On generalized numerical range of the Aluthge transformation

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This report is based on the following paper:

ABSTRACT
In this report the authors show that the Aluthge transformation $\tilde{T}$ of a matrix $T$ and a polynomial $f$ satisfy the inclusion relation $W_c(f(\tilde{T})) \subset W_c(f(T))$ for the generalized numerical range if $C$ is a Hermitian matrix or a rank-one matrix.

1. THE ALUTHGE TRANSFORMATION

In the development of operator theory, Aluthge [1] introduced a transformation $\tilde{T}$ for a bounded linear operator $T$ on a complex Hilbert space $H$ with the help of the polar decomposition $T = V|T|$ as follows:

Definition 1 (Aluthge transformation [1]). Let $T = V|T|$ be the polar decomposition of a bounded linear operator $T$. Then the Aluthge transformation $\tilde{T}$ of $T$ is defined by

$$\tilde{T} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}.$$ 

We remark that $\tilde{T}$ is defined by using a partial isometry $V$ and $|T|$ with $T = V|T|$ and $N(V) = N(|T|)$. But in fact, $\tilde{T}$ does not depend on the choice of $V$ (see [19]), for example, if $T = U|T|$ is a matrix with unitary $U$, then $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$.

As properties of $\tilde{T}$, the following assertions are well known:
\( \sigma(T) = \sigma(\tilde{T}) \), where \( \sigma(T) \) means the spectrum of an operator \( T \).

(ii) \( \|T\| \geq \|\tilde{T}\| \).

(i) has been shown in [9], and we can obtain (ii) easily as follows:

\[
\|\tilde{T}\| \leq \| |T|^{1/2} \| \cdot \|V\| \cdot \| |T|^{1/2} \| \leq \|T\|.
\]

Recently, many authors discuss the nth iterated Aluthge transformation which is denoted by \( \tilde{T}_n \), i.e.,

\[
\tilde{T}_n = (\tilde{T}_{n-1}) \quad \text{and} \quad \tilde{T}_0 = T,
\]

and the following interesting property is shown in [20].

\[
\lim_{n \to \infty} \|\tilde{T}_n\| = r(T),
\]

where \( r(T) \) is the spectral radius of \( T \).

2. Numerical range

In this section, we shall introduce the numerical range and a result on that of the Aluthge transformation.

**Definition 2** (Numerical range). For an operator \( T \), the numerical range \( W(T) \) of \( T \) is the subset of the complex numbers \( \mathbb{C} \), given by,

\[
W(T) = \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \}.
\]

The following properties of the numerical range are well known.

(i) \( W(T) \) is a convex set (Hausdorff-Toeplitz).

(ii) \( \sigma(T) \subset \overline{W(T)} \).

As a result on the numerical range of \( \tilde{T} \), the following result has been shown.

**Theorem 2.A** ([18]). Let \( T \) be a bounded linear operator, then the following inclusion relation holds.

\[
(2.1) \quad \overline{W(\tilde{T})} \subset \overline{W(T)}.
\]

Theorem 2.A was firstly shown in [10] in case \( T \) is a 2 \( \times \) 2 matrix (in this case, \( W(\tilde{T}) \) and \( W(T) \) are closed subsets of the complex number \( \mathbb{C} \)). Then one of the authors [19] proved that (2.1) holds if \( T \) admits a decomposition \( T = U |T| \) for an isometry operator \( U \). This condition is always satisfied if \( T \) is an \( n \times n \) matrix, or \( H \) is finite dimensional.

In [19], the relation (2.1) is shown by using the property of the numerical range

\[
(2.2) \quad \overline{W(T)} = \bigcap_{\lambda \in \mathbb{C}} \{ z \in \mathbb{C} : |z - \lambda| \leq w(T - \lambda I) \},
\]
where $w(T)$ is the numerical radius of $T$, that is, 

$$w(T) = \sup \{|z| : z \in W(T)\}$$

and the following characterization of $w(T) \leq 1$ by Berger and Stampfli [3]:

(2.3) $w(T) \leq 1$ if and only if $\|T - zI\| \leq 1 + \sqrt{1 + |z|^2}$ for all $z \in \mathbb{C}$.

In a recent paper [18], Wu showed that the inclusion (2.1) holds for every bounded linear operator $T$ on a Hilbert space $H$. He showed this result by using the previous result shown in [19] and some properties of numerical range and Aluthge transformation, so this proof is not easy. In this report, we shall obtain a simplified proof of Theorem 2.A in Section 4.

3. $C$-NUMERICAL RANGE

As a generalization of the numerical range, for $n \times n$ matrices $C$ and $T$, the $C$-numerical range of $T$ is defined in [7] as follows:

**Definition 3** ($C$-numerical range [7]). For $n \times n$ matrices $C$ and $T$, the $C$-numerical range $W_C(T)$ of $T$ is the compact subset of complex number $\mathbb{C}$, given by,

$$W_C(T) = \{ \text{tr}(C U^* T U) : U \text{ is a unitary matrix} \}.$$

Put $C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, then $W_C(T) = W(T)$, so we can regard $W_C(T)$ as a generalization of $W(T)$. But $W_C(T)$ is not always convex as follows:

**Example** ([17]). Let 

$$T = C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}.$$ 

Put unitary matrices $U_1$ and $U_2$ as follows:

$$U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$ 

Then we have

$$CU_1^* TU_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad CU_2^* TU_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix},$$

that is, $1, 2i \in W_C(T)$. But put a unitary matrix $U_3$ as follows:

$$U_3 = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}.$$
\[
\text{tr}(CU^*_3TU_3) = |u_{22}|^2 - |u_{33}|^2 + i(|u_{23}|^2 + |u_{32}|^2).
\]
Assume that \(\frac{1+2i}{2}\in W_C(T)\). Then the following relations hold:
\[
\begin{align*}
|u_{22}|^2 - |u_{33}|^2 &= \frac{1}{2}, \\
|u_{23}|^2 + |u_{32}|^2 &= 1.
\end{align*}
\]
So \(U_3\) can not be unitary, and it is a contradiction. Hence \(\frac{1+2i}{2}\not\in W_C(T)\).

In fact, it is known that \(W_C(T)\) is star-shaped as follows:

**Theorem 3.A ([4]).** For any \(n \times n\) matrices \(C\) and \(T\), the range \(W_C(T)\) is star-shaped with star center at \(y = \frac{1}{n}\text{tr}(C)\text{tr}(T)\), i.e., if \(x \in W_C(T)\), then
\[
\lambda x + (1 - \lambda)y \in W_C(T) \quad \text{for all } \lambda \in [0, 1].
\]

Especially, when \(C\) is a Hermitian matrix or a rank-one matrix, the range \(W_C(T)\) is a convex set (cf. [17] and [16]). In these cases, we can rephrase them in the following ways:

The case that \(C\) is a Hermitian matrix. We assume that the spectrum of the Hermitian matrix \(C\) is the set
\[
c = (c_1, c_2, \ldots, c_n).
\]
Since \(C\) is a Hermitian matrix, there is a unitary matrix \(U\) such that
\[
U^*CU = \begin{pmatrix}
c_1 & 0 & \cdots & 0 \\
0 & c_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_n
\end{pmatrix}.
\]
Hence the set \(W_C(T)\) can be rewritten as follows:
\[
W_C(T) = \left\{ \sum_{j=1}^{n} c_j \langle Tx_j, x_j \rangle : \{x_1, x_2, \ldots, x_n\} \text{ is an orthonormal basis of } \mathbb{C}^n \right\},
\]
which is denoted by \(W_c(T)\) and we call \(W_c(T)\) the \(c\)-numerical range of \(T\). Poon [14] gave an alternative proof of the convexity of \(W_c(T)\) using some type of majorization property (cf. [8, page 87-88]).

The case that \(C\) is a nonzero \(n \times n\) matrix of rank one. We assume that the operator norm of \(C\) is 1. Then there exists a unitary matrix \(U\) such that
\[
U^*CU = \begin{pmatrix}
q \sqrt{1-|q|^2} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix},
\]
where $q$ is an eigenvalue of $C$ with $|q| \leq 1$. Hence the set $W_C(T)$ can be rewritten as follows:

$$W_C(T) = \{(Tx, y) : x, y \in \mathbb{C}^n, \|x\| = \|y\| = 1, \langle x, y \rangle = q\},$$

which is denoted by $W_q(T)$ and we call $W_q(T)$ the $q$-numerical range of $T$.

In this report, firstly, we shall obtain the direct proof of Theorem 2.A without using (2.2) and (2.3). Secondly, we shall generalize this result to $c$-numerical range in Section 5 as follows:

$$(3.1) \quad W_c(f(\tilde{T})) \subset W_c(f(T))$$

holds for all polynomial $f$. Lastly, we shall show the same relation (3.1) holds for $q$-numerical range.

4. SIMPLIFIED PROOF OF THEOREM 2.A

In this section, we shall obtain a direct proof of Theorem 2.A without using (2.2) and (2.3). To prove the this result, we prepare an obvious lemma.

**Lemma 4.A ([9]).** Let $A$ be a self-adjoint operator and $B$ be an operator. Then $AB$ is invertible if and only if $BA$ is invertible. Hence $\sigma(AB) = \sigma(BA)$.

We denote the real part of an operator $A$ by $\Re(A) = \frac{A + A^*}{2}$.

**Simplified proof of Theorem 2.A.** Let $T = V|T|$ be a polar decomposition of $T$. Since

$$\Re(|T|V) = \frac{|T|V + V^*|T|}{2} = V^*|T|V + V^*|T||V^*V$$

$$= \frac{V^*|T| + |T||V^*V}{2} = V^*\Re(T)V,$$

we have

$$\langle \Re(|T|V)x, x \rangle = \langle V^*\Re(T)Vx, x \rangle$$

$$= \langle \Re(T)Vx, Vx \rangle$$

$$= \langle \Re(T)\frac{Vx}{\|Vx\|}, \frac{Vx}{\|Vx\|} \rangle \langle Vx, Vx \rangle.$$

Hence

$$(4.1) \quad W(\Re(|T|V)) \subset W(\Re(T))W(V^*V).$$
If \(0 \in W(V^*V)\), then \(0 \in W\left(\mathbb{R}(T)\right)\). By \(W(V^*V) = [0,1]\) and Hausdorff-Toeplitz Theorem, we obtain

\[
W\left(\mathbb{R}(|T|V)\right) \subset W\left(\mathbb{R}(T)\right)W(V^*V) \quad \text{by (4.1)}
\]

\[
= \left\{ \alpha \left(\mathbb{R}(T)x, x\right) : \|x\| = 1, \alpha \in [0,1] \right\}
\]

\[
= W\left(\mathbb{R}(T)\right).
\]

If \(0 \notin W(V^*V)\), then \(V\) is an isometry, so \(W\left(\mathbb{R}(|T|V)\right) \subset W\left(\mathbb{R}(T)\right)\) holds by (4.1).

On the other hand, for any two operators \(H\) and \(K\), the following relation is easily obtained:

\[
\mathbb{R}\{\mathbb{R}(H)K\} = \frac{1}{2}\{\mathbb{R}(HK) + \mathbb{R}(K^*H)\}.
\]

Therefore we have

\[
\overline{W\left(\mathbb{R}(\overline{T})\right)} = \overline{W\left(|T|^\frac{1}{2}\mathbb{R}(V)|T|^\frac{1}{2}\right)}
\]

\[
= \text{conv}\sigma\left(|T|^\frac{1}{2}\mathbb{R}(V)|T|^\frac{1}{2}\right)
\]

\[
= \Re\text{conv}\sigma\left(|T|^\frac{1}{2}\mathbb{R}(V)|T|^\frac{1}{2}\right)
\]

\[
= \Re\text{conv}\sigma\left(\mathbb{R}(V)|T|\right) \quad \text{by Lemma 4.A}
\]

\[
\subset \Re\overline{W\left(\mathbb{R}(V)|T|\right)}
\]

\[
= \overline{\Re\{\mathbb{R}(\mathbb{R}(V)|T|)\}}
\]

\[
= \overline{\frac{1}{2}\left(\Re\overline{W\left(\mathbb{R}(V)|T|\right)} + \Re\overline{W\left(|T|V\right)}\right)} \quad \text{by (4.3)}
\]

\[
\subset \frac{1}{2}\left\{ \overline{W\left(\mathbb{R}(T)\right)} + \overline{W\left(\mathbb{R}(|T|V)\right)} \right\}
\]

\[
\subset \frac{1}{2}\left\{ \overline{W\left(\mathbb{R}(T)\right)} + \overline{W\left(\mathbb{R}(T)\right)} \right\} \quad \text{by (4.2)}
\]

\[
= \overline{W\left(\mathbb{R}(T)\right)},
\]

where \(\text{conv}\sigma(T)\) means the convex hull of \(\sigma(T)\).

Since \((e^{i\theta}T) = e^{i\theta}\overline{T}\) holds for each \(\theta \in [0,2\pi]\), we have

\[
\overline{W\left(\mathbb{R}\{e^{i\theta}T\}\right)} \subset \overline{W\left(\mathbb{R}\{e^{i\theta}T\}\right)} \quad \text{for all } \theta \in [0,2\pi],
\]

so that we obtain (2.1).

\[\square\]

Remark. In our proof of Theorem 2.A, the equation (4.3) plays an important role. (4.3) is also useful to extend the relation (2.1) to \(c\)-numerical range or \(q\)-numerical range.
In this section, we shall generalize Theorem 2.A to $c$-numerical range and $T$ to $f(T)$ where $f$ is a polynomial.

**Theorem 5.1.** Let $T$ be an $n \times n$ matrix, $f$ be a polynomial and $c = (c_1, c_2, \ldots, c_n)$ be a finite real sequence. Then the following inclusion holds:

$$W_c(f\hat{T})) \subset W_c(f(T)).$$

In this result, we may assume that $c = (c_1, c_2, \ldots, c_n)$ is a finite real sequence arranged in the decreasing order $c_1 \geq c_2 \geq \ldots \geq c_n$ by the definition of $W_c(T)$.

To prove Theorem 5.1, we shall prepare the following results:

**Theorem 5.A ([12]).** Let $T$ be an $n \times n$ matrix and $c = (c_1, c_2, \ldots, c_n)$ is a finite real sequence arranged in the decreasing order $c_1 \geq c_2 \geq \ldots \geq c_n$. Then

$$\max \{\Re(ze^{i\theta}) : z \in W_c(T)\} = \sum_{j=1}^{n} c_j \lambda_j \left( e^{i\theta} T \right),$$

holds for every $0 \leq \theta \leq 2\pi$, where $\lambda_j(S)$ means the $j$th eigenvalue of an $n \times n$ Hermitian matrix $S$:

$$\lambda_1(S) \geq \lambda_2(S) \geq \ldots \geq \lambda_n(S).$$

**Lemma 5.B ([6], [13, page 237]).** Suppose that $T$ is an $n \times n$ complex matrix and $\{\Re\lambda_1(T), \Re\lambda_2(T), \ldots, \Re\lambda_n(T)\}$ denotes the set of real parts of eigenvalues of $T$ arranged in the decreasing order. Then the inequality

$$\sum_{j=1}^{k} \Re\lambda_j(T) \leq \sum_{j=1}^{k} \lambda_j \left( \Re(T) \right)$$

holds for every $1 \leq k \leq n - 1$.

**Lemma 5.C ([5], [13, page 241]).** Suppose that $G$ and $H$ are $n \times n$ Hermitian matrices. Then the inequality

$$\sum_{j=1}^{k} \lambda_j(G + H) \leq \sum_{j=1}^{k} \left\{ \lambda_j(G) + \lambda_j(H) \right\}$$

holds for every $1 \leq k \leq n - 1$.

**Lemma 5.2.** Let $A$ be a positive invertible matrix and $X$ be an arbitrary matrix. Then for each polynomial $f$ and real $\theta$, there exists a matrix $S$ such that

$$e^{i\theta} f(A^{1/2}XA^{1/2}) = A^{1/2}SA^{1/2},$$

$$e^{i\theta} f(XA) = SA,$$

$$e^{i\theta} f(AX) = AS.$$
Proof. Let \( f(z) = f(0) + g(z)z \), where \( g(z) \) is also a polynomial. By using the equation

\[
(A^{\frac{1}{2}} X A^{\frac{1}{2}})^n = A^{\frac{1}{2}} (XA)^{n-1} X A^{\frac{1}{2}},
\]
we obtain the following equation:

\[
f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) = f(0)I + g(XA)XA^{\frac{1}{2}}
\]

(5.2)

\[
= f(0)I + A^{\frac{1}{2}} g(XA)XA^{\frac{1}{2}}
\]

\[
= A^{\frac{1}{2}} \{ f(0)A^{-1} + g(XA)X \} A^{\frac{1}{2}}.
\]

By setting

\[
S = e^{i\theta} \{ f(0)A^{-1} + g(XA)X \},
\]
we have

\[
e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) = e^{i\theta} A^{\frac{1}{2}} \{ f(0)A^{-1} + g(XA)X \} A^{\frac{1}{2}} \quad \text{by (5.2)}
\]

\[
= A^{\frac{1}{2}} S A^{\frac{1}{2}},
\]

\[
SA = e^{i\theta} \{ f(0)A^{-1} + g(XA)X \} A
\]

\[
= e^{i\theta} \{ f(0)I + g(XA)XA \}
\]

\[
= e^{i\theta} f(XA)
\]

and

\[
AS = e^{i\theta} A \{ f(0)A^{-1} + g(XA)X \}
\]

\[
= e^{i\theta} \{ f(0)I + Ag(XA)X \}
\]

\[
= e^{i\theta} f(AX) \quad \text{by } A(XA)^n X = (AX)^{n+1}.
\]

Hence the proof is complete. \(\Box\)

**Proof of Theorem 5.1.** We use a polar decomposition \( T = U|T| \) where \( U \) is a unitary matrix. Put \( A = |T| \geq 0 \) and \( X = U \). By perturbing \( A \) to \( A + \epsilon I \) for small \( \epsilon > 0 \), we need only to prove Theorem 5.1 for a positive invertible \( A \). By Theorem 5.1A, we shall show the following inequality

\[
\sum_{j=1}^{n} c_j \lambda_j \left( \Re \{ e^{i\theta} f(\tilde{T}) \} \right) \leq \sum_{j=1}^{n} c_j \lambda_j \left( \Re \{ e^{i\theta} f(T) \} \right)
\]

(5.3)
for every $0 \leq \theta \leq 2\pi$. Moreover by the following equations

\[
\sum_{j=1}^{n} c_j \lambda_j \left( \Re \{e^{i\theta} f(T)\} \right) = \sum_{j=1}^{n-1} (c_j - c_{j+1}) \sum_{k=1}^{j} \lambda_k \left( \Re \{e^{i\theta} f(T)\} \right) + c_n \sum_{k=1}^{n} \lambda_k \left( \Re \{e^{i\theta} f(T)\} \right),
\]

\[
\sum_{j=1}^{n} \lambda_j \left( \Re \{e^{i\theta} f(T)\} \right) = \sum_{j=1}^{n-1} (c_j - c_{j+1}) \sum_{k=1}^{j} \lambda_k \left( \Re \{e^{i\theta} f(T)\} \right) + c_n \sum_{k=1}^{n} \lambda_k \left( \Re \{e^{i\theta} f(T)\} \right),
\]

\[
\sum_{k=1}^{n} \lambda_k \left( \Re \{e^{i\theta} f(T)\} \right) = \sum_{k=1}^{n} \lambda_k \left( \Re \{e^{i\theta} (AXA^{\frac{1}{2}})\} \right)
\]

\[
= \sum_{k=1}^{n} \lambda_k \left( \Re (AXA^{\frac{1}{2}}) \right) \text{ by Lemma 5.2}
\]

\[
= \text{tr} \left( \Re (AXA^{\frac{1}{2}}) \right) = \Re \left\{ \text{tr}(AXA^{\frac{1}{2}}) \right\} = \Re \left( \text{tr}(SA) \right)
\]

\[
= \text{tr} \left( \Re (SA) \right) = \sum_{k=1}^{n} \lambda_k \left( \Re \{e^{i\theta} f(AX)\} \right) \text{ by Lemma 5.2}
\]

\[
= \sum_{k=1}^{n} \lambda_k \left( \Re \{e^{i\theta} f(T)\} \right),
\]

it is sufficient to prove the inequality

\[
\sum_{j=1}^{k} \lambda_j \left( \Re \{e^{i\theta} f(T)\} \right) \leq \sum_{j=1}^{k} \lambda_j \left( \Re \{e^{i\theta} f(T)\} \right)
\]

holds for $0 \leq \theta \leq 2\pi$ and every $k = 1, 2, \ldots, n - 1.$
By using Lemma 5.2 and Fan’s two inequalities, we have
\[
\sum_{j=1}^{k} \lambda_j \left( \Re \{ e^{i\theta} f(\tilde{T}) \} \right)
\]
\[
= \sum_{j=1}^{k} \lambda_j \left( \Re \{ e^{i\theta} f(A^{\frac{1}{2}} X A^{\frac{1}{2}}) \} \right)
\]
\[
= \sum_{j=1}^{k} \lambda_j \left( \Re (A^{\frac{1}{2}} S A^{\frac{1}{2}}) \right) \text{ by Lemma 5.2}
\]
\[
= \sum_{j=1}^{k} \lambda_j \left( A^{\frac{1}{2}} \Re (S) A^{\frac{1}{2}} \right)
\]
\[
= \sum_{j=1}^{k} \Re \lambda_j \left( A^{\frac{1}{2}} \Re (S) A^{\frac{1}{2}} \right) \text{ by Lemma 4.A}
\]
\[
\leq \sum_{j=1}^{k} \lambda_j \left( \Re \{ \Re (S) A \} \right) \text{ by Lemma 5.B}
\]
\[
= \frac{1}{2} \sum_{j=1}^{k} \lambda_j \left( \Re (S A) + \Re (A S) \right) \text{ by (4.3)}
\]
\[
\leq \frac{1}{2} \left\{ \sum_{j=1}^{k} \lambda_j \left( \Re (S A) \right) + \sum_{j=1}^{k} \lambda_j \left( \Re (A S) \right) \right\} \text{ by Lemma 5.C}
\]
\[
= \frac{1}{2} \left\{ \sum_{j=1}^{k} \lambda_j \left( \Re \{ e^{i\theta} f(X A) \} \right) + \sum_{j=1}^{k} \lambda_j \left( \Re \{ e^{i\theta} f(A X) \} \right) \right\} \text{ by Lemma 5.2}
\]
\[
= \frac{1}{2} \left\{ \sum_{j=1}^{k} \lambda_j \left( \Re \{ e^{i\theta} f(T) \} \right) + \sum_{j=1}^{k} \lambda_j \left( U^* \Re \{ e^{i\theta} f(T) \} U \right) \right\}
\]
\[
= \sum_{j=1}^{k} \lambda_j \left( \Re \{ e^{i\theta} f(T) \} \right).
\]

Hence the proof of Theorem 5.1 is complete. \( \square \)

The case \( f(z) = z \), we obtain the following corollary.

**Corollary 5.3.** Let \( T \) be an \( n \times n \) matrix and \( c = (c_1, c_2, \ldots, c_n) \) be a finite real sequence.
Then the following inclusion holds:
\[
W_c(\tilde{T}) \subset W_c(T).
\]
6. \textit{q-Numerical range of the Aluthge transformation of a matrix}

It is known that there is a close relationship between the family of \textit{q-numerical ranges} $W_q(T)$ ($0 \leq q \leq 1$) of a matrix $T$ and the Davis-Wielandt shell $W(T, T^*T)$ of $T$. The latter is defined by

$$W(T, T^*T) = \{ (\langle Tx, x \rangle, \langle T^*Tx, x \rangle) \in \mathbb{C} \times \mathbb{R} : x \in \mathbb{C}^n, \|x\| = 1 \}.$$  

It is shown that the range $W(T, T^*T)$ is convex if $T$ is an $n \times n$ matrix for $n \geq 3$ in [2]. In the case $T$ is a $2 \times 2$ matrix, the range $W(T, T^*T)$ is convex if its affine hull is 2-dimensional, and the range $W(T, T^*T)$ is the boundary of a convex set if its affine hull is 3-dimensional. The following lemma provides a tool to compare the \textit{q-numerical ranges} of two matrices.

\textbf{Lemma 6.A ([11, page 389, Theorem 2.1])}. Suppose that $A$ is an $n \times n$ matrix and $B$ is an $m \times m$ matrix. Then the following two conditions are mutually equivalent:

\begin{enumerate}[(i)]
    \item The inclusion $W_q(B) \subset W_q(A)$ holds for every $0 \leq q \leq 1$.
    \item The inclusion $W(B) \subset W(A)$ and the inequality

$$\max\{h : (z, h) \in W(B, B^*B)\} \leq \max\{h : (z, h) \in W(A, A^*A)\}$$

hold for every $z \in W(B)$.
\end{enumerate}

In this section, we shall prove the following theorem.

\textbf{Theorem 6.1}. Suppose that $T$ is an $n \times n$ matrix and $f(z)$ is a polynomial in $z$. Then the inclusion

$$W_q(f(\tilde{T})) \subset W_q(f(T))$$

holds for every complex number $q$ with $|q| \leq 1$.

To prove Theorem 6.1, we have an alternative condition of (ii) in the above Lemma 6.A.

\textbf{Lemma 6.2}. Suppose that $A$ is an $n \times n$ matrix and $B$ is an $m \times m$ matrix. Then the following two conditions are mutually equivalent:

\begin{enumerate}[(i)]
    \item The inclusion $W_q(B) \subset W_q(A)$ holds for every $0 \leq q \leq 1$.
    \item The inequality

$$\lambda_1(B^* B + k \mathbb{R}(e^{i\theta} B)) \leq \lambda_1(A^* A + k \mathbb{R}(e^{i\theta} A))$$

hold for every $0 \leq \theta \leq 2\pi$ and $k \geq 0$.
\end{enumerate}
Proof. We prove the equivalence of condition (ii) of Lemma 6.A and condition (iii) of Lemma 6.2. We compare the following two compact convex sets:

\[ A_0 = \{(z, t) \in \mathbb{C} \times \mathbb{R} : z \in W(A), 0 \leq t \leq \max \{h : (z, h) \in W(A, A^*A)\}\} \]

and

\[ B_0 = \{(z, t) \in \mathbb{C} \times \mathbb{R} : z \in W(B), 0 \leq t \leq \max \{h : (z, h) \in W(B, B^*B)\}\}. \]

For every \(0 \leq \theta \leq 2\pi\), we consider the projection \(\Pi = \Pi_{\theta}\) given by

\[(z, t) = (\mathcal{R}(z), \mathcal{S}(z), t) \rightarrow \mathcal{R}(e^{i\theta}z) + it = (\cos \theta \mathcal{R}(z) - \sin \theta \mathcal{S}(z)) + it.\]

Then (ii) of Lemma 6.A holds if and only if the condition \(B_0 \subset A_0\) holds, and also this condition is equivalent to

\[(6.2) \quad \Pi_{\theta}(B_0) \subset \Pi_{\theta}(A_0) \]

for every \(0 \leq \theta \leq 2\pi\), where the compact convex sets \(\Pi_{\theta}(A_0)\) and \(\Pi_{\theta}(B_0)\) are characterized by

\[\Pi_{\theta}(A_0) = \text{conv}(W(\mathcal{R}(e^{i\theta}A)), W(\mathcal{R}(e^{i\theta}A) + iA^*A))\]

and

\[\Pi_{\theta}(B_0) = \text{conv}(W(\mathcal{R}(e^{i\theta}B)), W(\mathcal{R}(e^{i\theta}B) + iB^*B)).\]

Each of these sets contains its projection onto the real line. These sets are contained in the closed upper half plane \(\Im(z) \geq 0\). Thus, for each \(0 \leq \theta \leq 2\pi\), the inclusion relation (6.2) is equivalent to the inequality

\[(6.3) \quad \max \{\Im(z) + k \Re(z) : z \in W(\mathcal{R}(e^{i\theta}B) + iB^*B)\} \leq \max \{\Im(z) + k \Re(z) : z \in W(\mathcal{R}(e^{i\theta}A) + iA^*A)\} \]

for every \(k \in \mathbb{R}\) (cf. [15, page 81, Theorem A]). By basic properties of the numerical range, we have

\[\max \{\Im(z) + k \Re(z) : z \in W(\mathcal{R}(e^{i\theta}A) + iA^*A)\} = \max W(A^*A + k \mathcal{R}(e^{i\theta}A)) = \lambda_1(A^*A + k \mathcal{R}(e^{i\theta}A))\]

and

\[\max \{\Im(z) + k \Re(z) : z \in W(\mathcal{R}(e^{i\theta}B) + iB^*B)\} = \lambda_1(B^*B + k \mathcal{R}(e^{i\theta}B))\]

(cf. [8, page 9-11]), so that (6.3) is equivalent to

\[\lambda_1(B^*B + k \mathcal{R}(e^{i\theta}B)) \leq \lambda_1(A^*A + k \mathcal{R}(e^{i\theta}A))\]
for every $0 \leq \theta \leq 2\pi$ and $k \in \mathbb{R}$. By replacing $\theta$ by $\theta + \pi$, we may restrict the range of $k$ as $k \geq 0$. Thus the condition (ii) of Lemma 6.1 and the condition (iii) of Lemma 6.2 are equivalent.

\[ \square \]

\textit{Proof of Theorem 6.1.} Since the equation

\[ W_{cq}(S) = cW_q(S) \]

holds for any complex numbers $c, q$ with $|c| = 1$ and $|q| \leq 1$, it is sufficient to prove (6.1) for $0 \leq q \leq 1$. Therefore we have only to prove the inequality

\[ (6.4) \quad \lambda_1 \left( f(T)^* f(T) + k \Re \{ e^{i\theta} f(T) \} \right) \leq \lambda_1 \left( f(T)^* f(T) + k \Re \{ e^{i\theta} f(T) \} \right) \]

for every $0 \leq \theta \leq 2\pi$ and $k \geq 0$ by Lemma 6.2.

To prove the inequality (6.4), we shall prove that the following inequality holds for a positive matrix $A$ and an arbitrary $X$:

\[ \lambda_1 \left( f(A^{1/2}XA^{1/2})^* f(A^{1/2}XA^{1/2}) + k \Re \{ e^{i\theta} f(A^{1/2}XA^{1/2}) \} \right) \leq \frac{1}{2} \lambda_1 \left( f(XA)^* f(XA) + k \Re \{ e^{i\theta} f(XA) \} \right) + \frac{1}{2} \lambda_1 \left( f(AX)^* f(AX) + k \Re \{ e^{i\theta} f(AX) \} \right). \]

By perturbing $A$ to $A + \epsilon I$ for small $\epsilon > 0$, it suffices to prove (6.5) for a positive invertible matrix $A$. 

using Lemma 5.2, the Cauchy-Schwarz inequality and the Arithmetic-Geo inequality, we have

\[
\lambda_1 \left( f(A^{\frac{1}{2}}XA^{\frac{1}{2}})^* f(A^{\frac{1}{2}}XA^{\frac{1}{2}}) + k\Re\{e^{i\theta} f(A^{\frac{1}{2}}XA^{\frac{1}{2}})\} \right)
\]

\[
= \lambda_1 \left( A^{\frac{1}{2}} S^* A S A^{\frac{1}{2}} + k\Re\{ A^{\frac{1}{2}} \Re(S) A^{\frac{1}{2}} \} \right) \quad \text{by Lemma 5.2}
\]

\[
= \Re \lambda_1 \left( S^* A S A + k\Re(S)A \right) \quad \text{by Lemma 4.A}
\]

\[
\leq \lambda_1 \left( \Re\{S^* A S A + k\Re(S)A\} \right) \quad \text{by Lemma 5.B}
\]

\[
= \max_{x \in \mathbb{C}^n, ||x||=1} \left[ \Re \left< S^* A S Ax, x \right> + \frac{k}{2} \left< \{\Re(SA) + \Re(AS)\} x, x \right> \right] \quad \text{by (4.3)}
\]

\[
= \max_{x \in \mathbb{C}^n, ||x||=1} \left[ \Re \left< SAx, ASx \right> + \frac{k}{2} \left< \{\Re(SA) + \Re(AS)\} x, x \right> \right]
\]

\[
\leq \max_{x \in \mathbb{C}^n, ||x||=1} \left[ \left< A^2 S Ax, x \right> \frac{1}{2} \left< S^* A^2 S x, x \right> \frac{1}{2} + \frac{k}{2} \left< \{\Re(SA) + \Re(AS)\} x, x \right> \right]
\]

\[
\leq \max_{x \in \mathbb{C}^n, ||x||=1} \left[ \left< f(XA)^* f(XA)x, x \right> + \frac{1}{2} \left< f(AX)^* f(AX)x, x \right> + \frac{k}{2} \left< \Re\{e^{i\theta} f(XA)\} x, x \right> \right]
\]

\[
\leq \frac{1}{2} \max_{x \in \mathbb{C}^n, ||x||=1} \left[ \left< f(XA)^* f(XA)x, x \right> + k \left< \Re\{e^{i\theta} f(XA)\} x, x \right> \right]
\]

\[
\leq \frac{1}{2} \lambda_1 \left( f(XA)^* f(XA) + k\Re\{e^{i\theta} f(XA)\} \right)
\]

\[
= \frac{1}{2} \lambda_1 \left( f(AX)^* f(AX) + k\Re\{e^{i\theta} f(AX)\} \right)
\]

shall use a polar decomposition \( T = U|T|\) where \( U \) is a unitary matrix \( A = |T|, X = U \) in (6.5). Since \( (|T|U)^n = U^* T^n U \) for every integer \( n \geq \)
have the equation $f(|T|U) = U^*f(T)U$ for any polynomial $f$, so that
\begin{align*}
\lambda_1 \left( f(\bar{T})^*f(\bar{T}) + k\Re\{e^{i\theta}f(T)\} \right) \\
\leq \frac{1}{2} \lambda_1 \left( f(T)^*f(T) + k\Re\{e^{i\theta}f(T)\} \right) \\
+ \frac{1}{2} \lambda_1 \left( f(|T|U)^*f(|T|U) + k\Re\{e^{i\theta}f(|T|U)\} \right) \\
= \frac{1}{2} \lambda_1 \left( f(T)^*f(T) + k\Re\{e^{i\theta}f(T)\} \right) \\
+ \frac{1}{2} \lambda_1 \left( U^*f(T)^*f(T)U + kU^*\Re\{e^{i\theta}f(T)\}U \right) \\
= \lambda_1 \left( f(T)^*f(T) + k\Re\{e^{i\theta}f(T)\} \right).
\end{align*}
Hence the proof of Theorem 6.1 is complete. \(\square\)

In particular, by putting $q = 1$ in Theorem 6.1, we have the following relation.

**Corollary 6.3.** If $T$ is an $n \times n$ matrix. Then

$$W(f(\bar{T})) \subset W(f(T))$$
holds for all polynomial $f$.

Moreover, we obtain the inequalities on the numerical radius and the spectral norm.

**Corollary 6.4.** Let $T$ be an $n \times n$ matrices. Then the following assertions hold:

- (i) $w(f(\bar{T})) \leq w(f(T))$ for all polynomials $f$.
- (ii) $\|f(\bar{T})\| \leq \|f(T)\|$ for all polynomials $f$,

where $\| \cdot \|$ means the spectral norm.

Corollary 6.4 is easily obtained by the following Proposition 6.5.

**Proposition 6.5.** Let $A$ and $B$ be $n \times n$ matrices. Then the following assertions are mutually equivalent:

- (i) $W(f(A)) \subset W(f(B))$ for all polynomials $f$.
- (ii) $w(f(A)) \leq w(f(B))$ for all polynomials $f$.
- (iii) $\|f(A)\| \leq \|f(B)\|$ for all polynomials $f$,

where $\| \cdot \|$ means the spectral norm.

**Proof.** Proofs of (ii) $\implies$ (i) and (iii) $\implies$ (i) are obvious by

$$W(A) = \bigcap_{\mu \in \mathbb{C}} \{ z : |z - \mu| \leq w(A - \mu) \}$$

and

$$W(A) = \bigcap_{\mu \in \mathbb{C}} \{ z : |z - \mu| \leq \|A - \mu\| \}.$$
Proof of (i) $\Rightarrow$ (ii) is also obvious. Hence we shall show (i) $\Rightarrow$ (iii). In fact, we have only to show that
\[ W(f(A)) \subset W(f(B)) \quad \text{for all polynomials } f \quad \Rightarrow \quad \|A\| \leq \|B\|. \]
So we shall show
\[ \|B\| < 1 \Rightarrow \|A\| \leq 1. \]

Let $r(A)$ be the spectral radius of $A$. Since
\[ r(A) \leq w(A) \leq w(B) \leq \|B\| < 1 \]
hold, the inverses $(1+A)^{-1}$ and $(1+B)^{-1}$ exist, and we can consider the Cayley transform of $A$ and $B$ as follows:
\[ \Phi(A) \equiv (1-A)(1+A)^{-1}, \quad \Phi(B) \equiv (1-B)(1+B)^{-1}. \]

On the other hand, setting
\[ g_n(z) \equiv 1 + 2\sum_{k=1}^{n}(-1)^k z^k, \]
we have
\[ \Phi(A) = \lim_{n \to \infty} g_n(A), \quad \Phi(B) = \lim_{n \to \infty} g_n(B) \]
since
\[ \frac{1-z}{1+z} = 1 + 2\sum_{k=1}^{\infty}(-1)^k z^k \]
holds. By the assumption, we have
\[ W(g_n(A)) \subset W(g_n(B)) \quad (n = 1, 2, \ldots), \]
then we obtain
\[ W(\Phi(A)) \subset W(\Phi(B)). \]

On the other hand, since $B$ is a contraction, we have $\Re(\Phi(B)) \geq 0$, that is, $W(\Phi(B))$ is included in the right-half plane. Then $W(\Phi(A))$ is also included in the right-half plane, that is, $\Re(\Phi(A)) \geq 0$ holds. Therefore, $1 + \Phi(A)$ is invertible, and $A = \Phi(\Phi(A))$ is a contraction, so that the proof is complete.\[ \square \]

**Proof of Corollary 6.4.** Put $A = \tilde{T}$ and $B = T$ in Proposition 6.5, and we have Corollary 6.4 by Corollary 6.3.\[ \square \]

Lastly, we summarize Theorems 5.1 and 6.1 as follows:
Theorem 6.6. Suppose that $T$ and $C$ are $n \times n$ complex matrices and $f$ is a complex polynomial. If $C$ is a Hermitian matrix or a rank-one matrix, then the following inclusion relation holds:

$$W_{C}(f(\bar{T})) \subseteq W_{C}(f(T)).$$

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