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ON JOINT SPECTRA OF NON-COMMUTING HYPONORMAL OPERATORS[1]

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Let H be a complex Hilbert space and let $\mathcal{B}(H)$ denote the Banach algebra of all (bounded linear) operators on H.

For n-tuple $T = (T_1, \dots, T_n)$ of operators on H a *spectral set* $\gamma(T)$ is defined as follows:

$$\gamma(T) = \Big\{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n\colon \sum_{j=1}^{n}(T_j - \lambda_j)^2 \text{ is not invertible in } \mathcal{B}(H)\Big\}.$$

(Here we write as usual $T_j - \lambda_j$ instead of $T_j - \lambda_j \operatorname{id}_H$.) This set was introduced by McIntosh and Pryde ([1, 2]) and has proved useful not only in the spectral theory of self-adjoint operators but also in comparing various types of joint spectra of commuting families of operators (see [3]). One advantage of the set $\gamma(T)$ over other joint spectra is that it can be easily computed. In [4] it was shown that this set is also useful in the multiparameter spectral theory of normal operators.

We recall some necessary definitions. An operator $T \in \mathcal{B}(H)$ is *hyponormal* (*cohyponormal*) if $\|T^*x\| \leq \|Tx\|$ ($\|Tx\| \leq \|T^*x\|$ respectively) for all $x \in H$. Clearly if an operator T is hyponormal, then T^* is cohyponormal. Moreover an operator T is normal if it is both hypo- and cohyponormal.

Let $T = (T_1, \dots, T_n)$ be an n-tuple of operators. A point $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is not in the *left* (*joint*) *spectrum* of T if there exist operators $U_1, \dots, U_n \in \mathcal{B}(H)$ such that $\sum_{j=1}^{n} U_j(T_j - \lambda_j) = \operatorname{id}_H$. The left spectrum of T will be denoted by $\sigma_l(T)$. The *right spectrum*, $\sigma_r(T)$, is defined analogously. The *Harte spectrum* of T (in $\mathcal{B}(H)$), denoted by $\sigma_H(T)$, is the union of the left and right joint spectra, i.e.

$$\sigma_H(T) = \sigma_l(T) \cup \sigma_r(T).$$

All these spectra are compact (possibly empty) subsets of $\mathbb{C}^n$. Notice that for a single operator T the Harte spectrum $\sigma_H(T)$ coincides with the usual spectrum $\sigma(T)$. It is well-known that

$$\sigma_l(T) = \Big\{\lambda \in \mathbb{C}^n\colon \inf_{\|x\|=1} \sum_{j=1}^{n} \|(T_j - \lambda_j)x\| = 0\Big\}$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX (the *approximate point spectrum*) and

$$\sigma_r(T) = \left\{\lambda \in \mathbb{C}^n \colon \sum_{j=1}^{n} \big((T_j - \lambda_j)(H)\big) \neq H\right\}$$

(the *defect spectrum*). Let us introduce the following notation. For a single operator T symbols $\operatorname{Re} T$ and $\operatorname{Im} T$ will denote as usual its real and imaginary part. Hence $T = \operatorname{Re} T + i \operatorname{Im} T$. If $T = (T_1, \dots, T_n)$ is an n-tuple of operators, then $\operatorname{Re} T = (\operatorname{Re} T_1, \dots, \operatorname{Re} T_n)$, $\operatorname{Im} T = (\operatorname{Im} T_1, \dots, \operatorname{Im} T_n)$, and $\Pi(T) = (\operatorname{Re} T, \operatorname{Im} T)$. Letter p will denote the polynomial map $p(z_1, \dots, z_{2n}) = (z_1 + iz_{n+1}, \dots, z_n + iz_{2n})$.

We present a generalisation of one of the results proved in [4] to n-tuples of (not necessarily commuting) hyponormal operators. The result is as follows:

Theorem. *If $T = (T_1, \dots, T_n)$ is an arbitrary n-tuple of hyponormal (cohyponormal) operators, then*

$$\sigma_l(T) = p\big(\gamma(\Pi(T))\big)$$

(*and respectively*

$$\sigma_r(T) = p\big(\gamma(\Pi(T))\big) \text{).}$$

It is easy to see that one cannot replace in the theorem the left spectrum (or the right spectrum) by the Harte spectrum if the operators T_j are not normal.

References

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