Artin component and Weyl group for dihedral singularity

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Abstract
In this paper, we give the Artin components for dihedral singularities explicitly.

1 Introduction

Let $(X, x)$ be a rational surface singularity and let $\chi \rightarrow \text{Def}(X)$ be a versal deformation of $X$. In the sequel, we denote a base space of the versal deformation of $X$ by $\text{Def}(X)$. Riemenschneider [8] showed that a versal deformation (relatively to the exceptional set) $\tilde{\chi} \rightarrow T = \text{Def}(\tilde{\chi})$ of the minimal resolution $\tilde{X}$ of $X$ can be blow down simultaneously to a deformation $\chi \rightarrow T$ of $X$. Then from versality, we have a mapping $T = \text{Def}(\tilde{X}) \rightarrow \text{Def}(X)$. About the mapping, Artin[1], Lipman[6], Wahl[17] proved that

$$(\text{Def}(X))_{\text{art}} \cong \text{Def}(\tilde{X})/\prod W_j,$$

where $(\text{Def}(X))_{\text{art}}$ is a component of $\text{Def}(X)$, called the Artin component and $W_j$ are the finite Weyl groups belonging to the maximal connected $(-2)$-configurations, i.e. the rational double point configurations supported by the exceptional set of $\tilde{X}$. This theorem was first considered by Brieskorn[2],[3] and Tyurina[15] for a singularity of type ADE, using the theory of simple complex Lie groups and their Weyl groups of corresponding type. In the case of all cyclic quotient singularities, by following the method of Riemenschneider [9], one can construct a canonical candidate for the full Weyl group explicitly.

In this paper, we consider the case of dihedral singularities.

This result will be useful for consideration of deformations in terms of representations of quivers. The quiver-theoretic approach was proposed by P. Kronheimer [5] and then Ebeling, Slodowy[14] and Cassens[4] constructed the versal deformation using representations of the quivers for Kleinian singularity. Riemenschneider [11] generalized the method to yield the monodromy covering of the Artin component for all cyclic quotients using the special representations by the work of Wurram[18] and Riemenschneider[12]. To extend the Riemenschneider’s result to the dihedral singularity, we need to calculate the Artin component, and which is the motivation of this paper.

In Section 2, we obtain generators of the graded affined algebra of its minimal resolution (Theorem 4) using Pinkham result[7]. Then in Section 3, we extend these generators to the deformation space of the minimal resolution (Theorem 5) and the results give the Artin components.
2 Generators of the graded affined algebra

In what follows, we denote \( b_1 - \frac{1}{1} \) by \( b_1 - \underline{1} \) and let \( \{ r \} \) always be the least integer greater than or equal to \( r \).

Let \( n, q \) be positive integers with \( 1 < q < n \) and \( \gcd(n, q) = 1 \). We define \( 2 \times 2 \) matrices \( \varphi, \phi, \tau \) by \( \varphi = \begin{pmatrix} \eta_{2q} & 0 \\ 0 & \eta_{2q}^{-1} \end{pmatrix} \), \( \phi = \begin{pmatrix} \eta_{2m} & 0 \\ 0 & \eta_{2m} \end{pmatrix} \) and \( \tau = \begin{pmatrix} 0 & \eta_4 \\ \eta_4 & 0 \end{pmatrix} \) where \( \eta_k = \exp(2\pi i/k) \). The group \( D_{n,q} \subset \text{GL}(2, \mathbb{C}) \) generated by \( \{ \varphi, \phi, \tau \} \) if \( m = n - q \equiv 1 \pmod{2} \), \( \varphi, \phi \circ \tau \), if \( m = n - q \equiv 0 \pmod{2} \), determines the dihedral singularity of the quotient \( \mathbb{C}^2/D_{n,q} \).

We set \( n/q = b - \tilde{q}/\tilde{n} = b - \underline{1} \rfloor \overline{b_1} - \cdots - \lrcorner 1 \overline{b_r} \) with \( b, b_1, b_2, \ldots, b_r \geq 2 \).

The minimal resolution for the dihedral singularity of type \( D_{n,q} \) is written down explicitly by the union of \( 4 + r \) copies of \( \mathbb{C}^2 \);

\[
\begin{align*}
    u &= \frac{1}{u_0}, \\
    v &= u_0^2 v_0, \\
    u_0' &= \frac{1}{u}, \\
    v_0' &= u^2(v - 1),
\end{align*}
\]

\( u_1 = v^b u, \quad v_1 = \frac{1}{v}, \quad u_2 = v_1^b u_1, \quad v_2 = u_2^b u_2, \)

\( v_3 = \overline{v_2}, \quad u_3 = v_3^b u_2, \quad \cdot \cdot \cdot \)

and its dual graph is

\[
\begin{array}{c}
\begin{array}{c}
-2 \quad -b \quad -b_1 \quad -b_{r-1} \quad -b_r \\
\end{array}
\end{array}
\]

Let \( A \) be the graded affine algebra of \( X \). We denote by \( D \) the divisor of positive degree \( b = -(E_0 \cdot E_0) \) on \( E_0 \), where \( E_0 \) is the central curve of the resolution. Let \( D^{(l)} \) be the divisor

\[ D^{(l)} = lD - \left\{ \frac{l}{2} \right\} y_1 - \left\{ \frac{l}{2} \right\} y_2 - \left\{ \frac{q_l}{\tilde{n}} \right\} y_3 \]

where \( y_i \) are the intersection points of \( E_0 \) with the other components of the exceptional set. By applying Pinkham result[7] to the our case, we have \( A = \bigoplus_{l \geq 0} A_l \) with \( A_l = H_0(E_0, \mathcal{O}_E(D^{(l)})) \). By taking three points \( y_1 = 0, y_2 = 1, y_3 = \infty \), one sees immediately that generators of \( A \) in degree \( l \) are, for example,

\[ u^t v^s (v - 1)^t \] for \( \{ l/2 \} \leq t \leq bl - \{ q_l/\tilde{n} \} - \{ l/2 \}, s = \{ l/2 \} \).
To get minimal generators, we use the following theorem in Riemenschneider [9].

Let \( \tilde{n}/(\tilde{n} - \tilde{q}) = a_1 - \frac{1}{1} a_2 - \cdots - \frac{1}{p} a_p \) with \( a_1, a_2, \cdots, a_p \geq 2 \) and let

\[
\begin{align*}
&i_0 = \tilde{n}, \quad i_1 = \tilde{n} - \tilde{q}, \quad i_e = a_{e-1}i_{e-1} - i_{e-2} \\
j_0 = 0, \quad j_1 = 1, \quad j_e = a_{e-1}j_{e-1} - j_{e-2} \\
k_0 = 1, \quad k_1 = 1, \quad k_e = a_{e-1}k_{e-1} - k_{e-2} \quad \text{for } 2 \leq e \leq p + 1.
\end{align*}
\]

**Theorem 1 (Riemenschneider[9])** Then

1. \( p = 1 + \sum_{i=1}^{r} (b_i - 2) \).
2. \( i_0 = \tilde{n} > i_1 = \tilde{n} - \tilde{q} > i_3 > \cdots > i_p = 1 > i_{p+1} = 0 \).
3. \( i_e + \tilde{q}j_e = \tilde{n}k_e \), i.e., \( \{\tilde{q}j_e/\tilde{n}\} = k_e \) for \( e \geq 1 \).
4. For integers \( i, j, k \), if \( i + \tilde{q}j = \tilde{n}k \), \( 0 \leq i < \tilde{n} \) and \( j_e < j < j_{e+1} \), then it holds that \( i \geq i_e \).
5. For \( e = \sum_{m=1}^{m_0-1} (b_m - 2) + h \) and \( b_{m_0} \geq 3 \),

\[
a_e = \begin{cases} 
 m_0 - m_1 + 2 & \text{if } h = 1, 1 \leq m_0 \leq r, \\
2 & \text{if } 2 \leq h \leq b_{m_0} - 2, 1 \leq m_0 \leq r, \\
r - m_1 + 2 & \text{if } h = 1, m_0 = r + 1, \\
& \text{and if } b_1 = b_2 = \cdots = b_r = 2, \text{then } m_1 = 1, \\
& \text{otherwise } \exists b_{m_1} \neq 2, \\
b_{m_1+1} = b_{m_1+2} = \cdots = b_{m_0-1} = 2 & \text{if } b_1 \geq 3 \text{ and } b_{m_0} \leq 2.
\end{cases}
\]

**Corollary 1** If \( j < j_2 \) then \( \{\tilde{q}j/\tilde{n}\} = j \) and if \( j \geq j_2 \) then \( \{\tilde{q}j/\tilde{n}\} \leq j - 1 \).

**Corollary 2** If \( j_{e-1} < j < j_e \) then \( \{\tilde{q}j/\tilde{n}\} = \{\tilde{q}j_1/\tilde{n}\} + \{\tilde{q}(j - j_e)/\tilde{n}\} \). Moreover, for any \( l_1 \geq 1 \) and \( l_2 \geq 1 \) such that \( l_1 + l_2 = j_e \), it holds that \( \{\tilde{q}j_e/\tilde{n}\} = \{\tilde{q}l_1/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\} + 1 \). Q.E.D.

In order to clear the relation between \( a_e \) and \( \mathrm{C}(u_m, v_m) \), let us prepare the following Lemma.

**Theorem 2** Assume that \( b_1 = b_2 = \cdots = b_{m_0-1} = 2, b_{m_0} \neq 2 \) (if \( b_1 \neq 2 \), then \( m_0 = 1 \)) and that two sequences of numbers \( s_e \) (\( 1 \leq e \leq p + 1 \)) and \( l_m^{(h_m)} \) (\( 1 \leq m \leq r, 1 \leq h_m \leq b_m \)) satisfy

\[
\begin{align*}
s_{e+1} &= a_es_e - s_{e-1} \\
l_m^{(1)} &= l_{m-1}^{(b_{m-1}-1)}, \\
l_m^{(h_m)} &= l_{m}^{(h_{m-1})} + l_{m-1}^{(b_{m-1})} \quad \text{for } h_m = 2, \cdots, b_m.
\end{align*}
\]

Then if \( s_1 = l_1^{(1)} \) and \( s_2 = l_2^{(2)} \), it holds that \( s_e = l_m^{(h_m)} \) where \( e = \sum_{i=1}^{m-1} (b_i - 2) + h_m, 2 \leq h_m \leq b_m - 1, m = 1, \cdots, r \).

Also it holds that \( s_{p+1} = l_r^{(b_r)} \).
Let us prove it by induction. Suppose we have \( s_{e'} = l_{m}^{(h_{m'})} \) for \( e' = \sum_{i=1}^{m-1} (b_{i} - 2) + h_{m'} \).

If \( h_{m} \geq 3 \), then \( s_{e-1} = l_{m}^{(h_{m}-1)} \), \( s_{e-2} = l_{m}^{(h_{m}-2)} \), and \( s_{e-1} - s_{e-2} = l_{m}^{(h_{m}-1)} \). Since \( b_{m} \geq 4 \), we have \( a_{e-1} = 2 \) using Theorem 1 (5). Thus, we have \( s_{e} = 2s_{e-1} - s_{e-2} = l_{m}^{(h_{m}-1)} + l_{m-1}^{(b_{m-1})} = l_{m}^{(h_{m})} \).

If \( h_{m} = 2 \) and \( m = m_{0} \), then \( e = \sum_{i=1}^{m-1} (b_{i} - 2) + h_{m} = 2 \).

If \( h_{m} = 2 \), \( b_{m_{1}} \neq 2 \) and \( b_{m_{1}+1} = b_{m_{1}+2} = \cdots = b_{r} = 2 \), then \( s_{e-1} = l_{m}^{(h_{m}-1)}, s_{e-2} = l_{m}^{(h_{m}-2)} \) and \( s_{e-1} - s_{e-2} = l_{m-1}^{(b_{m-1})} \).

Since \( b_{m} \geq 4 \), we have \( a_{e-1} = 2 \) using Theorem 1 (5). Thus, we have \( s_{e} = 2s_{e-1} - s_{e-2} = l_{m}^{(h_{m}-1)} + l_{m-1}^{(b_{m-1})} \). Therefore,

\[
\begin{align*}
    s_{e} &= a_{e-1} s_{e-1} - s_{e-2} = (m - m_{1} + 2)l_{m_{1}}^{(b_{m_{1}}-1)} - l_{m_{1}}^{(b_{m_{1}}-2)} \\
    &= (m - m_{1} + 1)l_{m_{1}}^{(b_{m_{1}}-1)} + l_{m_{1}}^{(b_{m_{1}}-1)} - l_{m_{1}}^{(b_{m_{1}}-2)} \\
    &= (m - m_{1} + 1)l_{m_{1}}^{(b_{m_{1}}-1)} + l_{m_{1}-1}^{(b_{m_{1}-1})} = l_{r}^{(b_{r})}. \\
\end{align*}
\]

That is, \( s_{e} = l_{m}^{(h_{m})} \).

Let us define two sequences of integers \( n_{m}, q_{m} \) by

\[
\begin{align*}
n_{0} &= 1, q_{0} = 0, n_{m}/q_{m} = b_{1} - 1/b_{2} - \cdots - 1/b_{m}, \quad \gcd(n_{m}, q_{m}) = 1
\end{align*}
\]

for \( m = 1, 2, \ldots, r \). Those numbers \( n_{m}, q_{m} \) satisfy that

\[
\begin{align*}
n_{0} &= 1, n_{1} = b_{1}, \quad n_{m} = b_{m}n_{m-1} - n_{m-2}, \quad n_{m} > n_{m-1}, \\
q_{0} &= 0, q_{1} = 1, q_{2} = b_{2}, \quad q_{m} = b_{m}q_{m-1} - q_{m-2}, \quad q_{m} > q_{m-1},
\end{align*}
\]

for \( m = 2, \ldots, r \).

**Corollary 3** We have \( j_{e} = h_{m}n_{m-1} - n_{m-2} \) and \( k_{e} = h_{m}q_{m-1} - q_{m-2} \) where \( m = 1, \ldots, r \), \( 2 \leq h_{m} \leq b_{m} - 1 \) and \( e = \sum_{i=1}^{m-1} (b_{i} - 2) + h_{m} \). Also \( j_{p+1} = n_{r} = b_{r}n_{r-1} - n_{r-2} \) and \( k_{p+1} = q_{r} = b_{r}q_{r-1} - q_{r-2} \).
(Proof.) Assume that $b_1 = b_2 = \cdots = b_{m_0-1} = 2$, $b_{m_0} \neq 2$. Let $n_{-1} = 0$ and $l_m^{(h_m)} = h_m n_{m-1} - n_{m-2}$ ($1 \leq h_m \leq b_m - 1$). Then, we have $1 = j_1 = n_0 = l_1^{(1)}$ and $j_2 = a_1 = m_0 + 1 = 2n_{m_0-1} - n_{m_0-2} = l_2^{(2)}$ by $n_m = 2n_{m-1} - n_{m-2} = m + 1$ for $m = 1, \cdots, m_0$. Also we have

$$l_m^{(h_m)} = b_m n_{m-1} - n_{m-2} = n_m$$

$$l_m^{(1)} = n_{m-1} - n_{m-2} = b_{m-1} n_{m-2} - n_{m-3} - n_{m-2} = l_{m-1}^{(b_{m-1}-1)}$$

$$l_m^{(h_m)} = h_m n_{m-1} - n_{m-2} = (h_m - 1) n_{m-1} - n_{m-2} + n_{m-1} = l_{m-1}^{(h_m-1)} + l_{m-1}^{(b_{m-1})}$$

for $h_m = 2, \cdots, b_m$.

Therefore by Theorem 2, we have the proof for $j_e$.

By setting $q_{-1} = 0$ and $l_m^{(h_m)} = h_m q_{m-1} - q_{m-2}$ ($1 \leq h_m \leq b_m - 1$), similarly we have the proof for $k_e$.

Q.E.D.

**Corollary 4** Consider the coordinate system $C(u_m, v_m)$ defined by the equations (1). We have $u^{j_e} v^{b_e-k_e} = u^{j_1} v^{k} = \begin{cases} u_m v_m^{h_m} & m \text{ odd} \\ v_m u_m^{h_m} & m \text{ even} \end{cases}$ for $m = 1, \cdots, r$, $2 \leq h_m \leq b_m - 1$ and $e = \Sigma_{s=1}^{m-1} (b_s - 2) + h_m$.

**Theorem 3** Minimal generators of holomorphic functions on $\tilde{X}$ defined everywhere are the following.

Let $s_1 = ((-1)^{j_2+1} + 1)/2$, $s_2 = \{j_2/2\}$, $s_e = a_{e-1}s_{e-1} - s_{e-2}$ for $3 \leq e \leq p + 1$.

1. When $b = b_1 = \cdots = b_{m_0} = 2$, (if $b_1 > 0$, then we put $m_0 = 0$)

$$u^l (v(v-1))^s (v-1/2)^t$$

- $l = 2$, $s = 1$, $t = 0$
- $l = j_2 + 1$, $s = \{(j_2 + 1)/2\}$, $t = ((-1)^{j_2+1} + 1)/2$
- $l = j_e$, $s = s_e$, $t = 2j_e - k_e - 2s_e$

and the relation is

$$\text{rank} \begin{pmatrix} 1, & g_0, & g_1, & f_{e-1}^{a_{e-1}-1} f_{e-2}^{a_{e-2}-2} \cdots f_3^{a_3-2} f_2^{a_2-2} \\ U, & g_1, & f_2 + (-g_0)^{a_1} 4, & f_e \end{pmatrix} < 2,$$

where $U = u^{j_2-1}(v(v-1))^{((j_2-1)/2)}(v-1/2)((-1)^{j_2-1}+1)/2$ and $3 \leq e \leq p + 1$.

2. When $b > 2$,

$$u^l (v(v-1))^s (v-1/2)^t$$

- $l = 1$, $s = 1$, $t = 0, \cdots, b - 3$
- $l = 2$, $s = 1$, $t = 0, 1$
- $l = j_e$, $s = j_e$, $t = bj_e - k_e - 2j_e$

and the relation is

$$\text{rank} \begin{pmatrix} f_1^{(l+2)}, & g_0, & g_1, & f_{e-1}^{a_{e-1}-1} f_{e-2}^{a_{e-2}-2} \cdots f_3^{a_3-2} f_2^{a_2-2} \\ U, & g_1, & f_2 + (-g_0)^{a_1} 4, & f_e \end{pmatrix} < 2,$$

where $U = u^{j_2-1}(v(v-1))^{((j_2-1)/2)}(v-1/2)((-1)^{j_2-1}+1)/2$ and $3 \leq e \leq p + 1$. 
and the relation is
\[
\begin{align*}
\text{rank} & \left( \begin{array}{cccc}
1, & g_0, & g_1, & f_1^{(2)}, \\
v - \frac{1}{2}, & g_1, & (f_1^{(2)})^2 - \frac{g_0}{4}, & f_1^{(3)}, \\
& f_2, & f_{e-1}^{a_{e-1}-1}f_{e-2}^{a_{e-2}-2} \cdots f_2^{a_2-2}(f_1^{(b-2)})^{a_1-1} & f_e
\end{array} \right) < 2 \\
\text{rank} & \left( \begin{array}{c}
1, \\
v - \frac{1}{2}, \\
f_2, \\
f_{e-1}^{a_{e-1}-1}f_{e-2}^{a_{e-2}-2} \cdots f_2^{a_2-2}(f_1^{(b-2)})^{a_1-1}
\end{array} \right) < 2
\end{align*}
\]
for \(3 \leq e \leq p + 1\).

To prove the theorem, we prepare two lemmas.

In general, \(A_{l_1} \cdot A_{l_2} \subset A_{l_1 + l_2}\) since \(\{m\} + \{n\} \geq \{m + n\}\). On the other hand,

**Lemma 1** \(u^l v^t (v - 1)^s \in A_l\) is an element of \(A_{l_1} \cdot A_{l_2}\) if and only if \(l = l_1 + l_2, t \geq \{l_1/2\} + \{l_2/2\}, s \geq \{l_1/2\} + \{l_2/2\}\), \(t + s \leq bl - \{\tilde{q}l_1/\tilde{n}\} - \{\tilde{q}l_2/\tilde{n}\}\).

(Proof.) Recall that \(A_l = \{u^l v^t (v - 1)^s; t, s \geq \{l/2\}, t + s \leq bl - \{\tilde{q}l/\tilde{n}\}\}\).

Q.E.D.

**Lemma 2** For \(u^l v^t (v - 1)^s \in A_{l_1}\), we have the following.

1. For \(l \geq 3\), \(t + s \leq bl - \{\tilde{q}l/\tilde{n}\} - 1\), we have \(u^l v^t (v - 1)^s \in A_{l_1} \cdot A_{l_2}\).

2. We assume that \(b = b_1 = \cdots = b_{m_0} = 2, l = l_1 + l_2\) and \(\{\tilde{q}l_1/\tilde{n}\} = \{\tilde{q}l_2/\tilde{n}\}\) \((l_1 \geq 1, l_2 \geq 1)\). For \(s = \{l/2\}\) and \(t + s = bl - \{\tilde{q}l_1/\tilde{n}\} - \{\tilde{q}l_2/\tilde{n}\}\), we have

   (a) if \(2 \leq l_1 < j_2 = m_0 + 2\), then \(u^l v^t (v - 1)^s \in A_{l_1} \cdot A_{l_2}\).

   (b) if \(l_1, l_2 \geq j_2\), then \(u^l v^t (v - 1)^s \in A_{l_1} \cdot A_{l_2} + A_{l_1} \cdot A_{l_2}\).

   (c) if \(l_1 = 1\) and \(l_2 \geq j_2 + 1\), then \(u^l v^t (v - 1)^s \in A_{l_1} \cdot A_{l_2}\).

3. We assume that \(b > 2, l = l_1 + l_2 \geq 3\) and \(\{\tilde{q}l_1/\tilde{n}\} = \{\tilde{q}l_1/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\}\) \((l_1 \geq 1, l_2 \geq 1)\). For \(s = \{l/2\}\) and \(t + s = bl - \{\tilde{q}l_1/\tilde{n}\}\), we have \(u^l v^t (v - 1)^s \in A_{l_1} \cdot A_{l_2} + A_{l_1} \cdot A_{l_2}\).

(Proof.)

1. It follows from Lemma 1 and the fact \(\{\tilde{q}l_1/\tilde{n}\} + 1 \geq \{\tilde{q}(l - 1)/\tilde{n}\} + \{\tilde{q}2/\tilde{n}\}\) and \(\{l - 2)/2\} + \{2/2\} = \{l/2\}\).

2. \(l_1 = \{\tilde{q}l_1/\tilde{n}\} = 2 + l_1 - 2 = \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l_1 - 2)/\tilde{n}\}\) by Corollary 1. Thus,

\[
\begin{align*}
\{\tilde{q}l_1/\tilde{n}\} &= \{\tilde{q}l_1/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\} = \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l_1 - 2)/\tilde{n}\} + \{\tilde{q}l_2/\tilde{n}\} \\
&\geq \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l_1 + l_2 - 2)/\tilde{n}\} = \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l - 2)/\tilde{n}\} \\
&\geq \{\tilde{q}l/\tilde{n}\},
\end{align*}
\]

that is, \(\{\tilde{q}l/\tilde{n}\} = \{\tilde{q}2/\tilde{n}\} + \{\tilde{q}(l - 2)/\tilde{n}\}\). Therefore by Lemma 1, \(u^l v^t (v - 1)^s \in A_{l_1} \cdot A_{l_2}\).
By Corollary 1, \( \{\tilde{q}l_{1}/\tilde{n}\} \leq l_{1} - 1 \) and \( \{\tilde{q}l_{2}/\tilde{n}\} \leq l_{2} - 1 \). Thus, since for \( s' = \{l_{1}/2\} + \{l_{2}/2\} \), we have \( t' = 2l - \{\tilde{q}l/\tilde{n}\} - s' \geq \{l_{1}/2\} + \{l_{2}/2\} \). By Lemma 1, \( u^{t'}(v - 1)^{s'} \in A_{l_{1}} \cdot A_{l_{2}} \).

If \( l_{1}l_{2} \) is even, \( t' = t \) and \( s' = s \). If \( l_{1} \) and \( l_{2} \) are odd, \( s' = \{l_{1}/2\} + \{l_{2}/2\} = \{l/2\} + 1 = s + 1 \). Then, using Lemma 2 (1) and

\[
\begin{align*}
(4) & \quad u^{t'}(v - 1)^{s} = u^{t'-1}(v - 1)^{s+1} + u^{t'-1}(v - 1)^{s},
\end{align*}
\]

we have \( u^{t'}(v - 1)^{s} \in A_{l_{1}} \cdot A_{l_{2}} \).

(c) By Corollary 1, \( \{\tilde{q}(j_{2} - 1)/\tilde{n}\} = j_{2} - 1 = k_{2} = \{\tilde{q}j_{2}/\tilde{n}\} \). Thus, \( 1 + \{\tilde{q}l_{2}/\tilde{n}\} = \{\tilde{q}(1+l_{2})/\tilde{n}\} \leq \{\tilde{q}j_{2}/\overline{n}\} + \{\overline{q}(l_{2}-j_{2}+1)/\tilde{n}\} \leq 1 + \{\tilde{q}l_{2}/\tilde{n}\} \), that is, \( \{\tilde{q}(j_{2}-1)/\mathrm{n}\} + \{\tilde{q}(l_{2}-j_{2}+1)/\tilde{n}\} = \{\tilde{q}(l_{2}+1)/\tilde{n}\} \). It comes back to (a).

3. For \( s' = \{l_{1}/2\} + \{l_{2}/2\} \), we have \( t' = bl - \{\tilde{q}l/\tilde{n}\} - s' \geq \{l_{1}/2\} + \{l_{1}/2\} \). By Lemma 1, \( u^{t'}(v - 1)^{s'} \in A_{l_{1}} \cdot A_{l_{2}} \). Using Lemma 2(1) and the equation (4) again, we have \( u^{t'}(v - 1)^{s} \in A_{l_{1}} \cdot A_{l_{2}} + A_{l-2} \cdot A_{2} \).

Q.E.D.

(\text{The proof of Theorem 3.)}

In the case of \( b = b_{1} = \cdots = b_{m_{0}} = 2 \), one sees \( A_{j} = 0 \) for \( 1 \leq 2j - 1 < j_{2} \), since \( b(2j-1) - \{\tilde{q}(2j-1)/\tilde{n}\} = 2(2j-1)-(2j-1) = 2j-1 < 2j = 2\{(2j-1)/2\} \) by Corollary 1. Also one sees that \( j_{2} + 1 \) has only \( l_{1} = 1 \) and \( l_{2} = j_{2} \) such as \( j_{2} + 1 = l_{1} + l_{2} \) and \( \{\tilde{q}(j_{2}+1)/\overline{n}\} = \{\tilde{q}l_{1}/\tilde{n}\} + \{\tilde{q}l_{2}/\tilde{n}\} \). Therefore, by Lemma 2 and Corollary 2, we have

\[
\begin{align*}
A_{2} & = \{u^{2}v(v - 1)\}, \\
A_{j_{2}+1} & \ni u^{2j_{2}+1}v^{2(j_{2}+1)-\{\tilde{q}(j_{2}+1)/\overline{n}\}}(v-1)^{(j_{2}+1)/2}, \\
A_{je} & \ni u^{j_{e}}v^{t}(v - 1)^{s}, \quad e \geq 2, s = \{j_{e}/2\}, t = 2j_{e} - \{\tilde{q}j_{e}/\tilde{n}\} - s
\end{align*}
\]

are minimal generators. Since one sees easily \( s_{e} \geq j_{e}/2, 2j_{e} - k_{e} - s_{e} \geq j_{e}/2 \), we have generators, substituting \( s = s_{e} \) for \( s = \{j_{e}/2\} \), using the equation (4). Finally by

\[
(5) \quad u^{t}(v(v - 1))^{s}(v - \frac{1}{2})^{t} = u^{t}v^{s+t}(v - 1)^{s} + \cdots + (-\frac{1}{2})^{t}u^{t}v^{s}(v - 1)^{s},
\]

the proof is completed for \( b = 2 \).

In the case of \( b \geq 3 \), easy computation yields that \( A_{1} \) and \( A_{2} \) are generated by \( uv^{t}(v - 1), t = 1, \cdots, b - 2 \) and \( u^{2}v^{t}(v - 1), t = 1, 2 \). By Lemma 2 and Corollary 2, we have

\[
\begin{align*}
A_{1} & \ni uv^{t}(v - 1), t = 1, \cdots, b - 2, \\
A_{2} & \ni u^{2}v^{t}(v - 1), t = 1, 2, \\
A_{je} & \ni u^{j_{e}}v^{t}(v - 1)^{s}, \quad e \geq 2, s = \{j_{e}/2\}, t = bj_{e} - \{\tilde{q}j_{e}/\tilde{n}\} - s
\end{align*}
\]
are minimal generators. Again using the equation (4), we have generators, substituting $s = j_e$ for $s = \{j_e/2\}$ and then the equation (5) completes the proof for $b > 2$.

The relations are obtained using $j_e = a_{e-1}j_{e-1} - j_{e-2} = (a_{e-1} - 1)j_{e-1} + j_{e-1} - j_{e-2} = \cdots = (a_{e-1} - 1)j_{e-1} + (a_{e-2} - 2)j_{e-2} + (a_{e-3} - 2)j_{e-3} + \cdots + (a_1 - 2)j_1 + j_1 - j_0$, etc.

Q.E.D.

**Theorem 4** Let $n/(n-q) = a_1' - \cdots - a_{p+1}'$ with $a_1', a_2', \cdots, a_p' \geq 2$ and let

\[
\begin{align*}
    j_0' &= 0, \\
    j_1' &= 1, \\
    j_e' &= a_{e-1}'j_{e-1}' - j_{e-2}' \\
    k_0' &= 1, \\
    k_1' &= 1, \\
    k_e' &= a_{e-1}'k_{e-1}' - k_{e-2}' \\
\end{align*}
\]

for $2 \leq e \leq p' + 1$.

Also let $s_1' = ((-1)^{k_2'+1} + 1)/2, s_2' = \{k_2'/2\}, s_e' = a_{e-1}'s_{e-1}' - s_{e-2}'$ for $3 \leq e \leq p' + 1$.

Minimal generators of holomorphic functions on $\tilde{X}$ defined everywhere are

\[
\begin{align*}
    &u^l(v(v-1)^s(v-1/2)^t) \\
    &= g_0, \quad l = 2, \quad s = 1, \quad t = 0 \\
    &= g_1, \quad l = k_2'+1, \quad s = \{(k_2'+1)/2\}, \quad t = ((-1)^{k_2'+1} + 1)/2 \\
    &= f_e', \quad l = k_e', \quad s = s_e', \quad t = j_e' - 2s_e' \\
\end{align*}
\]

and the relations are given by all the $2 \times 2$-minors of the matrices;

\[
\begin{pmatrix}
    g_0 & g_1 \\
    f_2' + \frac{(-s_0')s_1'-1}{4} & f_e'
\end{pmatrix}
\]

for $3 \leq e$ and given by all the generalized $2 \times 2$-minors of the quasi-matrix;

\[
\begin{pmatrix}
    f_2' & f_3' & f_4' & \cdots & f_p' \\
    f_3' & f_4' & \cdots & f_{p+1}'
\end{pmatrix}
\]

(Proof.) When $b = 2$, we have $a_1' = a_1 + 1, a_e' = a_e$ for $e = 2, \cdots, p+1 = p' + 1$ by Theorem 1 (5). Thus $j_1' = 1 = 2j_1 - k_1, j_2' = a_2' = a_1 + 1 = j_2' + 1 = 2j_2 - k_2$ and $j_e' = 2j_e - k_e$ for $e = 3, \cdots, p+1 = p' + 1$. Also $k_1' = 1 = j_1, k_2' = a_1' - 1 = a_1 = j_2$ and $k_e' = j_e$ for $e = 3, \cdots, p+1 = p' + 1$.

When $b > 2$, we have $a_1' = \cdots = a_{b-2}' = 2, a_{b-1}' = a_1 + 1, a_e' = a_{e-b+2}$ for $e = b, \cdots, p'+1 = p + b - 1$ by Theorem 1 (5). Thus $j_1' = 1, j_2' = 2, \cdots, j_{b-1}' = b - 1, j_b' = a_{b-1}'j_{b-1}' - j_{b-2}' = (a_1 + 1)j_{b-1}' - j_{b-2}' = bj_b - k_2$ and $j_e' = bj_e - k_{e-b+2}$ for $e = b, \cdots, p'+1 = p + b - 1$. Also $k_1' = k_2' = \cdots = k_{b-1}' = 1, k_b' = a_{b-1}'k_{b-1}' - k_{b-2}' = a_1 + 1 - 1 = j_2$ and $k_e' = j_e-b+2$ for $e = b, \cdots, p'+1 = p + b - 1$.

Q.E.D

### 3 Extended Function

Next let us extend these generators to holomorphic functions on the deformation space of the minimal resolution.
For variables $T = \{ t^{(1)}_1, t^{(1)}_2, t^{(1)}_0, \ldots, t^{(b-1)}_0, t^{(i)}_j : j = 1, \ldots, r; i = 1, \ldots, b_j - 1 \}$, consider the versal deformation space of the minimal resolution:

\[
\begin{align*}
    u = \frac{1}{u_{0}}, & \quad v = u^{\nu_2}v_{0} + t^{(1)}_1 u_{0}, \\
    u'_{0} = \frac{1}{u}, & \quad v'_{0} = u^2(v - 1) + t^{(1)}_2 u, \\
    v_1 = \frac{1}{v}, & \quad u_1 = v^b u + t^{(1)}_0 v^{b-1} + \cdots + t^{(b-1)}_0 v, \\
    u_2 = \frac{1}{u_1}, & \quad v_2 = u_{1}^{b} v_{1} + t^{(1)}_1 v_{1}^{b-1} + \cdots + t^{(b-1)}_1 v_{1}, \\
    v_3 = \frac{1}{v_2}, & \quad u_3 = v_{2}^{b_{2}} u_{2} + t^{(1)}_2 v_{2}^{b_{2}-1} + \cdots + t^{(b-2)}_2 v_{2}, \\
    & \ldots
\end{align*}
\]

Let

\[
\begin{align*}
    H_{0}^{(0)} &= u, \\
    H_{0}^{(1)} &= uv + t^{(1)}_0, \\
    H_{0}^{(2)} &= (uv + t^{(1)}_0)v + t^{(2)}_0, \\
    \vdots \\
    H_{0}^{(b-1)} &= \cdots (uv + t^{(1)}_0)v + \cdots + t^{(b-2)}_0 v + t^{(b-1)}_0, \\
    H_{1}^{(1)} &= H_{0}^{(b-1)} + t^{(1)}_1 = u_{1}v_{1} + t^{(1)}_1, \\
    H_{1}^{(2)} &= H_{1}^{(1)}H_{0}^{(b-1)}v + t^{(2)}_1 = (u_{1}v_{1} + t^{(1)}_1)u_{1} + t^{(2)}_1, \\
    \vdots \\
    H_{1}^{(b-1)} &= H_{1}^{(b_{1}-2)}H_{1}^{(b_{1}-1)}v + t^{(b-1)}_1, \\
    H_{2}^{(1)} &= H_{1}^{(b-1)} + t^{(2)}_2 = u_{2}v_{2} + t^{(2)}_1, \\
    H_{2}^{(2)} &= H_{2}^{(1)}H_{1}^{(b-1)}H_{0}^{(b-1)}v + t^{(2)}_2 = (u_{2}v_{2} + t^{(1)}_2)v_{2} + t^{(2)}_2, \\
    \vdots \\
    \vdots \\
    H_{r}^{(b_r)} &= H_{r}^{(b_r-1)}H_{r-1}^{(b_r-1)} \cdots H_{0}^{(b-1)}v
\end{align*}
\]

These functions are holomorphic defined everywhere on $\mathbb{C}^2$ of the coordinate systems $C(u, v)$ and $C(u_m, v_m)$ $(m \geq 1)$, which are introduced by Riemenschneider [9] as "extended functions" of generators $u^{j_{e}}v^{k_{e}}$ for corresponding cyclic quotient singularity $C_{n,q}$.

Using these functions we will construct extended functions for $D_{n,q}$ singularities, which become extremely more complicated than those for cyclic singularities.

Let $w_{-2} = -(t^{(1)}_n + t^{(1)}_o)/2$, $w_{-1} = (t^{(1)}_n - t^{(1)}_1)/2$, $w_{0} = t^{(1)}_0 - (t^{(1)}_2 - t^{(1)}_1)/2$, $w_{l} = t^{(1)}_l + t^{(1)}_0 + \cdots + t^{(1)}_1 + (-t^{(1)}_n + t^{(1)}_1)/2$ for $l = 1, \ldots, k_{2}' - 1$. Also let us denote $\sum_{i_0 \leq j_1 < j_2 < \cdots < j_k \leq l} w_{j_1}w_{j_2} \cdots w_{j_k}$ by $\sum_{i_0}^{i_l} w_{J_{k}}$.

Let

\[
\begin{align*}
    G_{0} &= (u(v - 1) + w_{-1} - w_{-2})(uv + w_{-1} + w_{-2}) + w^{2}_{-2}, \\
    X_{0} &= v(u(v - 1) + w_{-1} + w_{0}) - 1/2 \sum_{-2}^{0} w_{J_{1}}.
\end{align*}
\]
\[ X_1 = (G_0 - w_0^2)(v - 1/2) + (w_0 + w_1)X_0 - 1/2 \prod_{k=-2}^{k} w_k \]

Inductively, let \( X_l \) \((l \geq 2)\) be

\[ X_l = X_{l-2}\{G_0 - (w_{l-1})^2\} + (w_{l-1} + w_l)X_{l-1} - \frac{1}{2} \prod_{k=-2}^{l-2} w_k \]

\[ = v(u(v - 1) \sum_{k=0}^{(l-1)/2} G_0^k \sum_{0}^{l} w_{J_{l-2k}} + \sum_{k=0}^{l/2} G_0^k \sum_{-1}^{l-1} w_{J_{l+1-2k}}) \]

\[ - \frac{1}{2} \sum_{k=0}^{(l/2)} G_0^k \sum_{-2}^{l} w_{J_{l+1-2k}} \]

**Lemma 3** \( G_0 \) is a holomorphic extended function of \( g_0 \).

(Proof.) Because it is clear that \( G_0|_{T=0} = g_0 \), we need to show that \( G_0 \) is holomorphic on each coordinate system and it is proved by

\[ G_0 = (u''_0 v''_0 - w_{-2})^2 - v''_0 \]

\[ = (u'_0 v'_0 + w_{-2})^2 + v'_0 \]

\[ = (H_0^{(1)} - w_0)^2 - H_0^{(0)}(H_0^{(1)} - w_0 + w_{-2}). \]

**Lemma 4** \( X_l \) is an extended function of

\[ u^{l+1}(v(v - 1))^{(l+1)/2}(v - 1/2)^{((-1)^{l+1}+1)/2}. \]

Moreover \( X_{k_2-1}, X_{k_2}'|_{w_{k_2}=0} \) are holomorphic, and \( X_{k_2-1}, X_{k_2}'|_{w_{k_2}=0} \) are extended functions of \( g_1, f'_2 \).

Remark: The variables \( w_l \) are defined at \( l \leq k'_2 - 1 \). So \( X_{k_2}'|_{w_{k_2}=0} \) means \( X_{k_2-2}\{G_0 - (w_{k_2-1})^2\} + w_{k_2-1}X_{k_2-1} - \frac{1}{2} \prod_{k=-2}^{k_2-2} w_k \).

(Proof.) By easy computation, we have

\[ X_l|_{T=0} = u^{l+1}(v(v - 1))^{(l+1)/2}(v - 1/2)^{((-1)^{l+1}+1)/2}. \]

Since

\[ X_0 = u''_0((u''_0 v''_0 - w_{-2} - w_{-1})(u''_0 v''_0 - w_{-2} + w_0) - v''_0) \]

\[ + (w_{-2} + w_{-1} - w_0)/2 \]

\[ = u'_0((u'_0 v'_0 + w_{-2} - w_{-1})(u'_0 v'_0 + w_{-2} + w_0) + v'_0) \]

\[ + (w_{-2} - w_{-1} + w_0)/2 \]

and the definitions of \( X_l \) \((9)\), all \( X_l \) are holomorphic on \( \mathbb{C}^2 \) of the coordinate systems \( C(u''_0, v''_0) \) and \( C(u'_0, v'_0) \).
Let $\tilde{X}_l = X_l - H_0^{(1)}H_1^{(1)}\cdots H_l^{(1)}v$.

Inductively, it is proved that $\tilde{X}_l$ are expressed by polynomials of $G_0, uv = H_0^{(1)} - w_{-1} - w_0$ and parameters $w_i$ by

\[
\tilde{X}_0 = -uv - 1/2 \sum_{-2}^0 w_J,
\]
\[
\tilde{X}_1 = -(uv)^2 - uv \sum_{-2}^1 w_J - \frac{1}{2} \sum_{k=0}^{1} G_0^k \sum_{-2}^1 w_{J-2-k},
\]
\[
\tilde{X}_l = -H_0^{(1)}H_1^{(1)} \cdots H_l^{(1)}uv(uv + w_{-2} + w_{-1}) + \tilde{X}_{l-2}(G_0 - (w_{l-1})^2) + \tilde{X}_{l-1}(w_{l-1} + w_{l}) - \frac{1}{2} \prod_{k=-2}^{l-2} w_k
\]

Thus since $G_0, uv = H_0^{(1)} - w_{-1} - w_0$ are holomorphic on $\mathbb{C}^2$ of the coordinate systems $C(u_m, v_m) (m \geq 1)$, $X_l$ are holomorphic on the whole space if and only if $H_0^{(1)}H_1^{(1)} \cdots H_l^{(1)}v$ are holomorphic on $\mathbb{C}^2$ of the coordinate systems $C(u_m, v_m) (m \geq 1)$.

If $b = b_1 = \cdots = b_{l-1} = 2, b_l > 2, H_l^{(2)} = H_0^{(1)}H_1^{(1)} \cdots H_l^{(1)}v + t_l^{(2)}$ is holomorphic on $C(u_m, v_m) (m \geq 1)$ and if $b > 2, H_0^{(2)} = H_l^{(1)}v + t_0^{(2)}$ is holomorphic on $C(u_m, v_m) (m \geq 1)$.

Therefore $X_{k_2-1}$ and $X_{k_2}|_{w_{k_2}'=0}$ are holomorphic on the whole space.

Q.E.D.

**Lemma 5** Let $b_0 = b$. Assume that the sequence of functions satisfy for $m \geq 0, h_m = 2, \cdots, b_m$,

\[
(10) \quad f_m^{(1)} = f_{m-1}^{(b_m-1)}, \quad f_m^{(h_m)} = f_m^{(h_m-1)}f_{m-1}^{(b_m-1)},
\]
\[
f_m^{(h)}(k_2-2) = f_{m-1}^{(h)}(k_2-2) / f_1^{(1)} \quad \text{and} \quad f_m^{(h)}(k_2-2) = f_2^{(2)}.
\]

Then it holds that $f_e' = f_m^{(h_m)}$ where $e = \sum_{i=0}^{m-1}(b_i - 2) + h_m, 2 \leq h_m \leq b_m - 1$ and $m = 0, \cdots, r$.

Also it holds that $f_{p+1}^{(h)} = f_r^{(b_r)}$.

(Proof.) Since $f_m^{(2)}(k_2-2) = f_0^{(1)}f_{k_2-2}^{(2)}$, we have $f_0^{(1)} = f_1^{(1)}$ and Theorem 2 completes the proof.

Q.E.D.

**Lemma 6** Let $b_0 = b$. There exist functions $F_m^{(2)}(k_2-2), F_m^{(h_m)}(k_2-1 \leq m \leq r, 1 \leq h_m \leq b_m)$ of $u, v$ with parameters in $T$ and polynomials $C_m^{(h_m)}$ of $t_1^{(1)}, t_2^{(1)}, t_m^{(h_m)} (m' < m)$ in $T$ such that

\[
F_m^{(2)}(k_2-2) = \begin{cases} X_{k_2-2}^{(2)} & k_2 \geq 2 \\ v - 1/2 & k_2 = 1 \end{cases}
\]
\[
F_m^{(2)}(k_2-1) = X_{k_2-1} + t_{k_2-1}^{(2)} - C_{k_2-1}^{(2)}
\]
with $C_m^{(b_m)} = 0$ and that for $k'_2 - 1 \leq m \leq r, 1 \leq h_m \leq b_m - 1$, $F_m^{(h_m)}$ are holomorphic on the whole space with $F_m^{(h_m)}|_{T=0} = f'_e$ where $e = \sum_{i=0}^{m-1}(b_i - 2) + h_m$.

(Proof.)

For simple notation, we set $H_m = H_{m}^{(b_m-1)}$ and $L_m = (l_{-1}, \ldots, l_{m-1}, s, h) \in \{l_{-1}, \ldots, l_{m-1} \geq 0, 0 \leq s \leq m, 1 \leq h \leq b_m - 2\}$.

Since $X_{k'_2-1}$ is holomorphic on the whole space, the function has a polynomial expression with $H_0^{(0)}, H_0^{(h)}, H_{k'_2-1}^{(1)}, H_{k'_2-1}^{(2)}$ and parameters in $T$.

Using the relation $H_m^{(1)} = H_{m-1} + t_m^{(1)}$ we have

$$X_{k'_2-1} = H_{k'_2-1}^{(2)} - t_{k'_2-1}^{(2)} + \sum_{L_{k'_2-2}} C_{L_{k'_2-2}}^{(2)} (H_0^{(0)})^{l_{-1}} H_0^{l} H_s^{(h)} + D_{k'_2-2}^{(2)}$$

where $C_{L_{k'_2-2}}^{(2)}$ and $C_{k'_2-1}^{(2)}$ are polynomials of $t_{m'}^{(1)}, t_{m'}^{(h_{m'})}$ ($m' < k'_2 - 1$), $t_{k'_2-1}^{(1)}$.

We set $F_{k'_2-1}^{(2)} = X_{k'_2-1} + t_{k'_2-1}^{(2)} - C_{k'_2-1}^{(2)}$.

$F_{k'_2-2}^{(2)}$ is not holomorphic on $C(u_m, v_m)$ ($m \geq 1$) but similarly we have

$$F_{k'_2-2}^{(2)} = H_{k'_2-3} H_{k'_2-4} \cdots H_0 v + \sum_{L_{k'_2-2}} C_{L_{k'_2-2}}^{(2)} (H_0^{(0)})^{l_{-1}} H_0^{l} H_s^{(h)} + D_{k'_2-2}^{(2)}$$

where $C_{L_{k'_2-2}}^{(2)}$ and $D_{k'_2-2}^{(2)}$ are polynomials of $t_{m'}^{(1)}, t_{m'}^{(h_{m'})}$ ($m' \leq k'_2 - 2$).

Inductively using the relations such as

$$H_{m}^{(h+1)} = H_{m}^{(h)} H_{m-1} \cdots H_0 v + t_{m}^{(h+1)}$$

$$H_{m}^{(h)} H_{m'}^{(h')} = H_{m}^{(h)} (H_{m'}^{(h')}) - H_{m'}^{(h') - 1} H_{m'-1} \cdots H_{m+1} + H_{m}^{(h)} t_{m'}^{(h')}$$

$$H_{m}^{(h)} H_{m'}^{(h')} = H_{m}^{(h)} H_{m'-1} + H_{m'}^{(h')} t_{m}^{(h')}$$

$$H_{m}^{(1)} = H_{m-1} + t_{m}^{(1)}$$

we construct $F_m^{(h_m)}(k'_2 - 1 \leq m \leq r, 1 \leq h_m \leq b_m - 1)$ with expressions

$$(13) \quad F_{m}^{(1)} = F_{m-1}^{(b_m-1)} + t_{m}^{(1)}$$

$$(14) \quad F_{m}^{(h_m)} = F_{m-1}^{(b_m-1)} F_{m-1}^{(h_m)} + t_{m}^{(h_m)} - C_{m}^{(h_m)}$$
\[
F_m^{(b_m)} = F_m^{(b_m-1)}F_m^{(b_m-1)} = H_{m-1}H_{m-2} \cdots H_0 v + \sum_{L_m} C_{L_m}^{(b_m)}(H_0^{(0)})^{l_{-1}}H_0^{l_0}H_1^{l_1} \cdots H_{m-1}^{l_{m-1}}H_s^{(h)} + D_m^{(b_m)}
\]

where \(C_{L_m}^{(b_m)}, C_m^{(h_m)}, D_m^{(b_m)}\) are polynomials of \(t_1^{(m')}, t_2^{(m')}, t_m^{(h_m')} (m' < m), t_m^{(h_m)} (h_m' < h_m)\).

For \(k_{2}'-1 \leq m \leq r\), by equations (13) and (14), \(F_m^{(b_m)}\) are holomorphic on \(\mathbb{C}^2\) of the coordinate systems \(C(u_m, v_m) (m \geq 1)\), and by equations (11) and (12), \(F_m^{(h_m)}\) are holomorphic on \(\mathbb{C}^2\) of the coordinate systems \(C(u_0, v_0)\) and \(C(u_0', v_0')\).

By Lemma 5, we also have \(F_m^{(h_m)}|_{T=0} = f_e'\) where \(e = \sum_{i=0}^{m-1}(b_i - 2) + h_m\).

Q.E.D.

**Theorem 5** We set \(F_e = F_m^{(h_m)} - t_m^{(1)}B + C_m^{(h_m)}\) where \(e = \sum_{i=0}^{m-1}(b_i - 2) + h_m \geq 2\) and \(b_0 = b\).

Then \(G_0, B = X_{k_2'-1}, A = X_{k_2'} \mid w_{k_2} = 0\) and \(F_e (e \geq 2)\) are extended functions of \(g_0, f_2', g_1\) and \(f_e' (e \geq 2)\), respectively.

There exists the set of variables \(W = \{w_1^{(h_1)}, w_m^{(h_m)}; -2 \leq h_1 \leq a_1'-1, 2 \leq m \leq p', 1 \leq h_m \leq a_m'-1\}\) which is algebraic isomorphic to \(T\) such that the relations of the functions \(G_0, A, B, F_e, W\) are given by all the \(2 \times 2\)-minors of the matrices;

\[
\begin{pmatrix}
G_0 - (w_1^{(a_1'-2)})^2, & A + w_1^{(a_1'-2)}B - \frac{(\Pi_{h=1}^{a_1'-2}w_1^{(h)})^2}{4G_0}, \\
A - w_1^{(a_1'-2)}B + \frac{(\Pi_{h=1}^{a_1'-2}w_1^{(h)})^2}{2}, & B^2 + \frac{\Pi_{h=1}^{a_1'-2}(w_1^{(h)})^2 - (\Pi_{h=1}^{a_1'-2}w_1^{(h)})}{4G_0}A,
\end{pmatrix}
\]

\[
(F_e - w_1^{(a_1'-1)})\Pi_{m=2}^{\hat{m}=1}(F_m - w_m^{(h_m)})
\]

for \(3 \leq e\) and given by all the generalized \(2 \times 2\)-minors of the quasi-matrix;

\[
\begin{pmatrix}
F_2 & F_3 & F_4 & \cdots & F_p' \\
\Pi_{h=1}^{a_2'-2}(F_3 - w_3^{(h)}) & \Pi_{h=1}^{a_3'-2}(F_4 - w_4^{(h)}) & \cdots & F_{p'+1}
\end{pmatrix}
\]

(Proof.)

It follows by if \(b_{m+1} = \cdots = b_{\hat{m}}\),

\[
F_m^{(b_m-1)} \cdots F_m^{(b_{\hat{m}}-1)} = F_m^{(b_m-1)}(F_m^{(b_m-1)} + t_{m+1}^{(1)})(F_m^{(b_m-1)} + t_{m+1}^{(1)} + t_{m+2}^{(1)}) \cdots (F_m^{(b_m-1)} + t_{m+1}^{(1)} + \cdots + t_{\hat{m}}^{(1)})
\]

and

\[
\text{rank}
\begin{pmatrix}
1, & G_0 - w_2^{(k_2'-1)}, & A + w_2^{(k_2'-1)}B - \frac{\Pi_{k=2}^{k_2'-1}w_k^2}{2}, \\
X_{k_2'-2}, & A - w_2^{(k_2'-1)}B + \frac{\Pi_{k=2}^{k_2'-1}w_k^2}{2}, & B^2 + \frac{\Pi_{k=2}^{k_2'-1}(-G_0 + w_2^2) - \Pi_{k=2}^{k_2'-1}w_k^2}{4G_0}
\end{pmatrix}
\]
Moreover from those relations we can see a canonical candidate for the full Weyl group.

1. In the case of $b = b_1 = \cdots = b_r = 2$, i.e., a rational double point, the relation of these functions which was also shown in G. N. Tyurina [15], is

$$0 = A^2 - B^2G_0 + B\Pi_{k=-2}^{r}w_k - \frac{\Pi_{k=-2}^{r}(-G_0 + w_k^2) - \Pi_{k=-2}^{r}w_k^2}{4G_0}$$

The corresponding Weyl group is $S_{r+3} \times Z_2^{r+2}$.

2. In the case of $b = b_1 = \cdots = b_{a_1'-3} = 2$, $b_{a_1'-2} \tau^{\leq 2(a_1'-3)} \leq a_1'$, the corresponding Weyl group is $S_{a_1'} \times Z_2^{a_1'-1} \times S_{a_2'-2} \times \cdots \times S_{a_p'-2}$ and it is easy to see how to act the Weyl group.

3. In the case of $b \geq 3$, a part of the relations

$$\text{rank} \left( \begin{array}{c} G_0 - (w_1^{(0)})^2, \\ A - w_1^{(0)}B + \frac{w_1^{(-2)}w_1^{(-1)}}{2}, \\ A + w_1^{(0)}B - \frac{w_1^{(-2)}w_1^{(-1)}}{2}, \\ B^2 + \frac{G_0 - (w_1^{(-2)})^2 - (w_1^{(-1)})^2}{4} \end{array} \right) < 2$$

and the corresponding Weyl group is $S_2 \times S_2 \times S_{a_2'-2} \times \cdots \times S_{a_p'-2}$.

By putting $\tilde{w}_1^{(-2)} = w_1^{(-2)} + w_1^{(-1)}$ and $\tilde{w}_1^{(-1)} = w_1^{(-2)} - w_1^{(-1)}$, one can see how to act the Weyl group since $w_1^{(-2)} - w_1^{(-1)} = (\tilde{w}_1^{(-2)})^2/4 - (\tilde{w}_1^{(-1)})^2/4$ and $(w_1^{(-2)})^2 + (w_1^{(-1)})^2 = (\tilde{w}_1^{(-2)})^2/2 + (\tilde{w}_1^{(-1)})^2/2$.

When $r = 1$ and $b > 2$, the exact coefficients $C_m^{(b_n)}$ are calculated ([16]) but in general these values which are defined by induction are very complicated.

4 Appendix

The theorem is proved by the similar way of Theorem 1 but another proof is shown here.

**Theorem 6** For any integer $l$, there exist $0 \leq t_m \leq b_{m+1} - 1$ such as $l = \sum_{m=0}^{r} t_m n_m$.

Then

$$\left\{ \frac{\tilde{q}}{n} l \right\} = \sum_{m=1}^{r} t_m q_m + 1.$$

(Proof)

Let $s_m$ be a positive integer defined by

$$\frac{s_m}{q_m} = \frac{1}{b_2 - 1/b_3 - \cdots - 1/b_m}$$

for $m = 0$. 
\[ n_i(j) = (b_i - \frac{1}{b_{m+1}} \cdots - \frac{1}{b_j}) n_{i-1} - n_{j-2} \]

\[ q_i(j) = (b_i - \frac{1}{b_{m+1}} \cdots - \frac{1}{b_j}) q_{i-1} - q_{j-2}. \]

\[ n_{i+1}(j) \]
\[ = (b_{i+1} - \frac{1}{b_{i+2}} \cdots - \frac{1}{b_j}) n_i - n_{i-1} \]
\[ = (b_{i+1} - \frac{1}{b_{i+2}} \cdots - \frac{1}{b_j}) (b_{n_i-1} - n_{i-2}) - n_{i-1} \]
\[ = (b_{i+1} - \frac{1}{b_{i+2}} \cdots - \frac{1}{b_j}) (b_{i+1} - \frac{1}{b_{i+2}} \cdots - \frac{1}{b_j}) n_{i-1} \]
\[ = (b_{i+1} - \frac{1}{b_{i+2}} \cdots - \frac{1}{b_j}) ((b_{i+1} - \frac{1}{b_{i+2}} \cdots - \frac{1}{b_j}) n_{i-1} - n_{i-2}) \]
\[ = (b_{i+1} - \frac{1}{b_{i+2}} \cdots - \frac{1}{b_j}) n_i(r) \]

Also
\[ q_{i+1}(r) = (b_{i+1} - \frac{1}{b_{i+2}} \cdots - \frac{1}{b_j}) q_i(j) \]

We have
\[ \frac{\tilde{q}}{n} n_m = \frac{\tilde{q}}{n} (b_1 q_m - s_m) = \frac{b_1 q_m - s_m}{b_1 - \frac{1}{b_2} \cdots - \frac{1}{b_r}} \]
\[ = q_m + \frac{1}{b_2 - \frac{1}{b_3} \cdots - \frac{1}{b_r}} - \frac{1}{b_1 - \frac{1}{b_2} \cdots - \frac{1}{b_r}} \]
\[ = q_m + q_m \frac{b_2 - \frac{1}{b_3} \cdots - \frac{1}{b_r}}{b_1 - \frac{1}{b_2} \cdots - \frac{1}{b_r}} - \frac{1}{b_2 - \frac{1}{b_3} \cdots - \frac{1}{b_r}} \]
\[ = q_m + q_m \frac{b_2 - \frac{1}{b_3} \cdots - \frac{1}{b_r}}{b_1 - \frac{1}{b_2} \cdots - \frac{1}{b_r}} - \frac{1}{b_2 - \frac{1}{b_3} \cdots - \frac{1}{b_r}} \]
\[ + q_m \left\{ \frac{1}{b_1 (b_2 - \frac{1}{b_3} \cdots - \frac{1}{b_r}) - 1} \right\} \left\{ \frac{1}{b_2 - \frac{1}{b_3} \cdots - \frac{1}{b_r}} \right\} \]
\[ = q_m + q_m \frac{1}{b_3 - \frac{1}{b_4} \cdots - \frac{1}{b_r} - b_3 - \frac{1}{b_4} \cdots - \frac{1}{b_m}} n_2(r) q_2(m) \]
\[ = q_m + q_m \frac{1}{b_4 - \frac{1}{b_5} \cdots - \frac{1}{b_r} - b_4 - \frac{1}{b_5} \cdots - \frac{1}{b_m}} n_3(r) q_3(m) \]
\[ = q_m + q_m \frac{1}{b_{m+1} - \frac{1}{b_{m+2}} \cdots - \frac{1}{b_r}} n_m(r) q_m(m) \]
On the other hand,

\[
\frac{(b_r-1)}{n_r} + \sum_{m=0}^{r-2} \frac{b_{m+1}-2}{n_{m+1}(r)} = \frac{1}{n_{r-1}(r)} \left( \frac{(b_r-1)/b_r + b_{r-1}-2}{n_{r-1}(r)} + \sum_{m=0}^{r-3} \frac{b_{m+1}-2}{n_{m+1}(r)} \right)
\]

\[
= \frac{1}{n_{r-2}(r)} \left( \frac{b_{r-2} - \frac{1}{b_r}}{b_{r-1} - \frac{1}{b_r}} - \frac{1}{b_r} \right) + \sum_{m=0}^{r-4} \frac{b_{m+1}-2}{n_{m+1}(r)}
\]

\[
= \frac{1}{b_1 (b_2 - \frac{1}{b_3} - \cdots - \frac{1}{b_r}) - 1} \left( b_2 - \frac{1}{b_3} - \cdots - \frac{1}{b_r} - 1 \right)
\]

\[
= 1 - \frac{1}{b_1 - \frac{1}{b_2} - \cdots - \frac{1}{b_r}}.
\]

Hence by \( \tilde{n} = (b_r - 1)n_{r-1} + \sum_{m=1}^{r-1} (b_m - 2)n_m + (b_1 - 1)n_0 \), it holds that for \( 0 < l < \tilde{n} \),

\[
0 < \sum_{m=0}^{r-1} \frac{t_m}{n_{m+1}(r)} < 1, \text{ i.e., } \{ \frac{\tilde{q}}{\tilde{n}} \} = \sum_{m=1}^{r} t_m q_m + 1.
\]

References


