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Artin component and Weyl group for dihedral singularity

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Abstract
In this paper, we give the Artin components for dihedral singularities explicitly.

1 Introduction

Let (X, x) be a rational surface singularity and let $\chi \to \mathrm{Def}(X)$ be a versal deformation of X. In the sequel, we denote a base space of the versal deformation of X by $\mathrm{Def}(X)$. Riemenschneider [8] showed that a versal deformation (relatively to the exceptional set) $\tilde{\chi} \to T = \mathrm{Def}(\tilde{\chi})$ of the minimal resolution $\tilde{X}$ of X can be blow down simultaneously to a deformation $\chi \to T$ of X. Then from versality, we have a mapping $T = \mathrm{Def}(\tilde{X}) \to \mathrm{Def}(X)$. About the mapping, Artin[1], Lipman[6], Wahl[17] proved that

$$(\mathrm{Def}(X))_{\mathrm{art}} \cong \mathrm{Def}(\tilde{X})/\prod W_j,$$

where $(\mathrm{Def}(X))_{\mathrm{art}}$ is a component of $\mathrm{Def}(X)$, called the Artin component and W_j are the finite Weyl groups belonging to the maximal connected (-2)-configurations, i.e. the rational double point configurations supported by the exceptional set of $\tilde{X}$. This theorem was first considered by Brieskorn[2], [3] and Tyurina[15] for a singularity of type ADE, using the theory of simple complex Lie groups and their Weyl groups of corresponding type. In the case of all cyclic quotient singularities, by following the method of Riemenschneider [9], one can construct a canonical candidate for the full Weyl group explicitly.

In this paper, we consider the case of dihedral singularities.

This result will be useful for consideration of deformations in terms of representations of quivers. The quiver-theoretic approach was proposed by P. Kronheimer [5] and then Ebeling, Slodowy[14] and Cassens[4] constructed the versal deformation using representations of the quivers for Kleinian singularity. Riemenschneider [11] generalized the method to yield the monodromy covering of the Artin component for all cyclic quotients using the special representations by the work of Wunram[18] and Riemenschneider[12]. To extend the Riemenschneider's result to the dihedral singularity, we need to calculate the Artin component, and which is the motivation of this paper.

In Section 2, we obtain generators of the graded affined algebra of its minimal resolution (Theorem 4) using Pinkham result[7]. Then in Section 3, we extend these generators to the deformation space of the minimal resolution (Theorem 5) and the results give the Artin components.

2 Generators of the graded affined algebra

In what follows, we denote $b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_r}}}$ by $b_1 - \underline{1}\rceil\overline{b_2} - \cdots - \underline{1}\rceil\overline{b_r}$ and let $\{r\}$ always be the least integer greater than or equal to r.

Let n, q be positive integers with $1 < q < n$ and $\gcd(n, q) = 1$. We define 2×2 matrices φ, ϕ, τ by $\varphi = \begin{pmatrix} \eta_{2q} & 0 \\ 0 & \eta_{2q}^{-1} \end{pmatrix}$, $\phi = \begin{pmatrix} \eta_{2m} & 0 \\ 0 & \eta_{2m} \end{pmatrix}$ and $\tau = \begin{pmatrix} 0 & \eta_4 \\ \eta_4 & 0 \end{pmatrix}$ where $\eta_k = \exp(2\pi i/k)$. The group $D_{n,q} \subset \mathrm{GL}(2, \mathbf{C})$ generated by

$$\begin{cases} \varphi, \phi, \tau, & \text{if } m = n - q \equiv 1 \pmod 2 \\ \varphi, \phi \circ \tau, & \text{if } m = n - q \equiv 0 \pmod 2 \end{cases}$$

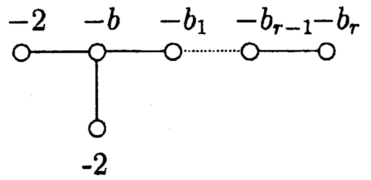
determines the dihedral singularity of the quotient $\mathbf{C}^2/D_{n,q}$.

We set $n/q = b - \tilde{q}/\tilde{n} = b - \underline{1}\rceil\overline{b_1} - \cdots - \underline{1}\rceil\overline{b_r}$ with $b, b_1, b_2, \cdots, b_r \geq 2$.

The minimal resolution for the dihedral singularity of type $D_{n,q}$ is written down explicitly by the union of $4 + r$ copies of $\mathbf{C}^2$;

$$\begin{array}{llll} u = \dfrac{1}{u''_0}, & v = u''^2_0 v''_0, & u'_0 = \dfrac{1}{u}, & v'_0 = u^2(v-1), \\ v_1 = \dfrac{1}{v}, & u_1 = v^b u, & & \\ u_2 = \dfrac{1}{u_1}, & v_2 = u_1^{b_1} v_1, & & \\ v_3 = \dfrac{1}{v_2}, & u_3 = v_2^{b_2} u_2, & & \\ & \vdots & & \end{array} \tag{1}$$

and its dual graph is



Let A be the graded affine algebra of X. We denote by D the divisor of positive degree $b = -(E_0 \cdot E_0)$ on E_0, where E_0 is the central curve of the resolution. Let $D^{(l)}$ be the divisor

$$D^{(l)} = lD - \{\frac{l}{2}\}y_1 - \{\frac{l}{2}\}y_2 - \{\frac{\tilde{q}l}{\tilde{n}}\}y_3$$

where y_i are the intersection points of E_0 with the other components of the exceptional set. By applying Pinkham result[7] to the our case, we have $A = \bigoplus_{l \geq 0} A_l$ with $A_l = H_0(E_0, \mathcal{O}_E(D^{(l)}))$. By taking three points $y_1 = 0$, $y_2 = 1$, $y_3 = \infty$, one sees immediately that generators of A in degree l are, for example,

$$u^l v^t (v-1)^s \quad \text{for} \quad \{l/2\} \leq t \leq bl - \{\tilde{q}l/\tilde{n}\} - \{l/2\}, s = \{l/2\}.$$

To get minimal generators, we use the following theorem in Riemenschneider [9]. Let $\tilde{n}/(\tilde{n}-\tilde{q}) = a_1 - \underline{1|}\overline{a_2} - \cdots - \underline{1|}\overline{a_p}$ with $a_1, a_2, \cdots, a_p \geq 2$ and let

$$\begin{array}{lll} i_0 = \tilde{n}, & i_1 = \tilde{n}-\tilde{q}, & i_e = a_{e-1}i_{e-1} - i_{e-2} \\ j_0 = 0, & j_1 = 1, & j_e = a_{e-1}j_{e-1} - j_{e-2} \\ k_0 = 1, & k_1 = 1, & k_e = a_{e-1}k_{e-1} - k_{e-2} \quad \text{for} \quad 2 \leq e \leq p+1. \end{array}$$

Theorem 1 (Riemenschneider[9]) *Then*

1. $p = 1 + \sum_{i=1}^{r}(b_i - 2)$.
2. $i_0 = \tilde{n} > i_1 = \tilde{n} - \tilde{q} > i_3 > \cdots > i_p = 1 > i_{p+1} = 0$.
3. $i_e + \tilde{q}j_e = \tilde{n}k_e$, *i.e.,* $\{\tilde{q}j_e/\tilde{n}\} = k_e$ *for* $e \geq 1$.
4. *For integers* i, j, k, *if* $i + \tilde{q}j = \tilde{n}k$, $0 \leq i < \tilde{n}$ *and* $j_e < j < j_{e+1}$, *then it holds that* $i \geq i_e$.
5. *For* $e = \sum_{m=1}^{m_0-1}(b_m - 2) + h$ *and* $b_{m_0} \geq 3$,

$$a_e = \begin{cases} m_0 - m_1 + 2 & \textit{if } h = 1, 1 \leq m_0 \leq r, \\ & \textit{and if } b_1 = b_2 = \cdots = b_{m_0-1} = 2, \textit{ then } m_1 = 1, \\ & \textit{otherwise } \exists b_{m_1} \neq 2, \\ & b_{m_1+1} = b_{m_1+2} = \cdots = b_{m_0-1} = 2. \\ 2 & \textit{if } 2 \leq h \leq b_{m_0} - 2, 1 \leq m_0 \leq r. \\ r - m_1 + 2 & \textit{if } h = 1, m_0 = r+1, \\ & \textit{and if } b_1 = b_2 = \cdots = b_r = 2, \textit{ then } m_1 = 1, \\ & \textit{otherwise } \exists b_{m_1} \neq 2, \\ & b_{m_1+1} = b_{m_1+2} = \cdots = b_{m_0-1} = 2. \end{cases}$$

Corollary 1 *If* $j < j_2$ *then* $\{\tilde{q}j/\tilde{n}\} = j$ *and if* $j \geq j_2$ *then* $\{\tilde{q}j/\tilde{n}\} \leq j - 1$.

Corollary 2 *If* $j_{e-1} < j < j_e$ *then* $\{\tilde{q}j/\tilde{n}\} = \{\tilde{q}j_e/\tilde{n}\} + \{\tilde{q}(j - j_e)/\tilde{n}\}$. *Moreover, for any* $l_1 \geq 1$ *and* $l_2 \geq 1$ *such that* $l_1 + l_2 = j_e$, *it holds that* $\{\tilde{q}j_e/\tilde{n}\} = \{\tilde{q}l_1/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\} + 1$.

(Proof.) By Theorem 1 (3) and (4), $\{\tilde{q}(j - j_e)/\tilde{n}\} = \{\tilde{q}j/\tilde{n}\} - k_e$. The second statement follows from Theorem 1 (2), (3) and (4). Q.E.D.

In order to clear the relation between a_e and $\mathbf{C}(u_m, v_m)$, let us prepare the following Lemma.

Theorem 2 *Assume that* $b_1 = b_2 = \cdots = b_{m_0-1} = 2, b_{m_0} \neq 2$ *(if* $b_1 \neq 2$, *then* $m_0 = 1$*) and that two sequences of numbers* s_e $(1 \leq e \leq p+1)$ *and* $l_m^{(h_m)}$ $(1 \leq m \leq r, 1 \leq h_m \leq b_m)$ *satisfy*

$$s_{e+1} = a_e s_e - s_{e-1} \tag{2}$$

$$l_m^{(1)} = l_{m-1}^{(b_{m-1}-1)}, \qquad l_m^{(h_m)} = l_m^{(h_m-1)} + l_{m-1}^{(b_m-1)} \text{ for } h_m = 2, \cdots, b_m. \tag{3}$$

Then if $s_1 = l_1^{(1)}$ *and* $s_2 = l_{m_0}^{(2)}$, *it holds that* $s_e = l_m^{(h_m)}$ *where* $e = \sum_{i=1}^{m-1}(b_i - 2) + h_m$, $2 \leq h_m \leq b_m - 1$, $m = 1, \cdots, r$.

Also it holds that $s_{p+1} = l_r^{(b_r)}$.

(Proof.) Let us prove it by induction. Suppose we have $s_{e'} = l_{m'}^{(h_{m'})}$ for $e' = \sum_{i=1}^{m'-1}(b_i - 2) + h_{m'} < \sum_{i=1}^{m-1}(b_i - 2) + h_m$.

If $h_m \geq 3$, then $s_{e-1} = l_m^{(h_m-1)}$, $s_{e-2} = l_m^{(h_m-2)}$ and $s_{e-1} - s_{e-2} = l_{m-1}^{(b_{m-1})}$. Since $b_m \geq 4$, we have $a_{e-1} = 2$ using Theorem 1 (5). Thus, we have $s_e = 2s_{e-1} - s_{e-2} = l_m^{(h_m-1)} + l_{m-1}^{(b_{m-1})} = l_m^{(h_m)}$.

If $h_m = 2$ and $m = m_0$, then $e = \sum_{i=1}^{m-1}(b_i - 2) + h_m = 2$.

If $h_m = 2$, $b_{m_1} \neq 2$ and $b_{m_1+1} = b_{m_1+2} = \cdots = b_{m-1} = 2$ then $s_{e-1} = l_m^{(1)} = l_{m-1}^{(1)} = \cdots = l_{m_1+1}^{(1)} = l_{m_1}^{(b_{m_1}-1)}$, $s_{e-2} = l_{m_1}^{(b_{m_1}-2)}$ and $l_{m-1}^{(2)} = l_{m-1}^{(1)} + l_{m-2}^{(2)} = 2l_{m-2}^{(1)} + l_{m-3}^{(2)} = \cdots = (m - m_1 - 1)l_{m_1+1}^{(1)} + l_{m_1}^{(b_{m_1})} = (m - m_1 - 1)l_{m_1}^{(b_{m_1}-1)} + l_{m_1}^{(b_{m_1})}$. Therefore,

$$\begin{aligned} s_e &= a_{e-1}s_{e-1} - s_{e-2} = (m - m_1 + 2)l_{m_1}^{(b_{m_1}-1)} - l_{m_1}^{(b_{m_1}-2)} \\ &= (m - m_1 + 1)l_{m_1}^{(b_{m_1}-1)} + l_{m_1}^{(b_{m_1}-1)} - l_{m_1}^{(b_{m_1}-2)} \\ &= (m - m_1 + 1)l_{m_1}^{(b_{m_1}-1)} + l_{m_1-1}^{(b_{m_1-1})} \end{aligned}$$

and

$$\begin{aligned} & l_m^{(2)} = l_m^{(1)} + l_{m-1}^{(2)} = l_{m_1}^{(b_{m_1}-1)} + (m - m_1 - 1)l_{m_1}^{(b_{m_1}-1)} + l_{m_1}^{(b_{m_1})} \\ =\ & (m - m_1)l_{m_1}^{(b_{m_1}-1)} + l_{m_1}^{(b_{m_1})} = (m - m_1 + 1)l_{m_1}^{(b_{m_1}-1)} + l_{m_1}^{(b_{m_1})} - l_{m_1}^{(b_{m_1}-1)} \\ =\ & (m - m_1 + 1)l_{m_1}^{(b_{m_1}-1)} + l_{m_1-1}^{(b_{m_1-1})}. \end{aligned}$$

That is, $s_e = l_m^{(2)}$.

If $h_r = b_r$, $m = r$, $b_{m_1} \neq 2$ and $b_{m_1+1} = b_{m_1+2} = \cdots = b_r = 2$, since $p = 1 + \sum_{i=1}^{r}(b_i - 2)$, we have

$$\begin{aligned} s_{p+1} &= (r - m_1 + 2)s_p - s_{p-1} = (r - m_1 + 2)l_{m_1}^{(b_{m_1}-1)} - l_{m_1}^{(b_{m_1}-2)} \\ &= (r - m_1 + 2)l_{m_1}^{(b_{m_1}-1)} + l_{m_1-1}^{(b_{m_1-1})} - l_{m_1}^{(b_{m_1}-1)} \\ &= (r - m_1 + 1)l_{m_1}^{(b_{m_1}-1)} + l_{m_1-1}^{(b_{m_1-1})} = l_r^{(b_r-1)} + l_{r-1}^{(b_{r-1})} \\ &= l_r^{(b_r)} \end{aligned}$$

Q.E.D.

Let us define two sequences of integers n_m, q_m by

$$n_0 = 1, q_0 = 0, n_m/q_m = b_1 - \underline{1}\rfloor\overline{b_2} - \cdots - \underline{1}\rfloor\overline{b_m}, \quad \gcd(n_m, q_m) = 1$$

for $m = 1, 2, \cdots, r$. Those numbers n_m, q_m satisfy that

$$\begin{array}{lll} n_0 = 1, n_1 = b_1, & n_m = b_m n_{m-1} - n_{m-2}, & n_m > n_{m-1}, \\ q_0 = 0, q_1 = 1, q_2 = b_2, & q_m = b_m q_{m-1} - q_{m-2}, & q_m > q_{m-1}, \end{array}$$

for $m = 2, \cdots, r$.

Corollary 3 *We have $j_e = h_m n_{m-1} - n_{m-2}$ and $k_e = h_m q_{m-1} - q_{m-2}$ where $m = 1, \cdots, r$, $2 \leq h_m \leq b_m - 1$ and $e = \sum_{s=1}^{m-1}(b_s - 2) + h_m$. Also $j_{p+1} = \tilde{n} = n_r = b_r n_{r-1} - n_{r-2}$ and $k_{p+1} = \tilde{q} = q_r = b_r q_{r-1} - q_{r-2}$.*

(Proof.) Assume that $b_1 = b_2 = \cdots = b_{m_0-1} = 2, b_{m_0} \neq 2$. Let $n_{-1} = 0$ and $l_m^{(h_m)} = h_m n_{m-1} - n_{m-2} (1 \leq h_m \leq b_m - 1)$. Then, we have $1 = j_1 = n_0 = l_1^{(1)}$ and $j_2 = a_1 = m_0 + 1 = 2n_{m_0-1} - n_{m_0-2} = l_{m_0}^{(2)}$ by $n_m = 2n_{m-1} - n_{m-2} = m + 1$ for $m = 1, \cdots, m_0$. Also we have

$$\begin{aligned} l_m^{(b_m)} &= b_m n_{m-1} - n_{m-2} = n_m \\ l_m^{(1)} &= n_{m-1} - n_{m-2} = b_{m-1} n_{m-2} - n_{m-3} - n_{m-2} \\ &= (b_{m-1} - 1) n_{m-2} - n_{m-3} = l_{m-1}^{(b_{m-1}-1)} \\ l_m^{(h_m)} &= h_m n_{m-1} - n_{m-2} = (h_m - 1) n_{m-1} - n_{m-2} + n_{m-1} \\ &= l_m^{(h_m-1)} + l_{m-1}^{(b_m-1)} \text{ for } h_m = 2, \cdots, b_m. \end{aligned}$$

Therefore by Theorem 2, we have the proof for j_e.

By setting $q_{-1} = 0$ and $\tilde{l}_m^{(h_m)} = h_m q_{m-1} - q_{m-2} (1 \leq h_m \leq b_m - 1)$, similarly we have the proof for k_e.

Q.E.D.

Corollary 4 *Consider the coordinate system $C(u_m, v_m)$ defined by the equations (1). We have* $u^{j_e} v^{bj_e - k_e} = u_1^{j_e} v_1^{k_e} = \begin{cases} u_m v_m^{h_m} & m \text{ odd} \\ v_m u_m^{h_m} & m \text{ even} \end{cases}$ *for $m = 1, \cdots, r$, $2 \leq h_m \leq b_m - 1$ and $e = \sum_{s=1}^{m-1} (b_s - 2) + h_m$.*

Theorem 3 *Minimal generators of holomorphic functions on $\tilde{X}$ defined everywhere are the following.*

Let $s_1 = ((-1)^{j_2+1} + 1)/2$, $s_2 = \{j_2/2\}$, $s_e = a_{e-1} s_{e-1} - s_{e-2}$ for $3 \leq e \leq p + 1$.

1. *When $b = b_1 = \cdots = b_{m_0} = 2$, (if $b_1 > 0$, then we put $m_0 = 0$)*

$$u^l (v(v-1))^s (v - 1/2)^t$$

$$\begin{cases} = g_0, & l = 2, & s = 1, & t = 0 \\ = g_1, & l = j_2 + 1, & s = \{(j_2 + 1)/2\}, & t = ((-1)^{j_2+1} + 1)/2 \\ = f_e, & l = j_e, & s = s_e, & t = 2j_e - k_e - 2s_e \\ & & 2 \leq e \leq p + 1 & \end{cases}$$

and the relation is

$$\text{rank} \begin{pmatrix} 1, & g_0, & g_1, & f_{e-1}^{a_{e-1}-1} f_{e-2}^{a_{e-2}-2} \cdots f_3^{a_3-2} f_2^{a_2-2} \\ U, & g_1, & f_2^2 + \frac{(-g_0)^{a_1}}{4}, & f_e \end{pmatrix} < 2,$$

where $U = u^{j_2-1} (v(v-1))^{\{(j_2-1)/2\}} (v - 1/2)^{((-1)^{j_2-1}+1)/2}$ and $3 \leq e \leq p + 1$.

2. *When $b > 2$,*

$$u^l (v(v-1))^s (v - 1/2)^t \quad \begin{cases} = f_1^{(t+2)}, & l = 1, & s = 1, & t = 0, \cdots, b - 3 \\ = g_0, g_1, & l = 2, & s = 1, & t = 0, 1 \\ = f_e, & l = j_e, & s = j_e, & t = bj_e - k_e - 2j_e \\ & & & 2 \leq e \leq p + 1, \end{cases}$$

and the relation is

$$\operatorname{rank}\begin{pmatrix} 1, & g_0, & g_1, & f_1^{(2)}, & \cdots, & f_1^{(b-2)} \\ v-\frac{1}{2}, & g_1, & (f_1^{(2)})^2-\frac{g_0}{4}, & f_1^{(3)}, & \cdots, & f_1^{(b-1)} \end{pmatrix} < 2$$

$$\operatorname{rank}\begin{pmatrix} 1, & (f_1^{(b-2)})^{a_1}, & f_{e-1}^{a_{e-1}-1} f_{e-2}^{a_{e-2}-2} \cdots f_2^{a_2-2} (f_1^{(b-2)})^{a_1-1} \\ v-\frac{1}{2}, & f_2, & f_e \end{pmatrix} < 2$$

for $3 \le e \le p+1$.

To prove the theorem, we prepare two lemmas.
In general, $A_{l_1} \cdot A_{l_2} \subset A_{l_1+l_2}$ since $\{m\}+\{n\} \ge \{m+n\}$. On the other hand,

Lemma 1 *$u^l v^t (v-1)^s \in A_l$ is an element of $A_{l_1} \cdot A_{l_2}$ if and only if $l = l_1 + l_2$, $t \ge \{l_1/2\} + \{l_2/2\}$, $s \ge \{l_1/2\} + \{l_2/2\}$, $t + s \le bl - \{\tilde{q}l_1/\tilde{n}\} - \{\tilde{q}l_2/\tilde{n}\}$.*

(Proof.) Recall that $A_l = \{u^l v^t (v-1)^s; t, s \ge \{l/2\}, t+s \le bl - \{\tilde{q}l/\tilde{n}\}\ \}$.

Q.E.D.

Lemma 2 *For $u^l v^t (v-1)^s \in A_l$, we have the following.*

1. *For $l \ge 3$, $t+s \le bl - \{\tilde{q}l/\tilde{n}\} - 1$, we have $u^l v^t (v-1)^s \in A_{l-2} \cdot A_2$.*
2. *We assume that $b = b_1 = \cdots = b_{m_0} = 2$, $l = l_1 + l_2$ and $\{\tilde{q}l/\tilde{n}\} = \{\tilde{q}l_1/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\}$ $(l_1 \ge 1, l_2 \ge 1)$. For $s = \{l/2\}$ and $t + s = bl - \{\tilde{q}l/\tilde{n}\}$, we have*
   (a) *if $2 \le l_1 < j_2 = m_0 + 2$, then $u^l v^t (v-1)^s \in A_2 \cdot A_{l-2}$.*
   (b) *if $l_1, l_2 \ge j_2$, then $u^l v^t (v-1)^s \in A_{l_1} \cdot A_{l_2} + A_{l-2} \cdot A_2$.*
   (c) *if $l_1 = 1$ and $l_2 \ge j_2 + 1$, then $u^l v^t (v-1)^s \in A_2 \cdot A_{l-2}$.*
3. *We assume that $b > 2$, $l = l_1 + l_2 \ge 3$ and $\{\tilde{q}l/\tilde{n}\} = \{\tilde{q}l_1/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\}$ $(l_1 \ge 1, l_2 \ge 1)$. For $s = \{l/2\}$ and $t+s = bl - \{\tilde{q}l/\tilde{n}\}$, we have $u^l v^t (v-1)^s \in A_{l_1} \cdot A_{l_2} + A_{l-2} \cdot A_2$.*

(Proof.)

1. It follows from Lemma 1 and the fact $\{\tilde{q}l/\tilde{n}\} + 1 \ge \{\tilde{q}(l-2)/\tilde{n}\} + \{\tilde{q}2/\tilde{n}\}$ and $\{(l-2)/2\} + \{2/2\} = \{l/2\}$.
2. (a) $l_1 = \{\tilde{q}l_1/\tilde{n}\} = 2 + l_1 - 2 = \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l_1-2)/\tilde{n}\}$ by Corollary 1. Thus,

$$\begin{aligned} \{\tilde{q}l/\tilde{n}\} &= \{\tilde{q}l_1/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\} = \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l_1-2)/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\} \\ &\ge \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l_1+l_2-2)/\tilde{n}\} = \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l-2)/\tilde{n}\} \\ &\ge \{\tilde{q}l/\tilde{n}\}, \end{aligned}$$

that is, $\{\tilde{q}l/\tilde{n}\} = \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l-2)/\tilde{n}\}$. Therefore by Lemma 1, $u^l v^t (v-1)^s \in A_2 \cdot A_{l-2}$.

(b) By Corollary 1, $\{\tilde{q}l_1/\tilde{n}\} \leq l_1 - 1$ and $\{\tilde{q}l_2/\tilde{n}\} \leq l_2 - 1$. Thus, since for $s' = \{l_1/2\} + \{l_2/2\}$, we have $t' = 2l - \{\tilde{q}l/\tilde{n}\} - s' \geq \{l_1/2\} + \{l_1/2\}$. By Lemma 1, $u^l v^{t'}(v-1)^{s'} \in A_{l_1} \cdot A_{l_2}$.
If $l_1 l_2$ is even, $t' = t$ and $s' = s$. If l_1 and l_2 are odd, $s' = \{l_1/2\} + \{l_2/2\} = \{l/2\} + 1 = s + 1$. Then, using Lemma 2 (1) and

$$u^l v^t (v-1)^s = u^l v^{t-1}(v-1)^{s+1} + u^l v^{t-1}(v-1)^s, \tag{4}$$

we have $u^l v^t (v-1)^s \in A_{l_1} \cdot A_{l_2} + A_{l-2} \cdot A_2$.

(c) By Corollary 1, $\{\tilde{q}(j_2 - 1)/\tilde{n}\} = j_2 - 1 = k_2 = \{\tilde{q}j_2/\tilde{n}\}$. Thus,

$$\begin{aligned} 1 + \{\tilde{q}l_2/\tilde{n}\} &= \{\tilde{q}(1+l_2)/\tilde{n}\} \leq \{\tilde{q}j_2/\tilde{n}\} + \{\tilde{q}(l_2 - j_2 + 1)/\tilde{n}\} \\ &= \{\tilde{q}(j_2 - 1)/\tilde{n}\} + \{\tilde{q}(l_2 - j_2 + 1)/\tilde{n}\} \leq 1 + \{\tilde{q}l_2/\tilde{n}\}, \end{aligned}$$

that is, $\{\tilde{q}(j_2 - 1)/\tilde{n}\} + \{\tilde{q}(l_2 - j_2 + 1)/\tilde{n}\} = \{\tilde{q}(l_2 + 1)/\tilde{n}\}$. It comes back to (a).

3. For $s' = \{l_1/2\} + \{l_2/2\}$, we have $t' = bl - \{\tilde{q}l/\tilde{n}\} - s' \geq \{l_1/2\} + \{l_1/2\}$. By Lemma 1, $u^l v^{t'}(v-1)^{s'} \in A_{l_1} \cdot A_{l_2}$. Using Lemma 2 (1) and the equation (4) again, we have $u^l v^t (v-1)^s \in A_{l_1} \cdot A_{l_2} + A_{l-2} \cdot A_2$.

Q.E.D.

(The proof of Theorem 3.)
In the case of $b = b_1 = \cdots = b_{m_0} = 2$, one sees $A_j = 0$ for $1 \leq 2j - 1 < j_2$, since $b(2j-1) - \{\tilde{q}(2j-1)/\tilde{n}\} = 2(2j-1) - (2j-1) = 2j - 1 < 2j = 2\{(2j-1)/2\}$ by Corollary 1. Also one sees that $j_2 + 1$ has only $l_1 = 1$ and $l_2 = j_2$ such as $j_2 + 1 = l_1 + l_2$ and $\{\tilde{q}(j_2+1)/\tilde{n}\} = \{\tilde{q}l_1/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\}$. Therefore, by Lemma 2 and Corollary 2, we have

$$\begin{aligned} A_2 &= \{u^2 v(v-1)\}, \\ A_{j_2+1} &\ni u^{j_2+1} v^{2(j_2+1) - \{\tilde{q}(j_2+1)/\tilde{n}\} - \{(j_2+1)/2\}} (v-1)^{\{(j_2+1)/2\}}, \\ A_{j_e} &\ni u^{j_e} v^t (v-1)^s, \quad e \geq 2, s = \{j_e/2\}, t = 2j_e - \{\tilde{q}j_e/\tilde{n}\} - s \end{aligned}$$

are minimal generators. Since one sees easily $s_e \geq j_e/2$, $2j_e - k_e - s_e \geq j_e/2$, we have generators, substituting $s = s_e$ for $s = \{j_e/2\}$, using the equation (4). Finally by

$$u^l (v(v-1))^s (v - \frac{1}{2})^t = u^l v^{s+t} (v-1)^s + \cdots + (-\frac{1}{2})^t u^l v^s (v-1)^s, \tag{5}$$

the proof is completed for $b = 2$.
In the case of $b \geq 3$, easy computation yields that A_1 and A_2 are generated by $uv^t(v-1), t = 1, \cdots, b-2$ and $u^2 v^t (v-1), t = 1, 2$. By Lemma 2 and Corollary 2, we have

$$\begin{aligned} A_1 &\ni uv^t(v-1), t = 1, \cdots, b-2, \\ A_2 &\ni u^2 v^t (v-1), t = 1, 2, \\ A_{j_e} &\ni u^{j_e} v^t (v-1)^s, \quad e \geq 2, s = \{j_e/2\}, t = bj_e - \{\tilde{q}j_e/\tilde{n}\} - s \end{aligned}$$

are minimal generators. Again using the equation (4), we have generators, substituting $s = j_e$ for $s = \{j_e/2\}$ and then the equation (5) completes the proof for $b > 2$.

The relations are obtained using $j_e = a_{e-1}j_{e-1} - j_{e-2} = (a_{e-1} - 1)j_{e-1} + j_{e-1} - j_{e-2} = \cdots = (a_{e-1} - 1)j_{e-1} + (a_{e-2} - 2)j_{e-2} + (a_{e-3} - 2)j_{e-3} + \cdots + (a_1 - 2)j_1 + j_1 - j_0$, etc..

Q.E.D.

Theorem 4 *Let* $n/(n-q) = a'_1 - \underline{1|}\, \overline{a'_2} - \cdots - \underline{1|}\, \overline{a'_p}$ *with* $a'_1, a'_2, \cdots, a'_p \geq 2$ *and let*

$$\begin{aligned} &j'_0 = 0, \quad j'_1 = 1, \quad j'_e = a'_{e-1}j'_{e-1} - j'_{e-2} \\ &k'_0 = 1, \quad k'_1 = 1, \quad k'_e = a'_{e-1}k'_{e-1} - k'_{e-2} \quad \text{for} \quad 2 \leq e \leq p' + 1. \end{aligned}$$

Also let $s'_1 = ((-1)^{k'_2+1} + 1)/2$, $s'_2 = \{k'_2/2\}$, $s'_e = a'_{e-1}s'_{e-1} - s'_{e-2}$ *for* $3 \leq e \leq p' + 1$. *Minimal generators of holomorphic functions on* $\tilde{X}$ *defined everywhere are* $u^l(v(v-1))^s(v-1/2)^t$

$$\left\{ \begin{array}{llll} = g_0, & l = 2, & s = 1, & t = 0 \\ = g_1, & l = k'_2 + 1, & s = \{(k'_2 + 1)/2\}, & t = ((-1)^{k'_2+1} + 1)/2 \\ = f'_e, & l = k'_e, & s = s'_e, & t = j'_e - 2s'_e \\ & 2 \leq e \leq p' + 1 & & \end{array} \right.$$

and the relations are given by all the 2×2*-minors of the matrices;*

$$\begin{pmatrix} g_0, & g_1, & f'^{a'_{e-1}-1}_{e-1} f'^{a'_{e-2}-2}_{e-2} \cdots f'^{a'_3-2}_3 f'^{a'_2-2}_2 \\ g_1, & f'^2_2 + \frac{(-g_0)^{a'_1-1}}{4}, & f'_e \end{pmatrix}$$

for $3 \leq e$ *and given by all the generalized* 2×2*-minors of the quasi-matrix;*

$$\begin{pmatrix} f'_2 & & f'_3 & & f'_4 & & \cdots & f'_{p'} \\ & f'^{a'_3-2}_3 & & f'^{a'_4-2}_4 & & \cdots & & \\ f'_3 & & f'_4 & & f'_5 & & \cdots & f'_{p'+1} \end{pmatrix}.$$

(Proof.) When $b = 2$, we have $a'_1 = a_1 + 1$, $a'_e = a_e$ for $e = 2, \cdots, p + 1 = p' + 1$ by Theorem 1 (5). Thus $j'_1 = 1 = 2j_1 - k_1$, $j'_2 = a'_2 = a_1 + 1 = j'_2 + 1 = 2j_2 - k_2$ and $j'_e = 2j_e - k_e$ for $e = 3, \cdots, p + 1 = p' + 1$. Also $k'_1 = 1 = j_1$, $k'_2 = a'_1 - 1 = a_1 = j_2$ and $k'_e = j_e$ for $e = 3, \cdots, p + 1 = p' + 1$.

When $b > 2$, we have $a'_1 = \cdots = a'_{b-2} = 2$, $a'_{b-1} = a_1 + 1$, $a'_e = a_{e-b+2}$ for $e = b, \cdots, p' + 1 = p + b - 1$ by Theorem 1 (5). Thus $j'_1 = 1$, $j'_2 = 2$, $\cdots$, $j'_{b-1} = b - 1$, $j'_b = a'_{b-1}j'_{b-1} - j'_{b-2} = (a_1 + 1)j'_{b-1} - j'_{b-2} = bj_2 - k_2$ and $j'_e = bj_{e-b+2} - k_{e-b+2}$ for $e = b, \cdots, p' + 1 = p + b - 1$. Also $k'_1 = k'_2 = \cdots = k'_{b-1} = 1$, $k'_b = a'_{b-1}k'_{b-1} - k'_{b-2} = a_1 + 1 - 1 = j_2$ and $k'_e = j_{e-b+2}$ for $e = b, \cdots, p' + 1 = p + b - 1$. Q.E.D

3 Extended function

Next let us extend these generators to holomorphic functions on the deformation space of the minimal resolution.

For variables $T = \{t"^{(1)}_1, t"^{(1)}_2, t^{(1)}_0, \ldots, t^{(b-1)}_0, t^{(i)}_j : j = 1, \ldots, r, i = 1, \ldots, b_j - 1\}$, consider the versal deformation space of the minimal resolution:

$$\begin{array}{ll}
u = \frac{1}{u"_0}, & v = u"^2_0 v"_0 + t"^{(1)}_1 u"_0, \\
u'_0 = \frac{1}{u}, & v'_0 = u^2(v-1) + t"^{(1)}_2 u, \\
v_1 = \frac{1}{v}, & u_1 = v^b u + t^{(1)}_0 v^{b-1} + \cdots + t^{(b-1)}_0 v, \\
u_2 = \frac{1}{u_1}, & v_2 = u_1^{b_1} v_1 + t^{(1)}_1 u_1^{b_1-1} + \cdots + t^{(b_1-1)}_1 u_1, \\
v_3 = \frac{1}{v_2}, & u_3 = v_2^{b_2} u_2 + t^{(1)}_2 v_2^{b_2-1} + \cdots + t^{(b_2-1)}_2 v_2, \\
 & \vdots
\end{array}$$

Let

$$\begin{array}{rcl}
H^{(0)}_0 & = & u \\
H^{(1)}_0 & = & uv + t^{(1)}_0 \\
H^{(2)}_0 & = & (uv + t^{(1)}_0)v + t^{(2)}_0 \\
 & \vdots & \\
H^{(b-1)}_0 & = & (\cdots(uv + t^{(1)}_0)v + t^{(2)}_0)v + \cdots + t^{(b-2)}_0)v + t^{(b-1)}_0 \\
H^{(1)}_1 & = & H^{(b-1)}_0 + t^{(1)}_1 = u_1 v_1 + t^{(1)}_1 \\
H^{(2)}_1 & = & H^{(1)}_1 H^{(b-1)}_0 v + t^{(2)}_1 = (u_1 v_1 + t^{(1)}_1)u_1 + t^{(2)}_1 \\
 & \vdots & \\
H^{(b_1-1)}_1 & = & H^{(b_1-2)}_1 H^{(b-1)}_0 v + t^{(b_1-1)}_1 \\
 & = & (\cdots(u_1 v_1 + t^{(1)}_1)u_1 + t^{(2)}_1)u_1 + \cdots + t^{(b_1-2)}_1)u_1 + t^{(b_1-1)}_1 \\
H^{(1)}_2 & = & H^{(b_1-1)}_1 + t^{(1)}_2 = u_2 v_2 + t^{(1)}_2 \\
H^{(2)}_2 & = & H^{(1)}_2 H^{(b_1-1)}_1 H^{(b-1)}_0 v + t^{(2)}_2 = (u_2 v_2 + t^{(1)}_2)v_2 + t^{(2)}_2 \\
H^{(3)}_2 & = & H^{(2)}_2 H^{(b_1-1)}_1 H^{(b-1)}_0 v + t^{(3)}_2 = ((u_2 v_2 + t^{(1)}_2)v_2 + t^{(2)}_2)v_2 + t^{(3)}_2 \\
 & \vdots & \\
 & \vdots & \\
H^{(b_r)}_r & = & H^{(b_r-1)}_r H^{(b_{r-1}-1)}_{r-1} \cdots H^{(b-1)}_0 v
\end{array}$$

These functions are holomorphic defined everywhere on $\mathbf{C}^2$ of the coordinate systems $C(u,v)$ and $C(u_m, v_m)$ $(m \geq 1)$, which are introduced by Riemenschneider [9] as "extended functions" of generators $u^{j_e}v^{k_e}$ for corresponding cyclic quotient singularity $C_{n,q}$.

Using these functions we will construct extended functions for $D_{n,q}$ singularities, which become extremely more complicated than those for cyclic singularities.

Let $w_{-2} = -(t"^{(1)}_2 + t"^{(1)}_1)/2$, $w_{-1} = (t"^{(1)}_2 - t"^{(1)}_1)/2$, $w_0 = t^{(1)}_0 - (t"^{(1)}_2 - t"^{(1)}_1)/2$, $w_l = t^{(1)}_0 + t^{(1)}_1 + \cdots + t^{(1)}_l + (-t"^{(1)}_2 + t"^{(1)}_1)/2$ for $l = 1, \cdots, k'_2 - 1$. Also let us denote $\sum_{i_0 \leq j_1 < j_2 < \cdots < j_k \leq i} w_{j_1} w_{j_2} \cdots w_{j_k}$ by $\sum^i_{i_0} \mathbf{w}_{J_k}$.

Let

$$G_0 = (u(v-1) + w_{-1} - w_{-2})(uv + w_{-1} + w_{-2}) + w^2_{-2}, \tag{6}$$

$$X_0 = v(u(v-1) + w_{-1} + w_0) - 1/2 \sum^0_{-2} \mathbf{w}_{J_1} \tag{7}$$

$$X_1 = (G_0 - w_0^2)(v - 1/2) + (w_0 + w_1)X_0 - 1/2 \prod_{k=-2}^{-1} w_k \tag{8}$$

Inductively, let X_l $(l \geq 2)$ be

$$\begin{aligned} X_l &= X_{l-2}\{G_0 - (w_{l-1})^2\} + (w_{l-1} + w_l)X_{l-1} - \frac{1}{2}\prod_{k=-2}^{l-2} w_k \\ &= v(u(v-1)\sum_{k=0}^{\{(l-1)/2\}} G_0^k \sum_{0}^{l} \mathbf{w}_{J_{l-2k}} + \sum_{k=0}^{\{l/2\}} G_0^k \sum_{-1}^{l} \mathbf{w}_{J_{l+1-2k}}) \\ &- \frac{1}{2}\sum_{k=0}^{\{l/2\}} G_0^k \sum_{-2}^{l} \mathbf{w}_{J_{l+1-2k}} \end{aligned} \tag{9}$$

Lemma 3 *G_0 is a holomorphic extended function of g_0.*

(Proof.) Because it is clear that $G_0|_{T=0} = g_0$, we need to show that G_0 is holomorphic on each coordinate system and it is proved by

$$\begin{aligned} G_0 &= (u"_0 v"_0 - w_{-2})^2 - v"_0 \\ &= (u'_0 v'_0 + w_{-2})^2 + v'_0 \\ &= (H_0^{(1)} - w_0)^2 - H_0^{(0)}(H_0^{(1)} - w_0 + w_{-2}). \end{aligned}$$

Lemma 4 *X_l is an extended function of*

$$u^{l+1}(v(v-1))^{\{(l+1)/2\}}(v - 1/2)^{((-1)^{l+1}+1)/2}.$$

Moreover $X_{k'_2-1}$, $X_{k'_2}|_{w_{k'_2}=0}$ are holomorphic, and $X_{k'_2-1}$, $X_{k'_2}|_{w_{k'_2}=0}$ are extended functions of g_1, f'_2.

Remark: The variables w_l are defined at $l \leq k'_2 - 1$. So $X_{k'_2}|_{w_{k'_2}=0}$ means $X_{k'_2-2}\{G_0 - (w_{k'_2-1})^2\} + w_{k'_2-1}X_{k'_2-1} - \frac{1}{2}\prod_{k=-2}^{k'_2-2} w_k$.

(Proof.) By easy computation, we have

$$X_l|_{T=0} = u^{l+1}(v(v-1))^{\{(l+1)/2\}}(v - 1/2)^{((-1)^{l+1}+1)/2}.$$

Since

$$\begin{aligned} X_0 &= u"_0((u"_0 v"_0 - w_{-2} - w_{-1})(u"_0 v"_0 - w_{-2} + w_0) - v"_0) \\ &\quad +(w_{-2} + w_{-1} - w_0)/2 \\ &= u'_0((u'_0 v'_0 + w_{-2} - w_{-1})(u'_0 v'_0 + w_{-2} + w_0) + v'_0) \\ &\quad +(w_{-2} - w_{-1} + w_0)/2 \end{aligned}$$

and the definitions of X_l (9), all X_l are holomorphic on $\mathbf{C}^2$ of the coordinate systems $C(u"_0, v"_0)$ and $C(u'_0, v'_0)$.

Let $\tilde{X}_l = X_l - H_0^{(1)} H_1^{(1)} \cdots H_l^{(1)} v$.

Inductively, it is proved that $\tilde{X}_l$ are expressed by polynomials of G_0, $uv = H_0^{(1)} - w_{-1} - w_0$ and parameters w_i by

$$\begin{aligned}
\tilde{X}_0 &= -uv - 1/2 \sum_{-2}^{0} \mathbf{w}_{J_1} \\
\tilde{X}_1 &= -(uv)^2 - uv \sum_{-2}^{1} \mathbf{w}_{J_1} - \frac{1}{2} \sum_{k=0}^{1} G_0^k \sum_{-2}^{1} \mathbf{w}_{J_{2-2k}} \\
\tilde{X}_l &= -H_0^{(1)} H_1^{(1)} \cdots H_{l-2}^{(1)} uv(uv + w_{-2} + w_{-1}) \\
&\quad + \tilde{X}_{l-2}\{G_0 - (w_{l-1})^2\} + \tilde{X}_{l-1}(w_{l-1} + w_l) - \frac{1}{2} \prod_{k=-2}^{l-2} w_k
\end{aligned}$$

Thus since G_0, $uv = H_0^{(1)} - w_{-1} - w_0$ are holomorphic on $\mathbf{C}^2$ of the coordinate systems $C(u_m, v_m)$ $(m \geq 1)$, X_l are holomorphic on the whole space if and only if $H_0^{(1)} H_1^{(1)} \cdots H_l^{(1)} v$ are holomorphic on $\mathbf{C}^2$ of the coordinate systems $C(u_m, v_m)$ $(m \geq 1)$.

If $b = b_1 = \cdots = b_{l-1} = 2$, $b_l > 2$, $H_l^{(2)} = H_0^{(1)} H_1^{(1)} \cdots H_l^{(1)} v + t_l^{(2)}$ is holomorphic on $C(u_m, v_m)$ $(m \geq 1)$ and if $b > 2$, $H_0^{(2)} = H_0^{(1)} v + t_0^{(2)}$ is holomorphic on $C(u_m, v_m)$ $(m \geq 1)$.

Therefore $X_{k'_2 - 1}$ and $X_{k'_2}|_{w_{k'_2} = 0}$ are holomorphic on the whole space.

Q.E.D.

Lemma 5 *Let $b_0 = b$. Assume that the sequence of functions satisfy for $m \geq 0$, $h_m = 2, \cdots, b_m$,*

$$f_m^{(1)} = f_{m-1}^{(b_{m-1}-1)}, \qquad f_m^{(h_m)} = f_m^{(h_m - 1)} f_{m-1}^{(b_{m-1})}, \tag{10}$$

$f_{k'_2-2}^{(2)} = f_{k'_2-1}^{(2)} / f'_1$ and $f_{k'_2-2}^{(2)} = f'_2$.

Then it holds that $f'_e = f_m^{(h_m)}$ where $e = \sum_{i=0}^{m-1}(b_i - 2) + h_m$, $2 \leq h_m \leq b_m - 1$ and $m = 0, \cdots, r$.

Also it holds that $f'_{p+1} = f_r^{(b_r)}$.

(Proof.) Since $f_{k'_2-1}^{(2)} = f_0^{(1)} f_{k'_2-2}^{(2)}$, we have $f_0^{(1)} = f'_1$ and Theorem 2 completes the proof.

Q.E.D.

Lemma 6 *Let $b_0 = b$. There exist functions $F_{k'_2-2}^{(2)}$, $F_m^{(h_m)}$ $(k'_2 - 1 \leq m \leq r, 1 \leq h_m \leq b_m)$ of u, v with parameters in T and polynomials $C_m^{(h_m)}$ of $t"_1^{(1)}$, $t"_2^{(1)}$, $t_{m'}^{(h_{m'})}$ $(m' < m)$ in T such that*

$$\begin{aligned}
F_{k'_2-2}^{(2)} &= \begin{cases} X_{k'_2-2} & k'_2 \geq 2 \\ v - \frac{1}{2} & k'_2 = 1 \end{cases} \\
F_{k'_2-1}^{(2)} &= X_{k'_2-1} + t_{k'_2-1}^{(2)} - C_{k'_2-1}^{(2)}
\end{aligned}$$

$$F_m^{(1)} = F_{m-1}^{(b_{m-1}-1)} + t_m^{(1)} \tag{11}$$

$$F_m^{(h_m)} = F_m^{(h_m-1)} F_{m-1}^{(b_m-1)} + t_m^{(h_m)} - C_m^{(h_m)} \tag{12}$$

with $C_m^{(b_m)} = 0$ *and that for* $k_2' - 1 \leq m \leq r, 1 \leq h_m \leq b_m - 1$, $F_m^{(h_m)}$ *are holomorphic on the whole space with* $F_m^{(h_m)}|_{T=0} = f_e'$ *where* $e = \sum_{i=0}^{m-1}(b_i - 2) + h_m$.

(Proof.)

For simple notation, we set $H_m = H_m^{(b_m-1)}$ and $L_m = (l_{-1}, \cdots, l_{m-1}, s, h) \in \{l_{-1}, \cdots, l_{m-1} \geq 0, 0 \leq s \leq m, 1 \leq h \leq b_m - 2\}$.

Since $X_{k_2'-1}$ is holomorphic on the whole space, the function has a polynomial expression with $H_0^{(0)}$, $H_0^{(h)}$, $H_{k_2'-1}^{(1)}$, $H_{k_2'-1}^{(2)}$ and parameters in T.

Using the relation $H_m^{(1)} = H_{m-1} + t_m^{(1)}$ we have

$$X_{k_2'-1} = H_{k_2'-1}^{(2)} - t_{k_2'-1}^{(2)} + \sum_{L_{k_2'-1}} C_{L_{k_2'-1}}^{(2)} (H_0^{(0)})^{l_{-1}} H_0^{l_0} H_s^{(h)} + C_{k_2'-1}^{(2)}$$

where $C_{L_{k_2'-1}}^{(2)}$ and $C_{k_2'-1}^{(2)}$ are polynomials of $t"_1^{(1)}$, $t"_2^{(1)}$, $t_{m'}^{(h_{m'})}$ $(m' < k_2' - 1)$, $t_{k_2'-1}^{(1)}$.

We set $F_{k_2'-1}^{(2)} = X_{k_2'-1} + t_{k_2'-1}^{(2)} - C_{k_2'-1}^{(2)}$.

$F_{k_2'-2}^{(2)}$ is not holomorphic on $C(u_m, v_m)$ $(m \geq 1)$ but similarly we have

$$F_{k_2'-2}^{(2)} = H_{k_2'-3} H_{k_2'-4} \cdots H_0 v + \sum_{L_{k_2'-2}} C_{L_{k_2'-2}}^{(2)} (H_0^{(0)})^{l_{-1}} H_0^{l_0} H_s^{(h)} + D_{k_2'-2}^{(2)}$$

where $C_{L_{k_2'-2}}^{(2)}$ and $D_{k_2'-2}^{(2)}$ are polynomials of $t"_1^{(1)}$, $t"_2^{(1)}$, $t_{m'}^{(h_{m'})}$ $(m' \leq k_2' - 2)$.

Inductively using the relations such as

$$\begin{aligned}
H_m^{(h+1)} &= H_m^{(h)} H_{m-1} H_{m-2} \cdots H_0 v + t_m^{(h+1)} \\
H_m^{(h)} H_{m'}^{(h')} &= H_m^{(h)} (H_{m'}^{(h'-1)} H_{h'-1} \cdots H_0 v + t_{m'}^{(h')}) \\
&= H_m^{(h+1)} H_{m'}^{(h'-1)} H_{m'-1} \cdots H_{m+1} \\
&\quad - t_m^{(h+1)} H_{m'}^{(h'-1)} H_{m'-1} \cdots H_{m+1} + H_m^{(h)} t_{m'}^{(h')} \\
H_m^{(h)} H_{m'}^{(1)} &= H_m^{(h)} H_{m'-1} + H_m^{(h)} t_{m'}^{(1)} \\
H_m^{(1)} &= H_{m-1} + t_m^{(1)},
\end{aligned}$$

we construct $F_m^{(h_m)} (k_2' - 1 \leq m \leq r, 1 \leq h_m \leq b_m - 1)$ with expressions

$$\begin{aligned}
F_m^{(1)} &= F_{m-1}^{(b_{m-1}-1)} + t_m^{(1)} \\
&= H_m^{(1)} + \sum_{L_m} C_{L_m}^{(1)} (H_0^{(0)})^{l_{-1}} H_0^{l_0} H_1^{l_1} \cdots H_{m-1}^{l_{m-1}} H_s^{(h)}
\end{aligned} \tag{13}$$

$$\begin{aligned}
F_m^{(h_m)} &= F_m^{(h_m-1)} F_{m-1}^{(b_m-1)} + t_m^{(h_m)} - C_m^{(h_m)} \\
&= H_m^{(h_m)} + \sum_{L_m} C_{L_m}^{(h_m)} (H_0^{(0)})^{l_{-1}} H_0^{l_0} H_1^{l_1} \cdots H_{m-1}^{l_{m-1}} H_s^{(h)}
\end{aligned} \tag{14}$$

$$\begin{aligned} F_m^{(b_m)} &= F_m^{(b_m-1)}F_{m-1}^{(b_{m-1})} = H_{m-1}H_{m-2}\cdots H_0 v \\ &+\sum_{L_m} C_{L_m}^{(b_m)}(H_0^{(0)})^{l_{-1}}H_0^{l_0}H_1^{l_1}\cdots H_{m-1}^{l_{m-1}}H_s^{(h)} + D_m^{(b_m)} \end{aligned}$$

where $C_{L_m}^{(h_m)}$, $C_m^{(h_m)}$ and $D_m^{(b_m)}$ are polynomials of $t"_1^{(1)}$, $t"_2^{(1)}$, $t_{m'}^{(h_{m'})}$ $(m' < m)$, $t_m^{(h'_m)}$ $(h'_m < h_m)$.

For $k'_2-1 \leq m \leq r, 1 \leq h_m \leq b_m-1$, by equations (13) and (14), $F_m^{(h_m)}$ are holomorphic on $\mathbf{C}^2$ of the coordinate systems $C(u_m, v_m)$ $(m \geq 1)$, and by equations (11) and (12), $F_m^{(h_m)}$ are holomorphic on $\mathbf{C}^2$ of the coordinate systems $C(u"_0, v"_0)$ and $C(u'_0, v'_0)$.

By Lemma 5, we also have $F_m^{(h_m)}|_{T=0} = f'_e$ where $e = \sum_{i=0}^{m-1}(b_i-2)+h_m$.

Q.E.D.

Theorem 5 *We set* $F_e = F_m^{(h_m)} - t_m^{(h_m)} + C_m^{(h_m)}$ *where* $e = \sum_{i=0}^{m-1}(b_i-2)+h_m \geq 2$ *and* $b_0 = b$.

Then G_0, $B = X_{k'_2-1}$, $A = X_{k'_2}|_{w_{k'_2}=0}$ *and* F_e $(e \geq 2)$ *are extended functions of* g_0, f'_2, g_1 *and* f'_e $(e \geq 2)$, *respectively.*

There exists the set of variables $W = \{w_1^{(h_1)}, w_m^{(h_m)}; -2 \leq h_1 \leq a'_1-1, 2 \leq m \leq p', 1 \leq h_m \leq a'_m - 1\}$ *which is algebraic isomorphic to* T *such that the relations of the functions* G_0, A, B, F_e *and* W *are given by all the* 2×2*-minors of the matrices;*

$$\begin{pmatrix} G_0 - (w_1^{(a'_1-2)})^2, & A + w_1^{(a'_1-2)}B - \frac{(\Pi_{h=-2}^{a'_1-3}w_1^{(h)})}{2}, \\ A - w_1^{(a'_1-2)}B + \frac{(\Pi_{h=-2}^{a'_1-3}w_1^{(h)})}{2}, & B^2 + \frac{\Pi_{h=-2}^{a'_1-3}(-G_0+(w_1^{(h)})^2)-\Pi_{h=-2}^{a'_1-3}(w_1^{(h)})^2}{4G_0}, \\ \multicolumn{2}{c}{(F_{e-1} - w_{e-1}^{(a_{e-1}-1)})\Pi_{m=2}^{e-1}\Pi_{h=1}^{a'_m-2}(F_m - w_m^{(h)})} \\ \multicolumn{2}{c}{F_e} \end{pmatrix}$$

for $3 \leq e$ *and given by all the generalized* 2×2*-minors of the quasi-matrix;*

$$\begin{pmatrix} F_2 & & F_3 & & F_4 & \cdots & F_{p'} \\ & \Pi_{h=1}^{a'_3-2}(F_3 - w_3^{(h)}) & & \Pi_{h=1}^{a'_4-2}(F_4 - w_4^{(h)}) & & \cdots & \\ F_3 & & F_4 & & F_5 & \cdots & F_{p'+1} \end{pmatrix}.$$

(Proof.)

It follows by if $b_{m+1} = \cdots = b_{\tilde{m}}$,

$$\begin{aligned} & F_m^{(b_m-1)}\cdots F_{\tilde{m}}^{(b_{\tilde{m}}-1)} \\ = \ & F_m^{(b_m-1)}(F_m^{(b_m-1)} + t_{m+1}^{(1)})(F_m^{(b_m-1)} + t_{m+1}^{(1)} + t_{m+2}^{(1)}) \\ & \cdots (F_m^{(b_m-1)} + t_{m+1}^{(1)} + \cdots + t_{\tilde{m}}^{(1)}) \end{aligned}$$

and

$$\mathrm{rank}\begin{pmatrix} 1, & G_0 - w_{k'_2-1}^2, & A + w_{k'_2-1}B - \frac{\Pi_{k=-2}^{k'_2-2}w_k}{2}, \\ X_{k'_2-2}, & A - w_{k'_2-1}B + \frac{\Pi_{k=-2}^{k'_2-2}w_k}{2}, & B^2 + \frac{\Pi_{k=-2}^{k'_2-2}(-G_0+w_k^2)-\Pi_{k=-2}^{k'_2-2}w_k^2}{4G_0}, \end{pmatrix}$$

$$\begin{array}{l} F_{k'_2-1}^{(b_{k'_2-1}-1)} \cdots F_{j-1}^{(b_{j-1}-1)} F_j^{(l-1)} \\ \qquad F_j^{(l)} - t_j^{(l)} + C_j^{(l)} \end{array} \Bigg) < 2.$$

Q.E.D.

Moreover from those relations we can see a canonical candidate for the full Weyl group.

1. In the case of $b = b_1 = \cdots = b_r = 2$, i.e., a rational double point, the relation of these functions which was also shown in G. N. Tyurina [15], is

$$0 = A^2 - B^2 G_0 + B\Pi_{k=-2}^{r} w_k - \frac{\Pi_{k=-2}^{r}(-G_0 + w_k^2) - \Pi_{k=-2}^{r} w_k^2}{4G_0}$$

   The corresponding Weyl group is $S_{r+3} \ltimes \mathbf{Z}_2^{r+2}$.

2. In the case of $b = b_1 = \cdots = b_{a'_1-3} = 2, b_{a'_1-2} \neq 2 (3 \leq a'_1)$, the corresponding Weyl group is $S_{a'_1} \ltimes \mathbf{Z}_2^{a'_1-1} \times S_{a'_2-2} \times \cdots \times S_{a'_{p'}-2}$ and it is easy to see how to act the Weyl group.

3. In the case of $b \geq 3$, a part of the relations

$$\operatorname{rank} \begin{pmatrix} G_0 - (w_1^{(0)})^2, & A + w_1^{(0)} B - \frac{w_1^{(-2)} w_1^{(-1)}}{2} \\ A - w_1^{(0)} B + \frac{w_1^{(-2)} w_1^{(-1)}}{2}, & B^2 + \frac{G_0 - (w_1^{(-2)})^2 - (w_1^{(-1)})^2}{4} \end{pmatrix} < 2$$

   and the corresponding Weyl group is $S_2 \times S_2 \times S_{a'_2-2} \times \cdots \times S_{a'_{p'}-2}$.

   By putting $\tilde{w}_1^{(-2)} = w_1^{(-2)} + w_1^{(-1)}$ and $\tilde{w}_1^{(-1)} = w_1^{(-2)} - w_1^{(-1)}$, one can see how to act the Weyl group since $w_1^{(-2)} w_1^{(-1)} = (\tilde{w}_1^{(-2)})^2/4 - (\tilde{w}_1^{(-1)})^2/4$ and $(w_1^{(-2)})^2 + (w_1^{(-1)})^2 = (\tilde{w}_1^{(-2)})^2/2 + (\tilde{w}_1^{(-1)})^2/2$.

When $r = 1$ and $b > 2$, the exact coefficients $C_m^{(h_m)}$ are calculated ([16]) but in general these values which are defined by induction are very complicated.

4 Appendix

The theorem is proved by the similar way of Theorem 1 but another proof is shown here.

Theorem 6 *For any integer l, there exist $0 \leq t_m \leq b_{m+1} - 1$ such as $l = \sum_{m=0}^{r} t_m n_m$. Then*

$$\{\frac{\tilde{q}}{\tilde{n}} l\} = \sum_{m=1}^{r} t_m q_m + 1.$$

(Proof)

Let s_m be a positive integer defined by $\frac{s_m}{q_m} = \frac{1}{b_2 - 1 \rceil b_3 - \cdots - 1 \rceil b_m}$ for $m =$

$$n_i(j) = (b_i - \frac{1|}{|b_{m+1}} - \cdots - \frac{1|}{|b_j})n_{i-1} - n_{j-2}$$

ad

$$q_i(j) = (b_i - \frac{1|}{|b_{m+1}} - \cdots - \frac{1|}{|b_j})q_{i-1} - q_{j-2}.$$

hen

$$\begin{aligned}
& n_{i+1}(j) \\
= \; & (b_{i+1} - \frac{1|}{|b_{i+2}} - \cdots - \frac{1|}{|b_j})n_i - n_{i-1} \\
= \; & (b_{i+1} - \frac{1|}{|b_{i+2}} - \cdots - \frac{1|}{|b_j})(b_i n_{i-1} - n_{i-2}) - n_{i-1} \\
= \; & (b_i(b_{i+1} - \frac{1|}{|b_{i+2}} - \cdots - \frac{1|}{|b_j}) - 1)n_{i-1} \\
& -(b_{i+1} - \frac{1|}{|b_{i+2}} - \cdots - \frac{1|}{|b_j})n_{i-2} \\
= \; & (b_{i+1} - \frac{1|}{|b_{i+2}} - \cdots - \frac{1|}{|b_j})(b_i - \frac{1|}{|b_{i+1}} - \cdots - \frac{1|}{|b_j})n_{i-1} \\
& -(b_{i+1} - \frac{1|}{|b_{i+2}} - \cdots - \frac{1|}{|b_j})n_{i-2} \\
= \; & (b_{i+1} - \frac{1|}{|b_{i+2}} - \cdots - \frac{1|}{|b_j})((b_i - \frac{1|}{|b_{i+1}} - \cdots - \frac{1|}{|b_j})n_{i-1} - n_{i-2}) \\
= \; & (b_{i+1} - \frac{1|}{|b_{i+2}} - \cdots - \frac{1|}{|b_j})n_i(r)
\end{aligned}$$

Also

$$q_{i+1}(r) = (b_{i+1} - \frac{1|}{|b_{i+2}} - \cdots - \frac{1|}{|b_j})q_i(j)$$

We have

$$\frac{\tilde{q}}{\tilde{n}} n_m = \frac{\tilde{q}}{\tilde{n}}(b_1 q_m - s_m) = \frac{b_1 q_m - s_m}{b_1 - \frac{1|}{|b_2} - \cdots - \frac{1|}{|b_r}}$$

$$= q_m + \frac{\dfrac{q_m}{b_2 - \frac{1|}{|b_3} - \cdots - \frac{1|}{|b_r}} - s_m}{b_1 - \frac{1|}{|b_2} - \cdots - \frac{1|}{|b_r}}$$

$$= q_m + q_m \frac{\dfrac{1}{b_2 - \frac{1|}{|b_3} - \cdots - \frac{1|}{|b_r}} - \dfrac{1}{b_2 - \frac{1|}{|b_3} - \cdots - \frac{1|}{|b_m}}}{b_1 - \frac{1|}{|b_2} - \cdots - \frac{1|}{|b_r}} = q_m$$

$$+ q_m \frac{\dfrac{1}{b_3 - \frac{1|}{|b_4} - \cdots - \frac{1|}{|b_r}} - \dfrac{1}{b_3 - \frac{1|}{|b_4} - \cdots - \frac{1|}{|b_m}}}{\left\{b_1\left(b_2 - \frac{1|}{|b_3} - \cdots - \frac{1|}{|b_r}\right) - 1\right\}\left\{b_2 - \frac{1|}{|b_3} - \cdots - \frac{1|}{|b_m}\right\}}$$

$$= q_m + q_m \frac{\dfrac{1}{b_3 - \frac{1|}{|b_4} - \cdots - \frac{1|}{|b_r}} - \dfrac{1}{b_3 - \frac{1|}{|b_4} - \cdots - \frac{1|}{|b_m}}}{n_2(r)q_2(m)}$$

$$= q_m + q_m \frac{\dfrac{1}{b_4 - \frac{1|}{|b_5} - \cdots - \frac{1|}{|b_r}} - \dfrac{1}{b_4 - \frac{1|}{|b_5} - \cdots - \frac{1|}{|b_m}}}{n_3(r)q_3(m)}$$

$$= q_m + q_m \frac{\dfrac{1}{b_{m+1} - \frac{1|}{|b_{m+2}} - \cdots - \frac{1|}{|b_r}}}{n_m(r)q_m(m)}$$

$$= q_m + q_m \frac{\overline{b_{m+1} - \frac{1|}{|b_{m+2}} - \cdots - \frac{1|}{|b_r}}}{n_m(r) q_m} = q_m + \frac{1}{n_{m+1}(r)}$$

On the other hand,

$$\frac{(b_r - 1)}{n_r} + \sum_{m=0}^{r-2} \frac{b_{m+1} - 2}{n_{m+1}(r)} = \frac{1}{n_{r-1}(r)} ((b_r - 1)/b_r + b_{r-1} - 2) + \sum_{m=0}^{r-3} \frac{b_{m+1} - 2}{n_{m+1}(r)}$$

$$= \frac{1}{n_{r-1}(r)} (b_{r-1} - 1/b_r - 1) + \sum_{m=0}^{r-3} \frac{b_{m+1} - 2}{n_{m+1}(r)}$$

$$= \frac{1}{n_{r-2}(r)} \left(\frac{b_{r-1} - \frac{1}{b_r} - 1}{b_{r-1} - \frac{1}{b_r}} + b_{r-2} - 2 \right) + \sum_{m=0}^{r-4} \frac{b_{m+1} - 2}{n_{m+1}(r)}$$

$$= \frac{1}{n_{r-2}(r)} \left(b_{r-2} - \frac{1}{b_{r-1} - \frac{1}{b_r}} - 1 \right) + \sum_{m=0}^{r-4} \frac{b_{m+1} - 2}{n_{m+1}(r)}$$

$$\cdots = \frac{1}{b_1 \left(b_2 - \frac{1|}{|b_3} - \cdots - \frac{1|}{|b_r}\right) - 1} \left(b_2 - \frac{1|}{|b_3} - \cdots - \frac{1|}{|b_r} - 1\right)$$

$$+ \frac{b_1 - 2}{n_1(r)} = \frac{1}{b_1 - \frac{1|}{|b_2} - \cdots - \frac{1|}{|b_r}} \left(b_1 - \frac{1|}{|b_2} - \cdots - \frac{1|}{|b_r} - 1\right)$$

$$= 1 - \frac{1}{b_1 - \frac{1|}{|b_2} - \cdots - \frac{1|}{|b_r}}.$$

Hence by $\tilde{n} = (b_r - 1)n_{r-1} + \sum_{m=1}^{r-2}(b_m - 2)n_m + (b_1 - 1)n_0$, it holds that for $0 < l < \tilde{n}$,

$$0 < \sum_{m=0}^{r-1} \frac{t_m}{n_{m+1}(r)} < 1, i.e., \{\frac{\tilde{q}}{\tilde{n}} l\} = \sum_{m=1}^{r} t_m q_m + 1.$$

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