame as

$$\Phi_{11}^{CH}(\boldsymbol{k}, 0) = \Phi_{22}^{CH}(\boldsymbol{k}, 0) = \frac{E(k)}{4\pi k^2}, \quad \Phi_{12}^{CH}(\boldsymbol{k}, 0) = 0. \tag{34}$$

Then we obtain the three-dimensional spectra in the Craya-Herring frame. The $\mathbf{e}_1$ (vortex mode) component spectrum becomes Integrating (8) in the whole spectral space, we obtain the kinetic energy of the vortex mode (E_V) as

$$\begin{aligned} E_V(t) &= \frac{1}{2}\overline{\phi_1^2} = \frac{1}{2}\int \Phi_{11}^{CH}(\boldsymbol{k},t)d\boldsymbol{k} \\ &= \frac{1}{2}KE_0 - \frac{1}{4}(KE_0 - 2PE_0) \\ &\times \int_0^\pi d\theta \frac{N^2 f^2}{a^4} \sin^3\theta \cos^2\theta (1-\cos at)^2 \end{aligned} \tag{35}$$

Similarly, the kinetic energy of the wave mode (E_W) can be calculated as

$$\begin{aligned} E_W(t) &= \frac{1}{2}\overline{\phi_2^2} = \frac{1}{2}\int \Phi_{22}^{CH}(\boldsymbol{k},t)d\boldsymbol{k} \\ &= \frac{1}{2}KE_0 - \frac{1}{8}(KE_0 - 2PE_0) \\ &\times \int_0^\pi d\theta \frac{N^2}{a^2} \sin^3\theta (1-\cos 2at), \end{aligned} \tag{36}$$

The steady components in the integrand of (37) shows that, when $KE_0 > 2PE_0$, E_W is not reduced by the components of $\theta \sim 0$ or π.

The potential energy becomes

$$\begin{aligned} PE(t) &= \frac{1}{2N^2}\overline{\rho^2} = \frac{1}{2N^2}\int \Phi_{\rho\rho}(\boldsymbol{k},t)d\boldsymbol{k} \\ &= PE_0 + \frac{1}{8}(KE_0 - 2PE_0) \\ &\times \int_0^\pi d\theta \left[\frac{N^2}{a^2}\sin^3\theta(1-\cos 2at)\right. \\ &+ \left.\frac{2N^2 f^2}{a^4}\sin^3\theta\cos^2\theta(1-\cos at)^2\right], \end{aligned} \tag{37}$$

which agrees with the results previously obtained in the usual Eulerian frame [3]. The density fluxes become

$$\overline{\rho\phi_1}(t) = \frac{1}{2}(KE_0 - 2PE_0)$$

$$
\begin{aligned}
&\times \int_0^\pi d\theta \frac{N^2 f}{a^4} \sin^2\theta\cos\theta(1-\cos at)(N^2\sin^2\theta\cos at + f^2\cos^2\theta) \\
&= 0, \qquad (38)
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\rho\phi_2}(t) &= \frac{1}{2}(KE_0 - 2PE_0) \\
&\times \int_0^\pi d\theta \frac{N^2}{a^3} \sin^2\theta \sin at(N^2\sin^2\theta\cos at + f^2\cos^2\theta). \qquad (39)
\end{aligned}
$$

Note that $\overline{\rho\phi_1} = 0$ holds identically because of the asymmetry of the integrand against $\theta = \pi/2$.

2.1 General case of $N \neq f$

If we separate the steady and unsteady components in the integrals (38)–(40), and integrate analytically only the steady components, we obtain

$$
\begin{aligned}
E_V &= \frac{1}{2}KE_0 + \frac{3}{8}\frac{f^2N^2}{(f^2-N^2)^2}(KE_0 - 2PE_0)\left(3 - \frac{f^2+2N^2}{f^2-N^2}I_A\right) \\
&+ \frac{1}{8}(KE_0 - 2PE_0) \\
&\times \int_0^\pi d\theta \frac{N^2f^2}{a^4}\sin^3\theta\cos^2\theta(4\cos at - \cos 2at), \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
E_W &= \frac{1}{2}KE_0 + \frac{1}{8}(KE_0 - 2PE_0) \\
&\times \left(\frac{2N^2}{f^2-N^2} - \frac{2N^2f^2}{(f^2-N^2)^2}I_A + \int_0^\pi d\theta \frac{N^2}{a^2}\sin^3\theta\cos 2at\right) \\
&= \frac{1}{2}KE_0 + \frac{1}{8}(KE_0 - 2PE_0) \\
&\times \left[\frac{2N^2}{f^2-N^2} - \frac{2N^2f^2}{(f^2-N^2)^2}I_A\right. \\
&+ \left.\int_0^\pi d\theta \left(\frac{N^4}{a^4}\sin^5\theta + \frac{N^2f^2}{a^4}\sin^3\theta\cos^2\theta\right)\cos 2at\right], \qquad (41)
\end{aligned}
$$

$$
PE(t) = PE_0 + \frac{1}{8}(KE_0 - 2PE_0)
$$

$$\times \left[-\frac{N^2(11f^2-2N^2)}{(f^2-N^2)^2}+\frac{N^2f^2(5f^2+4N^2)}{(f^2-N^2)^3}I_A \right.$$
$$\left. - \int_0^\pi d\theta \left(\frac{N^4}{a^4}\sin^5\theta\cos 2at+\frac{4N^2f^2}{a^4}\sin^3\theta\cos^2\theta\cos at\right)\right], \quad (42)$$

where

$$I_A = \int_0^1 \frac{dx}{x^2+N^2/(f^2-N^2)}$$
$$= \frac{(f^2-N^2)^{1/2}}{N}\tan^{-1}\frac{(f^2-N^2)^{1/2}}{N} \quad (f>N)$$

or

$$= \frac{(N^2-f^2)^{\frac{1}{2}}}{N}\log\frac{N-(N^2-f^2)^{1/2}}{f} \quad (f<N), \quad (43)$$

and the sign $\pm$ represents $+$ when $f>N$, and $-$ when $f<N$.

As is clear in (38)–(40), $\overline{\rho\phi_1}(=0)$ and $\overline{\rho\phi_2}$ do not have steady components. We note that there is an unsteady exchange of energy among E_V, E_W and PE. The echange between E_W and PE is due to stratification which exsts even when $f=0$, while the exchange $E_V \leftrightarrow E_W$ exists only when $f \neq 0$.

Note that if there is no rotation ($f=0$), E_W asymptotes to $E_W(t\to\infty)=(1/4)KE_0+(1/2)PE_0$, which is equivalent to the asymptotic value of the potential energy $PE(t\to\infty)=(1/4)KE_0+(1/2)PE_0$ [4]. This also agrees with the DNS for non-rotating stratified turbulence [7] which showed the equi-partition of energy between E_W and PE. It is also important to note that $E_W(t\to\infty)=PE(t\to\infty)$ holds irrespective of the initial energy partition between $KE_0(=E_V(0)+E_W(0))$ and PE_0, showing that this is a rather general feature of the stratified turbulence. In the previous studies Godeferd & Cambon(1994) ($PE_0=0$) and Métais & Herring (1989) ($PE_0=0.05KE_0 \ll KE_0$) argued $E_W(t\to\infty)=PE(t\to\infty)$ for small initial potential energy. The equi-partition is, however, violated by the system rotation and the parameter f/N alters the energy partition among E_V, E_W and PE (Hanazaki 2002).

2.2 Special case of $N = f$

In the special case of $N = f$ the integrations can be done exactly and the energy and the fluxes become

$$E_V = \frac{1}{5}(2KE_0 + PE_0) + (KE_0 - 2PE_0)\left(\frac{2}{15}\cos Nt - \frac{1}{30}\cos 2Nt\right), \quad (44)$$

$$E_W = \frac{1}{3}(KE_0 + PE_0) + \frac{1}{6}(KE_0 - 2PE_0)\cos 2Nt, \quad (45)$$

$$PE = \frac{4}{15}KE_0 + \frac{7}{15}PE_0 - \frac{2}{15}(KE_0 - 2PE_0)(\cos Nt + \cos 2Nt), \quad (46)$$

$$\overline{\rho\phi_1} = 0, \quad (47)$$

$$\overline{\rho\phi_2} = \frac{N\pi}{16}(KE_0 - 2PE_0)\left(\sin Nt + \frac{3}{2}\sin 2Nt\right), \quad (48)$$

which show non-decaying oscillation (Kaneda, 2000; Hanazaki, 2002). This is in contrast to the horizontal kinetic energy components $\overline{u_1^2}$ and $\overline{u_2^2}$ which had components proportional to $\cos Nt$ only and where clear distinction from the vertical components ($\propto \cos 2Nt$) could be made (Iida & Nagano 1999, Hanazaki 2002).

When $N = f$, $\overline{\rho\phi_2}$ does not agree with $\overline{\rho u_3}$ even in the long-time limit because there is no localization to $\theta \to \pi/2$ in this particular case.

3 Conclusions

Solutions of the RDT equations for the stratified rotating turbulence in the Craya-Herring frame have been obtained. The results for the non-rotating stratified turbulence showed the equi-partition of energy between E_W and PE in their final equilibrium state, as observed in the previous DNS. This is independent of the initial energy partition. It will be altered, however, with the system rotation, since the final steady values of E_V, E_W and PE will depend on the value of f/N.

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