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AROUND THE FURUTA INEQUALITY
フルタ不等式の周辺の不等式

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1. Chaotic order と Furuta inequality A と B は Hilbert space 上の positive operator とする. A が positive (resp. positive invertible) operator のとき $A \geq 0$ (resp. $A > 0$) と表す. 久保-安藤 [18] によって導入された A と B の α-power mean は次のように与えられる.

$$A \natural_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \quad \text{for} \quad 0 \leq \alpha \leq 1.$$

Furuta inequality [6] はこの α-power mean を用いることで次のように表すことができる ([2],[3],[12],[13],[14]).

Furuta inequality: *If $A \geq B \geq 0$, then*

$$\text{(F)} \qquad A^{u} \natural_{\frac{1-u}{p-u}} B^{p} \leq A \quad \text{and} \quad B \leq B^{u} \natural_{\frac{1-u}{p-u}} A^{p}$$

holds for $u \leq 0$ and $1 \leq p$.

この Furuta inequality は Löwner-Heinz inequality の歴史的拡張と絶賛されている.

$$\text{(LH)} \qquad If \quad A \geq B \geq 0, \quad \text{then} \quad A^{\alpha} \geq B^{\alpha} \quad \text{for} \quad 0 \leq \alpha \leq 1.$$

我々は [12] (cf.[7]) において (F) を一行にまとめて表せることを示した. これを satellite theorem of the Furuta inequality と呼ぶ:

If $A \geq B \geq 0$, then

$$\text{(SF)} \qquad A^{u} \natural_{\frac{1-u}{p-u}} B^{p} \leq B \leq A \leq B^{u} \natural_{\frac{1-u}{p-u}} A^{p}$$

holds for all $u \leq 0$ and $p \geq 1$.

A, $B > 0$ に対し, $\log A \geq \log B$ の時 $A \gg B$ と表し chaotic order ([3],[16],[17]) と呼んでいる. 次は chaotic order の特徴づけであるが, 今後 chaotic order の議論においての出発点となるものであることより, これを chaotic Furuta inequality [3] と命名しておく.

If $A \gg B$, then

$$\text{(CF)} \qquad A^{u} \natural_{\frac{-u}{p-u}} B^{p} \leq I \leq B^{u} \natural_{\frac{-u}{p-u}} A^{p}$$

for any $p \geq 0$ and $u \leq 0$.

satellite theorem (SF) は通常の順序 $A \geq B$ における Furuta inequality (F) と chaotic order $A \gg B$ の違いをはっきりと示している. 実際, 次のような結果を得ることができる ([16],[17]).

If $A \gg B$, then

$$\text{(SCF)} \qquad A^{u} \natural_{\frac{1-u}{p-u}} B^{p} \leq B \quad \text{and} \quad A \leq B^{u} \natural_{\frac{1-u}{p-u}} A^{p}$$

holds for any $p \geq 1$ and $u \leq 0$.

(CF) と (SCF) は更に次のように一般化することができる [16].

Theorem A. *For A, $B > 0$, if $A \gg B$, then the following* (1) *and* (2) *hold.*

$$(1)\qquad A^u \sharp_{\frac{\delta-u}{p-u}} B^p \leq B^\delta \text{ and } A^\delta \leq B^u \sharp_{\frac{\delta-u}{p-u}} A^p \text{ for } u \leq 0 \text{ and } 0 \leq \delta \leq p$$

$$(2)\qquad A^u \sharp_{\frac{\alpha-u}{p-u}} B^p \leq A^\alpha \text{ and } B^\alpha \leq B^u \sharp_{\frac{\alpha-u}{p-u}} A^p \text{ for } u \leq \alpha \leq 0 \text{ and } 0 \leq p.$$

2. Furuta inequality の一般化, Grand Furuta inequality. 古田は Furuta inequality の一般化を次のような形で示した [8]. これは 安藤-日合が与えた log majorization についての主結果と同値な不等式 [1] と Furuta inequality を繋ぐものとなっており, 我々はこれを grand Furuta inequality と呼んでいる [4],[5],[15].

The grand Furuta inequality: *If $A \geq B \geq 0$ and A is invertible, then for each $1 \leq p$ and $0 \leq t \leq 1$,*

$$\text{(GF)}\qquad A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A \text{ and } B \leq B^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^t \natural_s A^p)$$

holds for $t \leq r$ and $1 \leq s$.

これらの指数 $\frac{1-t+r}{(p-t)s+r}$ が best possible であるということについては棚橋によって示されている [19]. (GF) における指数 s を $\frac{\beta-t}{p-t}$ for $1 \leq p \leq \beta$ と置き換え, α-power mean を用いることで (F) の場合と同様に次のような satellite form を与えることができる [15].

If $A \geq B > 0$, then the following (SGF) *holds for $0 \leq t \leq 1 \leq p \leq \beta$ and $u \leq 0$.*

$$A^u \sharp_{\frac{1-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{1}{\beta}} \leq B^u \sharp_{\frac{1-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

上で使われた記号 $\natural$ は α-power mean を $\alpha \in R$ にまで拡張したものであり $\alpha \in [0, 1]$ に於いては $\sharp$ と一致するがそれ以外では作用素平均とはならない.
(SGF) における中心部分の不等式は [4], [5] において示しているように (SF) の視点からすれば (GF) の本質的な性質であろう. そこで 我々はこの進化型を次のように与えておく.

Theorem 1. *Let $A \geq B > 0$ and $0 \leq t \leq 1 \leq p$. Then*

$$H(\beta) = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}}$$

is a decreasing function with $\beta \geq p$ and in particular $H(\beta) \leq B^p$ for $\beta \geq p$.

Proof. First of all, suppose that $1 \leq \frac{\beta-t}{p-t} \leq 2$. Then

$$A^t \natural_{\frac{\beta-t}{p-t}} B^p = B^p \natural_{\frac{p-\beta}{p-t}} A^t = B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} A^{-t})B^p \leq B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} B^{-t})B^p = B^\beta$$

By (LH), we have $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p$.

Next we assume that $H(\beta) \le B^p$ for a given $\beta \ge p$. Since $p \ge 1$, we have $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \le B \le A$. If we take β_1 with $1 \le \frac{\beta_1-t}{\beta-t} \le 2$, then the preceding argument ensures that

$$A^t \natural_{\frac{\beta_1-t}{p-t}} B^p = A^t \natural_{\frac{\beta_1-t}{\beta-t}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) = A^t \natural_{\frac{\beta_1-t}{\beta-t}} B_1^{\beta} \le B_1^{\beta_1},$$

that is, $A^t \natural_{\frac{\beta_1-t}{p-t}} B^p \le (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\beta_1}{\beta}}$. So it follows from (LH) that

$$(A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{p}{\beta_1}} \le (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \le B^p,$$

which shows the monotonicity of $H(\beta)$.

3. 内山による Furuta innequality の一般化の試み.

最近, 内山は (GF) の一般化の方向として次のような形を与えた [20].

If $A \ge B \ge C > 0$, then for each $0 \le t \le 1 \le p$

$$\text{(U)} \qquad A^{1-t} \ge A^{-r} \sharp_{\frac{1-t+r}{(p-t)s+r}} (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^s$$

holds for $r \ge t$ and $s \ge 1$.

これに関して我々は (SGF) の観点からすれば「歪み」があるように感じ, 次のような形を提案した [5].

If $A,\ B,\ C > 0$ satisfy $A \gg B$ and $B \ge C$, then for each $0 \le t \le 1$

$$B \ge C \ge (B^t \natural_s C^p)^{\frac{1}{(p-t)s+t}} \ge A^{-r+t} \sharp_{\frac{1+r-t}{(p-t)s+r}} (B^t \natural_s C^p)$$

holds for all $p \ge 1,\ s \ge 1$ and $r \ge t$.

この不等式に於いて $A \ge B = C$ ならば (F) を得, $A = B \ge C$ とすれば (GF) となることは明らかであろう.

ところが古田は [9] (cf.[10]) において [11] を用いることで (U) のより一般的な結果が得られることを示した. このことについては Theorem 4 で触れることにして, ここでは古田の結果を chaotic order の観点から見直してみる.

Theorem 2. *For fixed $A,\ B,\ C > 0$ and $0 \le t \le 1 \le p$, if $A \gg D = (B^{-\frac{t}{2}} C^p B^{-\frac{t}{2}})^{\frac{1}{p-t}}$ is satisfied, then (1) holds for $\beta \ge p$ and $r \ge t$.*

$$\text{(1)} \qquad B^{\frac{t}{2}} A^{-t} B^{\frac{t}{2}} \sharp_{\frac{t}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \ge B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

Additionally, if $A \ge B$, then (2) holds.

$$\text{(2)} \qquad B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \ge B^{\frac{t}{2}} A^{-r} B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

Proof. Since $A \gg D$, (CF) implies that

$$(\dagger) \qquad (A^{\frac{t}{2}} D^{\beta-t} A^{\frac{t}{2}})^{\frac{t}{\beta}} \le A^t$$

and so $(A^{\frac{t}{2}}D^{\beta-t}A^{\frac{t}{2}})^{\frac{1}{\beta}} \ll A$. Therefore it follows from (SCF) that

$$A^{-r+t} \sharp_{\frac{1-(t-r)}{\beta-(t-r)}} \{(A^{\frac{t}{2}}D^{\beta-t}A^{\frac{t}{2}})^{\frac{1}{\beta}}\}^{\beta} \le (A^{\frac{t}{2}}D^{\beta-t}A^{\frac{t}{2}})^{\frac{1}{\beta}},$$

namely

$$A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} D^{\beta-t} \le A^{-t} \sharp_{\frac{1}{\beta}} D^{\beta-t}.$$

Since $B^{\frac{t}{2}}D^{\beta-t}B^{\frac{t}{2}} = B^t \natural_{\frac{\beta-t}{p-t}} C^p$, we have (1) by multiplying $B^{\frac{t}{2}}$ on both sides.

(2) is also shown as follows: Since $A^t \gg (A^{\frac{t}{2}}D^{\beta-t}A^{\frac{t}{2}})^{\frac{t}{\beta}}$ as in above, Theorem A (1) implies that

$$(A^t)^{-\frac{r-t}{t}} \sharp_{\frac{\frac{p}{t}+\frac{r-t}{t}}{\frac{\beta}{t}+\frac{r-t}{t}}} (A^{\frac{t}{2}}D^{\beta-t}A^{\frac{t}{2}})^{\frac{t}{\beta}\frac{\beta}{t}} \le (A^{\frac{t}{2}}D^{\beta-t}A^{\frac{t}{2}})^{\frac{t}{\beta}\frac{p}{t}},$$

that is,

$$A^{-r+t} \sharp_{\frac{p-t+r}{\beta-t+r}} A^{\frac{t}{2}}D^{\beta-t}A^{\frac{t}{2}} \le (A^{\frac{t}{2}}D^{\beta-t}A^{\frac{t}{2}})^{\frac{p}{\beta}}.$$

Multiplying $A^{-\frac{t}{2}}$ from the both sides of the above, we have

$$A^{-r} \sharp_{\frac{p+r-t}{\beta+r-t}} D^{\beta-t} \le A^{-t} \sharp_{\frac{p}{\beta}} D^{\beta-t} \le B^{-t} \sharp_{\frac{p}{\beta}} D^{\beta-t} = B^{-\frac{t}{2}}(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{p}{\beta}}B^{-\frac{t}{2}} \le B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}},$$

where the final inequality follows from Theorem 1. Again multiplying $B^{\frac{t}{2}}$ to each sides of this formula, we have

$$B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{p-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \le C^p.$$

Hence it follows that

$$\begin{aligned} & B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \\ = \; & B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} \{B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{p-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)\} \\ \le \; & B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p. \end{aligned}$$

次に Theorem A の応用として上と同様な不等式を与えておく.

Theorem 3. *If $A,\ B,\ C > 0$ satisfy $A \gg D = (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{1}{p-t}}$ for some $0 \le t \le 1 \le p$, the the following* (1) *and* (2) *hold for $\beta \ge p$ and $r \ge t$.*

(1) $$B^t \sharp_{\frac{1-t}{p-t}} C^p \ge B^{\frac{t}{2}}A^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \ge B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

(2) $$B^t \sharp_{\frac{1-t}{p-t}} C^p \ge B^{\frac{t}{2}}A^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{p}} C^p \ge B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \ge B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

Proof. (1) follows from Theorem A (2) and (1). Actually we have

$$\begin{aligned} & A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} D^{\beta-t} = D^{\beta-t} \sharp_{\frac{\beta-1}{\beta-t+r}} A^{-r} \\ = \; & D^{\beta-t} \sharp_{\frac{\beta-1}{\beta}} \{D^{\beta-t} \sharp_{\frac{\beta}{\beta-t+r}} A^{-r}\} \\ = \; & D^{\beta-t} \sharp_{\frac{\beta-1}{\beta}} \{A^{-r} \sharp_{\frac{-t+r}{\beta-t+r}} D^{\beta-t}\} \\ \le \; & D^{\beta-t} \sharp_{\frac{\beta-1}{\beta}} A^{-t} = A^{-t} \sharp_{\frac{1}{\beta}} D^{\beta-t} \\ = \; & A^{-t} \sharp_{\frac{1-t+t}{\beta-t+t}} (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{\beta-t}{p-t}} \le (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{1-t}{p-t}} \end{aligned}$$

So the conclusion is obtained by multiplying $B^{\frac{t}{2}}$ both sides of each term.

In addition, (2) except the final part is obtained by taking $\beta = p$ in (1). Moreover we have

$$A^{-r} \sharp_{\frac{1-t+r}{\beta-t+r}} D^{\beta-t} = A^{-r} \sharp_{\frac{1-t+r}{p-t+r}} \{A^{-r} \sharp_{\frac{p-t+r}{\beta-t+r}} D^{\beta-t}\} \le A^{-r} \sharp_{\frac{1-t+r}{p-t+r}} D^{p-t}$$

by Theorem A (2). Multiplying $B^{\frac{t}{2}}$ to each term from both sides, we attain the conclusion.

最後に古田 [9] による結果が上の Theorem 2 および Theorem 3 から簡単に導かれることを示そう. ここでも s を $\frac{\beta-t}{p-t}$ for $\beta \ge p$ と置き換えておく.

Theorm 4 (Furuta). *If $A \ge B \ge C > 0$ and $0 \le t \le 1 \le p$, then*

(1) $B \ge C \ge (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \ge B^{\frac{t}{2}}A^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \ge B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$

and

(2) $$B \ge C \ge B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \ge B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

hold for $\beta \ge p$ and $r \ge t$.

Proof. First of all, the assumption $B \ge C > 0$ ensures $(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \le B$ by (SGF). As in the proof of Theorem 2, (†) is the essential point, which is shown as follows:
Let $D = (B^{-\frac{t}{2}}C^pB^{-\frac{t}{2}})^{\frac{1}{p-t}}$ be as in Theorem 2. Then

$$A^{-t} \sharp_{\frac{t}{\beta}} D^{\beta-t} \le B^{-t} \sharp_{\frac{t}{\beta}} D^{\beta-t} = B^{-\frac{t}{2}}(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{t}{\beta}} B^{-\frac{t}{2}} \le B^{-\frac{t}{2}}B^tB^{-\frac{t}{2}} = I.$$

Since (†) is shown, (1) connects with Theorem 2 (1). Namely we have

$$\begin{aligned} & B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \le B^{\frac{t}{2}}A^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \\ \le \; & B^{\frac{t}{2}}B^{-t}B^{\frac{t}{2}} \sharp_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) = (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{\beta}} \le C \le B. \end{aligned}$$

Next we show (2). For this, we have only to check $B^{\frac{t}{2}}A^{-r}B^{\frac{t}{2}} \sharp_{\frac{1-t+r}{p-t+r}} C^p \le C \le B$ by (†) and Theorem 2 (2). Fortunately, it is obtained by taking $\beta = p$ in the former (1).

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