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Mass Normalization of Collapses in the Theory of Self-Interacting Particles

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1 Introduction

This paper is concerned with the elliptic-parabolic system of cross-diffusion,

$$\left.\begin{array}{l} u_t = \nabla \cdot (\nabla u - u\nabla v) \\ 0 = \Delta v - av + u \end{array}\right\} \quad \text{in} \quad \Omega \times (0,T)$$
$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,T)$$
$$u|_{t=0} = u_0(x) \quad \text{in} \quad \Omega, \tag{1}$$

where $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $a > 0$ is a constant, and ν is the outer unit normal vector on $\partial\Omega$. It is proposed by Nagai [8] as a simplified form of the ones given by Jäger and Luckhaus [6], Nanjundiah [12], Keller and Segel [7], and Patlak [14] to describe the chemotactic feature of cellular slime molds. It is also a description of the non-equilibrium mean field of self-attractive particles subject to the second law of thermodynamics. Actually, this physical principle is realized by introducing the friction and fluctuations of particles. See Bavaud [1] and Wolansky [23], [24]. On the other hand, the mathematical study has a long history, and we refer to [21] for the background, known results, and standard arguments.

Actually, it follows from Yagi [25] and Biler [2] that the unique classical solution exists locally in time if the initial value is smooth, and that the solution becomes positive if the initial value is non-negative and not identically

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zero. Letting $T_{\max} > 0$ to be the supremum of the existence time of the solution, we say that the solution blows-up in finite time if $T_{\max} < +\infty$. Then, it is proven in Senba and Suzuki [16] that in the case of $T_{\max} < +\infty$ there exists a finite set $\mathcal{S} \subset \overline{\Omega}$ and a non-negative function $f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$ such that

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \qquad \text{in} \quad \mathcal{M}(\overline{\Omega}) \tag{2}$$

as $t \uparrow T_{\max}$ with

$$m(x_0) \geq m_*(x_0) \qquad (x_0 \in \mathcal{S}), \tag{3}$$

where $\mathcal{M}(\overline{\Omega})$ denotes the set of measures on $\overline{\Omega}$, $\rightharpoonup$ the $*$-weak convergence there, and

$$m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

It follows from $T_{\max} < +\infty$ that

$$\lim_{t \to T_{\max}} \|u(t)\|_\infty = +\infty$$

and $\mathcal{S}$ is actually the blowup set of u. That is, $x_0 \in \overline{\Omega}$ belongs to $\mathcal{S}$ if and only if there exist $x_k \to x_0$ and $t_k \uparrow T_{\max}$ such that $u(x_k, t_k) \to +\infty$. Because

$$\|u(t)\|_1 = \|u_0\|_1 \tag{4}$$

holds for $t \in [0, T_{\max})$, inequality (2) with (3) implies that

$$2 \cdot \sharp(\Omega \cap \mathcal{S}) + \sharp(\partial\Omega \cap \mathcal{S}) \leq \|u_0\|_1 / (4\pi). \tag{5}$$

We have, furthermore, that $\mathcal{S} \neq \emptyset$ if $T_{\max} < +\infty$, and therefore, $\|u_0\|_1 < 4\pi$ implies $T_{\max} = +\infty$. This fact on the existence of the solution globally in time was proven independently by Nagai, Senba, and Yoshida [11], Biler [2], and Gajewski and Zacharias [4], while relation (2) was conjectured by Nanjundiah [12]. It is referred to as the formation of *chemotactic collapses*, and each collapse

$$m(x_0)\delta_{x_0}(dx)$$

is regarded as a spore created by the cellular slime molds.

In 1996, Herrero and Velázquez [5] constructed a family of radially symmetric blowup solutions by the method of matched asymptotic expansion, where it holds that $m(x_0) = m_*(x_0)$ with $x_0 = 0 \in \Omega \cap \mathcal{S}$. Also, Nagai [9] and Senba and Suzuki [17] showed that if

$$\|u_0\|_1 > 4\pi \qquad \text{and} \qquad \int_\Omega |x - x_0|^2 u_0(x) dx \ll 1$$

hold for $x_0 \in \partial\Omega$, then it follows that $T_{\max} < +\infty$. This means that the mass of collapses made by those solutions can be close to 4π as we like. However, it may be always 4π, and under those considerations it was suspected that $m(x_0) = m_*(x_0)$ for any $x_0 \in \mathcal{S}$.

This problem, referred to as the *mass normalization* in the present paper, is related to the blowup rate, and we say that $x_0 \in \mathcal{S}$ is of type (I) if

$$\limsup_{t \to T} \sup_{|x - x_0| \leq Cr(t)} r(t)^2 u(x,t) < +\infty$$

holds for any $C > 0$, and that it is of type (II) for the other case that

$$\limsup_{t \to T} \sup_{|x - x_0| \leq Cr(t)} r(t)^2 u(x,t) = +\infty$$

holds with some $C > 0$, where $T = T_{\max} < +\infty$ and $r(t) = (T - t)^{1/2}$. It is expected that type (I) blowup point never arises. Here, we shall show the following.

Theorem 1 *If $x_0 \in \mathcal{S}$ is of type (II), then the mass normalization $m(x_0) = m_*(x_0)$ occurs.*

2 Preliminaries

We suppose that $T = T_{\max} < +\infty$, and take the standard backward self-similar transformation

$$z(y,s) = (T - t)u(x,t)$$

for $y = (x - x_0)/(T - t)^{1/2}$ and $s = -\log(T - t)$, where $x_0 \in \mathcal{S}$ denotes the blowup point in consideration. The zero extension of $z(y,s)$ is always taken to the region where it is not defined.

The following fact is proven similarly to [20] concerning Jäger - Luckhaus model, where

$$\{m_*(y_0)\delta_{y_0}(dy) \mid y_0 \in \mathcal{B}\}$$

and $F(y)dy = \mu_{0a.c.}(dy)$ are called the sub-collapses and the residual term, respectively. It is referred to as the *formation of sub-collapses*, and the proof is quite similar to the one given in [19] concerning the blowup in infinite time for the pre-scaled system. Here and henceforth, $\mu_s(dy)$ and $\mu_{a.c.}(dy)$ denote the singular and the absolutely continuous parts of $\mu(dy) \in \mathcal{M}(\mathbf{R}^2)$ relative to the Lebesgue measure dy, respectively.

Lemma 2 *Any $s_n \to +\infty$ admits $\{s_n'\} \subset \{s_n\}$ such that*

$$z(y, s_n')dy \quad \rightharpoonup \quad \mu_0(dy)$$

as $n \to \infty$ in $\mathcal{M}(\mathcal{R}^2)$, where supp $\mu_0(dy) \subset \overline{L}$ and

$$\mu_0(dy) = \sum_{y_0 \in \mathcal{B}} m_*(y_0)\delta_{y_0}(dy) + F(y)dy \tag{6}$$

with

$$m_*(y_0) = \begin{cases} 8\pi & (y_0 \in L) \\ 4\pi & (y_0 \in \partial L), \end{cases}$$

$0 \leq F \in L^1(L) \cap C(\overline{L} \setminus \mathcal{B})$, *and*

$$L = \begin{cases} \mathbf{R}^2 & (x_0 \in \Omega) \\ H & (x_0 \in \partial\Omega). \end{cases}$$

Here, H denotes the half space in $\mathbf{R}^2$ with ∂H containig the origin and parallel to the tangent line of $\partial\Omega$ at x_0, and the case $\mathcal{B} = \emptyset$ is admitted.

On the other hand, the following fact is referred to as the existence of the *parabolic envelop.*

Lemma 3 *We have*

$$m(x_0) = \mu_0(\overline{L}) = \sum_{y_0 \in \mathcal{B}} m_*(y_0) + \int_L F(y)dy. \tag{7}$$

Proof: First, we take

$$\varphi = \varphi_{x_0, R', R}$$

for $x_0 \in \mathcal{S}$ and $0 < R' < R$ satisfying $0 \le \varphi \le 1$, supp $\varphi \subset B(x_0, R)$, $\varphi = 1$ on $B(x_0, R')$, and $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$. Then, we set

$$M_R(t) = \int_\Omega \psi(x)u(x,t)dx$$

for $\psi = \varphi^4_{x_0,R,2R}$. Relation (2) implies that

$$\lim_{R\downarrow 0}\lim_{t\to T} M_R(t) = m(x_0).$$

On the other hand, in [16] it is proven that

$$\left|\frac{d}{dt}M_R(t)\right| \le C\left(\lambda^2 R^{-2} + \lambda R^{-1}\right)$$

with a constant $C > 0$ determined by Ω, and hence we obtain

$$|M_R(T) - M_R(t)| \le C\left(\lambda^2 R^{-2} + \lambda R^{-1}\right)(T-t).$$

Putting

$$R = br(t) = b(T-t)^{1/2}$$

to this inequality with a constant $b > 0$, we get that

$$\left|M_{br(t)}(T) - M_{br(t)}(t)\right| \le C\left(\lambda^2 b^{-2} + \lambda b^{-1}(T-t)^{1/2}\right),$$

and therefore, for

$$\overline{m}_b(x_0) = \limsup_{t\to T} M_{br(t)}(t) \qquad \text{and} \qquad \underline{m}_b(x_0) = \liminf_{t\to T} M_{br(t)}(t)$$

it holds that

$$m(x_0) - C\lambda^2 b^{-2} \le \underline{m}_b(x_0) \le \overline{m}_b(x_0) \le m(x_0) + C\lambda^2 b^{-2}$$

by $m(x_0) = \lim_{t\to T} M_{br(t)}(T)$. We note that this inequality is indicated as

$$\overline{m}_b(x_0) - C\lambda^2 b^{-2} \le m(x_0) \le \underline{m}_b(x_0) + C\lambda^2 b^{-2}. \qquad (8)$$

Here, we have

$$\int_{B(x_0,R)\cap\Omega} u(x,t)dx \le M_R(t) \le \int_{B(x_0,2R)\cap\Omega} u(x,t)dx$$

and hence it follows that

$$\int_{B(0,b)} z(y,s)dy \leq M_{br(t)}(t) \leq \int_{B(0,2b)} z(y,s)dy.$$

Thus we obtain

$$\mu_0\left(B(0,b-1)\right) \leq \underline{m_b}(x_0) \leq \overline{m_b}(x_0) \leq \mu_0\left(B(0,2b+1)\right),$$

and hence it follows that

$$\lim_{b\to+\infty} \underline{m_b}(x_0) = \lim_{b\to+\infty} \overline{m_b}(x_0) = \mu_0\left(\mathbf{R}^2\right) = \mu_0\left(\overline{L}\right).$$

Then, (7) is obtained by (8).

3 Movement of Sub-collapses

Similarly to the pre-scaled system treated in [18], Lemma 2 is refined in the following way. Namely, any $s_n \to +\infty$ admits $\{s_n'\} \subset \{s_n\}$ such that

$$z(y, s+s_n')dy \quad \rightharpoonup \quad \mu(dy,s)$$

in $C_*\left((-\infty,+\infty), \mathcal{M}(\mathbf{R}^2)\right)$, where supp $\mu(dy,s) \subset \overline{L}$, $m(x_0) = \mu\left(\overline{L}, s\right)$, and

$$\mu_s(dy,s) = \sum_{y_0\in\mathcal{B}_s} m_*(y_0)\delta_{y_0}(dy)$$

with

$$8\pi \cdot \sharp\left(L\cap\mathcal{B}_s\right) + 4\pi\cdot\sharp\left(\partial L\cap\mathcal{B}_s\right) + \mu_{a.c.}(L,s) = m(x_0).$$

This $\mu(dy,s)$ becomes a weak solution to

$$\begin{array}{lll} z_s = \nabla\cdot(\nabla z - z\nabla p) & \text{in} & L\times(-\infty,\infty) \\ \frac{\partial z}{\partial s} = 0 & \text{on} & \partial L\times(-\infty,\infty), \end{array} \tag{9}$$

where $p = w + \frac{|y|^2}{4}$ and

$$\nabla_y w(y,s) = \int_L \nabla_y G_0(y,y')z(y',s)dy$$

with

$$G_0(y,y') = \begin{cases} \frac{1}{2\pi}\log\frac{1}{|y-y'|} & (x_0 \in \Omega) \\ \frac{1}{2\pi}\log\frac{1}{|y-y'|} + \frac{1}{2\pi}\log\frac{1}{|y-y'^*|} & (x_0 \in \partial\Omega) \end{cases}$$

for the reflection y'^* of y' with respect to ∂H. The proof is similar to the one for the pre-scaled case ([18]), and the precise notion of weak solution is not necessary for later arguments. However, let us note that the zero extension of $\mu(dy,s)$ to $\mathbf{R}^2 \setminus L$ is always taken in the case of $x_0 \in \partial\Omega$, following the agreement for $z(y,s)$, and furthermore, that if $\eta \in C_0(\overline{L}) \cap C^2(\overline{L})$ satisfies $\frac{\partial\eta}{\partial\nu}\Big|_{\partial L} = 0$, then the mapping

$$s \in [0,\infty) \quad \mapsto \quad \int_{\overline{L}} \eta(y)\mu(dy,s)$$

is locally absolutely continuous, where $C_0(\overline{L})$ is the set of continuous functions on $\overline{L}$ taking the value zero at infinity.

If $F(y,s)dy = \mu_{a.c.}(dy,s)$, then $F(y,s) \geq 0$ is smooth in

$$\mathcal{D} = \bigcup_{s\in\mathbf{R}} \left(\overline{L} \setminus \mathcal{B}_s\right) \times \{s\}.$$

Actually, this is a consequence of the parabolic and elliptic regularity, and $F(y,s)$ satisfies there that

$$F_s = \nabla \cdot (\nabla F - F\nabla p) \tag{10}$$

with smooth p. As a consequence, if $G \subset \overline{L}$ is relatively open, if $\eta \in C^2(G) \cap C(\overline{G})$ satisfies $\eta|_{\partial G} = 0$ and $\frac{\partial\eta}{\partial\nu}\Big|_{\partial L} = 0$, and if $\operatorname{supp}\,\mu_s(dy,s) \cap \partial G = \emptyset$ holds for $s \in J$ with the time interval $J \neq \emptyset$, then

$$s \in J \quad \mapsto \quad \int_{\overline{L}} \eta(y)\mu(dy,s)$$

is locally absolutely continuous.

First, we study a special case of Theorem 1, making use of

$$\left[\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \mu(dy,s')\right]_{s'=s}^{s'=s+\Delta s} \geq \int_s^{s+\Delta s} ds' \cdot \left\{\int_{B_R} (-4 - |y|^2)\mu(dy,s') + \frac{4}{m_*(y_0)}\mu(B_R,s')^2\right\}, \tag{11}$$

where $R > 0$, $B_R = B(0,R)$, and $0 \leq s < s + \Delta s$.

In fact, in use of the standard backward self-similar transformation given in the previous section,

$$z(y,s) = (T-t)u(x,t) \qquad \text{and} \qquad w(y,s) = v(x,t)$$

with $y = x/(T-t)^{1/2}$ and $s = -\log(T-t)$, it follows that

$$\begin{aligned} &\left.\begin{aligned} z_s &= \nabla\cdot(\nabla z - z\nabla w - yz/2) \\ 0 &= \Delta w + z - ae^{-s}w \end{aligned}\right\} \quad \text{in} \quad \mathcal{O} \\ &\frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on} \qquad \Gamma \\ &z\big|_{s=-\log T} = z_0 \end{aligned} \tag{12}$$

for $z_0(y) = Tu_0(x)$,

$$\mathcal{O} = \bigcup_{s>-\log T} e^{s/2}\left(\Omega - \{x_0\}\right)\times\{s\},$$

and

$$\Gamma = \bigcup_{s>-\log T} e^{s/2}\left(\partial\Omega - \{x_0\}\right)\times\{s\}.$$

Here, we have

$$\begin{aligned} w(y,s) = v(x,t) &= \int_\Omega G(x,x')u(x',t)dx' \\ &= \int_{\mathcal{O}(s)} G\left(e^{-s/2}y + x_0, e^{-s/2}y' + x_0\right) z(y',s)dy', \end{aligned}$$

and therefore, system (12) is reduced to

$$\begin{aligned} z_s &= \nabla\cdot(\nabla z - z\nabla p) && \text{in} \quad \mathcal{O} \\ \tfrac{\partial z}{\partial \nu} &= 0 && \text{on} \quad \Gamma \end{aligned}$$

with $p = w + \frac{|y|^2}{4}$, where $G = G(y,y')$ denotes the Green's function for $-\Delta + a$ in Ω with $\frac{\partial}{\partial\nu}\cdot\Big|_{\partial\Omega} = 0$.

Letting

$$\varphi = \left(R^2 - |y|^2\right)_+,$$

we have

$$\varphi|_{\partial B_R} = 0, \qquad \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial B_R} < 0, \qquad \text{and} \qquad \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial H} = 0$$

with the last case valid only for $x_0 \in \partial\Omega$. Let us note that

$$B_R = B(0,R) = \left\{ y \in \mathbf{R}^2 \mid \varphi(y) > 0 \right\}.$$

Then, from (12) we can deduce that

$$\frac{d}{ds}\int_{\mathbf{R}^2} \varphi(y) z(y,s) dy \geq \int_{B_R} (\Delta\varphi + \frac{y}{2}\cdot\nabla\varphi) z(y,s) dy + \frac{1}{2}\int\int_{B_R\times B_R} \rho^s_\varphi(y,y') z(y,s) z(y',s) dy dy' \qquad (13)$$

with

$$\rho^s_\varphi(y,y') = \nabla\varphi(y)\cdot\nabla_y G^s(y,y') + \nabla\varphi(y')\cdot\nabla_{y'} G^s(y,y')$$

and $G^s(y,y') = G\left(e^{-s/2}y + x_0, e^{-s/2}y' + x_0\right)$.

Here, we have

$$\Delta\varphi + \frac{y}{2}\cdot\nabla\varphi = -4 - |y|^2$$

in B_R. Also we have for $\theta \in (0,1)$ that

$$G(y,y') = G_0(y,y') + K_1(y,y')$$

with $K_1 \in C^{1+\theta}_{loc}(\Omega\times\overline{\Omega}) \cap C^{1+\theta}_{loc}(\overline{\Omega}\times\Omega)$. In the case of $x_0 \in \Omega$, those relations imply the continuity of ρ^s_φ as well as the uniform convergence $\rho^s_\varphi \to \rho^0$ as $s \to +\infty$ on $\overline{B_R}\times\overline{B_R}$, where

$$\rho^0(y,y') = \nabla\varphi(y)\cdot\nabla_y G_0(y,y') + \nabla\varphi(y')\cdot\nabla_{y'} G_0(y.y') = \frac{1}{\pi}. \qquad (14)$$

In the case of $x_0 \in \partial\Omega$, on the other hand, we can make use of

$$G(y,y') = G_0(X(y), X(y')) + G_0(X(y), X(y')^*) + K_2(y,y')$$

with $K_2 \in C^{\theta,1+\theta}(\Omega\cup\gamma\times\overline{\Omega}) \cap C^{1+\theta,\theta}(\overline{\Omega}\times\Omega\cup\gamma)$, where $X : \overline{\hat{\Omega}} \to \overline{\mathbf{R}^2_+}$ is the conformal mapping satisfying $X(x_0) = 0$, γ is the connected compornent of $\partial\Omega$ containig x_0, and $\hat{\Omega}$ is the domain defined by $\partial\hat{\Omega} = \gamma$. Then, the above conclusion follows similarly, with (14) replaced by

$$\rho^0(y,y') = \frac{2}{\pi}.$$

Now, inequality (11) follows from (13) with $z(y,s)$ replaced by $z(y,s+s'_n)$ and sending $n \to \infty$. Here, we refer to [16], [22] for those facts on the Green's function.

In terms of $\nu(dy,s) = \mu(dy,s) - m_*(y_0)\delta_0(dy)$, inequality (11) reads;

$$\left[\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s')\right]_{s'=s}^{s'=s+\Delta s} \geq \int_s^{s+\Delta s} ds' \left\{\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s') + I_R(s')\right\} \tag{15}$$

with

$$I_R(s) = m_*(y_0)R^2 - (R^2+4)\mu(B_R,s) + \frac{4}{m_*(y_0)}\mu(B_R,s)^2.$$

Here, $0 < R \leq 2$ and

$$\mu(B_R,s) > m_*(y_0) \tag{16}$$

imply $I_R(s) > 0$. On the other hand, (16) follows from

$$\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s) > 0.$$

We now show that

$$0 < R \leq 2 \qquad \text{with} \qquad \int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,0) > 0 \tag{17}$$

gives a contradiction. In fact, applying (15) with $s = 0$, we see that

$$\left\{s \in [0,\infty) \mid \int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s') > 0 \text{ on } s' \in [0,s]\right\}$$

is right-closed from the above consideration. Its right-openess follows from $\mu(dy,s) \in C_*\left((-\infty,\infty),\mathcal{M}(\mathbf{R}^2)\right)$, so that (17) induces

$$\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s) > 0$$

for any $s \in [0,\infty)$. Simultaneously, it also holds that $I_R(s) > 0$ for $s \in [0,\infty)$, and again (15) assures the monotone increasing of the mapping

$$s \in [0,\infty) \quad \mapsto \quad \int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s).$$

Therefore, for $n = 1, 2, \cdots$ we have

$$\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy, n+1) \geq \int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy, n)$$
$$+ \int_n^{n+1} ds' \cdot \int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy, s')$$
$$\geq 2 \int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy, n),$$

which implies that

$$\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy, n) \geq 2^n \int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy, 0).$$

However, this is impossible by $\mu(\mathbf{R}^2, s) = m(x_0) < +\infty$.

We have shown that (17) does not occur. If $0 \in \text{supp } \mu_s(dy, 0)$, then $\nu(dy, 0) \geq 0$ holds and this means that

$$\nu(dy, 0) = 0 \qquad \text{on} \qquad B(0, 2),$$

or equivalently, supp $\mu_s(dy, 0) \cap B(0, 2) = \{0\}$ and

$$F(y, 0) = 0 \qquad \text{for a.e.} \qquad y \in B(0, 2).$$

Recall the notation that $F(y, s)dy = \mu_{a.c.}(dy, s)$. Because $F(y, s) \geq 0$ satisfies the parabolic equation (10) with smooth coeficient p in $\mathcal{D} = \cup_{s \in \mathbf{R}} \left(\overline{L} \setminus \mathcal{B}_s\right) \times \{s\}$, the strong maximum principle quarantees $F(y, s) = 0$ there. Hence $\mu_{0.a.c.}(dy) = 0$ follows.

To treat the general case, we note that if $s \in [0, \infty) \mapsto y_0(s) \in \mathbf{R}^2$ is locally absolutely continuous, then inequality (11) is replaced by

$$\left[\int_{\mathbf{R}^2} \left(R^2 - |y - y_0(s')|^2\right)_+ \mu(dy, s')\right]_{s'=s}^{s'=s+\Delta s} \geq \int_s^{s+\Delta s} ds' \cdot$$
$$\{\int_{B(y(s),R)} (2y_0'(s') \cdot (y - y_0(s')) - 4 - y \cdot (y - y_0(s')))\, \mu(dy, s')$$
$$\frac{4}{m_*(y_0)} \mu\left(B(y_0(s'), R), s'\right)^2\}.$$

In terms of $\mu'(dy, s)$ defined by $\mu'(A, s) = \mu(A + \{y_0(s)\}, s)$, it is represented as

$$\left[\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \mu'(dy, s')\right]_{s'=s}^{s'=s+\Delta s} \geq \int_s^{s+\Delta s} ds' \cdot$$

$$\{\int_{B_R}\left(-4-|y|^2+(2y_0'(s)-y_0(s))\cdot y\right)\mu'(dy,s')$$
$$+\frac{4}{m_*(y_0)}\mu'(B_R,s')^2\}.$$

If we take

$$y_0(s)=y_0e^{s/2},$$

then it is reduced to (11):

$$\left[\int_{\mathbf{R}^2}\left(R^2-|y|^2\right)_+\mu'(dy,s')\right]_{s'=s}^{s'=s+\Delta s}\geq\int_s^{s+\Delta s}ds'\cdot$$
$$\left\{\int_{B_R}\left(-4-|y|^2\right)\mu'(dy,s')+\frac{4}{m_*(y_0)}\mu'(B_R,s')^2\right\}.$$

We see that $0\in\operatorname{supp}\mu_s'(dy,0)$, or equivalently $y_0\in\operatorname{supp}\mu_s(dy,0)$, implies $\mu_{a.c.}(dy,0)=0$ and

$$\operatorname{supp}\mu_s(dy,0)\cap B\left(y_0,2\right)=\left\{y_0\right\}.$$

If $x_0\in\mathcal{S}$ is of type (II), then there is $s_n\to+\infty$ such that $z(y,s_n)dy\rightharpoonup\mu_0(dy)$ in $\mathcal{M}(\mathbf{R}^2)$ with $\operatorname{supp}\mu_{0s}(dy)\neq\emptyset$. We now take $\{s_n'\}\subset\{s_n\}$ such that $z(y,s+s_n')dy\rightharpoonup\mu(dy,s)$ in $C_*\left((-\infty,\infty),\mathcal{M}(\mathbf{R}^2)\right)$ with $\mu(dy,s)$ being the weak solution to (9). Because of $\mu_s(dy,0)=\mu_{0s}(dy)\neq0$, it follows from the above argument that $\mu_{a.c.}(dy,s)\equiv0$. We also have $\mu(dy,s)\in C_*\left((-\infty,\infty),\mathcal{M}(\overline{L})\right)$ and $\mu\left(\{y_0\},s\right)=m_*(y_0)$ for any $y_0\in\operatorname{supp}\mu_s(dy,s)$, and therefore, it holds that

$$\mu(dy,s)=\sum_{i=1}^n m_*^i\delta_{y_i(s)}(dy),$$

with $s\in(-\infty,\infty)\mapsto y_i(s)\in\overline{L}$ being continuous, $y_i(s)\in L$ or $y_i(s)\in\partial L$ exclusively in $s\in\mathbf{R}$, and

$$m_i^*=\begin{cases}8\pi & (y_i(s)\in L)\\ 4\pi & (y_i(s)\in\partial L).\end{cases}$$

Then, again the above argument guarantees that

$$|y_i(s)-y_j(s)|\geq2\qquad(i\neq j,\ s\in\mathbf{R}).\tag{18}$$

We also have

$$m(x_0) = \sum_{i=1}^{n} m_*^i.$$

Now, we take $i = 1, \cdots, n$, $R \in (0,2)$, and the interval

$$J_i = \Big\{ s \in [0,\infty) \mid \text{supp } \mu_s(dy, s') \cap \overline{B(y_i(0)e^{s'/2}, R)} = \{y_i(s')\} \\ \text{for any } s' \in [0,s] \Big\},$$

which is a right neighbourhood of 0. Then, we repeat the same argument for $\nu(dy,s) = \mu'(dy,s) - m_*^i \delta_0(dy)$ with $\mu'(A,s) = \mu\left(A + \left\{y_i(0)e^{s/2}\right\}, s\right)$. This time, we have $I_R'(s) = 0$ for $s \in J_i$, where

$$I_R'(s) = m_*^i R^2 - (R^2+4)\mu'(B_R, s) + \frac{4}{m_*^i}\mu'(B_R,s)^2.$$

Furthermore,

$$s \in J_i \quad \mapsto \quad \int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s)$$

is locally absolutely continuous, and it holds by (15) that

$$\frac{d}{ds}\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s) \geq \int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s)$$

for a.e. $s \in J_i$. Therefore, because of

$$\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,0) = 0$$

we obtain

$$\int_{\mathbf{R}^2} \left(R^2 - |y|^2\right)_+ \nu(dy,s) \geq 0,$$

or equivalently

$$R^2 - \left|y_i(s) - y_i(0)e^{s/2}\right|^2 \geq R^2,$$

and hence $y_i(s) = y_i(0)e^{s/2}$ follows for $s \in J_i$.

This relation holds for each $i = 1, \cdots, n$, so that

$$d_i(s) = \min_{j \neq i} |y_i(s) - y_j(s)|$$

is increasing in s. We have $J_i = [0,\infty)$ and the relation $y_i(s) = y_i(0)e^{s/2}$ continues to hold for every $s \in [0,\infty)$. Now, we translate the time variable as $s \mapsto s-s_0$, repeat the same argument, and see that $y_i(s-s_0) = y_i(-s_0)e^{s/2}$ holds for any $s_0 \geq 0$. This implies $y_i(-s)e^s = y_i(0)$ for $s \geq 0$, so that

$$y_i(s) = y_i(0)e^{s/2} \qquad (s \in \mathbf{R})$$

holds. Consequently,

$$\lim_{s \to -\infty} y_i(s) = 0$$

follows for $i = 1, \cdots, n$. However, this contradicts to (18) in the case of $n \geq 2$. We get $n = 1$, $m(x_0) = m_*(x_0)$, and

$$\mu(dy, s) = m_*(x_0)\delta_{y_0 e^{s/2}}(dy) \qquad (s \in \mathbf{R}),$$

and the proof is complete.

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