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Asymptotic Expansions of Elastic Waves Reflected Totally by Boundaries

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Introduction

In this note we consider the (linear) elastic wave equation, and construct asymptotic expansions (solutions) of the waves reflected by boundaries. The simple and typical equation of the wave phenomena is the d'Alembert one. For this equation there are many results on construction of the solutions, formulation of scattering theories, examination of the energy decay, etc. The elastic equations are often regarded as an analogy to the d'Alembert one containing a little more complex phenomena, or as a concrete example of general hyperbolic systems.

The elastic equations, however, seem to possess their own characteristic properties different from the d'Alembert one and the general systems, and should be examined by methods suiting the elastic case. We know that in the elastic waves there exist several modes of the body waves (e.g., P-wave, S-wave, etc), the surface waves (e.g., the Rayleigh wave, the evanescent wave, etc.) and so on. In order to examine those waves, we should develop methods suitable to the elastic waves.

To express behavior of the waves, we often employ asymptotic expansions (solutions). Those expansions $u(t,x)$ mean functions of the form

$$u(t,x)=\sum_{j=0}^{N}\rho_j(t-\varphi(x))v^j(t,x). \tag{0.1}$$

Here, $\varphi(x)$ (the phase function) and v^j (the amplitude function) are chosen so that u satisfies the wave equation approximately. $\rho_0(s),\rho_1(s),\dots$ $(s\in$

$\mathbb{R}$) are scalar-valued functions or distributions satisfying

$$\frac{d\rho_j}{ds}(s) = \rho_{j-1}(s) \qquad (j = 1, 2, \dots),$$

e.g., $\rho_0(s) = \delta(s)$, $\rho_1(s) = h(s)$ (the Heaviside function) , ... or $\rho_0(s) = e^{i\sigma s}$, $\rho_1(s) = (i\sigma)^{-1}e^{i\sigma s}$, ... ($\sigma > 0$). Many authors have made the asymptotic solutions for elastic equations. Karal-Keller [1] made them (of the type $\rho_j(s) = (i\sigma)^{-j}e^{i\sigma s}$) for the isotropic equation in various cases of reflection. Soga [5] has dealt with the anisotropic equation concerning the type $\rho_j(s) = (i\sigma)^{-j}e^{i\sigma s}$.

The main goal in this note is to construct the asymptotic solutions of total reflection for the incident wave of the Dirac δ-function type (see Theorem 3.1 in §3). For the type $\rho_j(s) = (i\sigma)^{-j}e^{i\sigma s}$ those solutions have been made in Soga [7]. Because of elasticity we can accomplish the construction so that sum of the body wave and the evanescent wave coincides with any incident wave on the boundaries. In general hyperbolic systems this is not necessarily expected without aditional assumptions. For the proof we apply the theory of complex functions to the symbol of the elastic operator, which is developed in Soga [7], Kawashita-Ralston-Soga [2], etc. The main part of the result is described in Soga [9].

§1. Equations and Assumptions

Let Ω be a domain in $\mathbb{R}^n$, and consider the (linear) elastic wave equation

$$(\partial_t^2 - L(x, \partial_x))u(t, x) = 0 \quad \text{in } \mathbb{R} \times \Omega, \tag{1.1}$$

where u is the displacement vector and $L(x, \partial_x)$ is of the form

$$L(x, \partial_x) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b_i(x)\partial_{x_i} + c(x).$$

The coefficients a_{ij}, b_i and c are matrices whose components consist of bounded real-valued C^∞ functions with bounded derivatives. We assume that $a_{ij}(x)$ satisfy the following assumptions at each x.

(A.1) $\quad a_{ij}(x) = {}^t a_{ji}(x), \quad i, j = 1, 2, \dots, n.$

(A.2) $\quad L_0(x, \xi) \equiv \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$ is positive definite.

(A.3) The eigenvalues of $L_0(x,\xi)$ are of constant multiplicity on $\mathbb{R}^n \times (\mathbb{R}^n - \{0\})$.

There are several kinds of the elastic (body) waves, e.g., P-wave and S-wave in the isotropic case, and they are classified by means of the eigenvalues of $L_0(x,\xi)$. These kinds are called the modes. From the assumptions (A.2) and (A.3) we can denote the eigenvalues of $L_0(x,\xi)$ by the C^∞ functions $\lambda_j(x,\xi)$ on $\mathbb{R}^n \times (\mathbb{R}^n - \{0\})$ satisfying

$$(0 <)\lambda_1(x,\xi) < \lambda_2(x,\xi) < \cdots < \lambda_d(x,\xi).$$

Here, we add an assumption on the eigenvalues: The slowness surface $\Sigma_j = \{\xi : \lambda_j(x,\xi) = 1\}$ satisfies:

(A.4) Every Σ_j is convex and the Gaussian curvature does not vanish.

From this assumption it follows that for a vector $\eta \in \mathbb{R}^n - \{0\}$ with $\lambda_j(x,\eta) < 1$ and a unit vector ν normal to η, there exist two roots $z = z_\pm^j(x,\eta)$ of the equation $\lambda_j(x,\eta + z\nu) = 1$ which satisfy

$$\lambda_j(x,\eta + z_\pm(x,\eta)\nu) = 1, \qquad \pm\partial_\nu\lambda_j(x,\eta + z_\pm(x,\eta)\nu) > 0.$$

$z_+(x,\eta)$ $(z_-(x,\eta))$ is called the outgoing (incoming) root with respect to ν, which is connected with direction of the propagation of the waves. And we say that η is non-glancing if $\partial_\nu\lambda_k(x,\eta + z\nu) \neq 0$ for all k and real z satisfying $\lambda_k(x,\eta + z\nu) = 1$.

§2. Procedures of Construction of Asymptotic Expansions

In this section, assuming that the conditions (A.1)$\sim$(A.4) are satisfied, we describe the procedures of construction of the asymptotic expansions. This is a summary of the procedures in the papers by Soga [5,6,8]. The basic idea is similar to Lax's [4] for general hyperbolic systems, but fairly different concerning treatment of the amplitude functions, etc.

At first, we explain the basic profile functions (i.e., the functions $\rho_j(s)$ in (0.1)).

(i) δ-function type: Let $\delta(s)$ be the Dirac δ-function, and set

(2.1) $\rho_0(s) = \delta(s),$

$$\rho_j(s) = 0 \quad (s \le 0), \quad = s^{j-1}/(j-1)! \quad (s > 0) \quad \text{for } j = 1, 2, \ldots .$$

(ii) Oscillatory type: Let σ be a positive parameter, and set

$$\rho_j(s) = (i\sigma)^{-j} e^{i\sigma s} \quad (j = 0, 1, \ldots). \tag{2.2}$$

(iii) Logarithmic type: Extend the variable s to the complex one $z = s + ir$ $(s, r \in \mathbb{R})$, and set

$$\begin{aligned} \tilde{\rho}_0(z) &= (2\pi i)^{-1} z^{-1}, \qquad \tilde{\rho}_1(z) = (2\pi i)^{-1} \log z, \\ \tilde{\rho}_j(z) &= (2\pi i)^{-1} \int_0^z \frac{(z-\zeta)^{j-2}}{(j-2)!} \log \zeta \, d\zeta \quad (j = 2, 3, \ldots), \end{aligned} \tag{2.3}$$

where we take a branch (of log z) connected with arg $z = 0$ when Im $z = 0$ and Re $z > 0$, and the integral on a path linking 0 to z. We use the notation $\tilde{\rho}_j$ instead of ρ_j in the case of the logarithmic type.

In (2.1) and (2.3), the singularity of ρ_j and $\tilde{\rho}_j$ at s (and z) $= 0$ is smoother as j increases. In (2.2), the order of multiplication of $(i\sigma)^{-1}$ is larger when j is larger, and then $|\rho_j|$ tends to 0 faster as $\sigma \to \infty$. Thus, $|\rho_j|$ is closer to 0 in some senses when j is larger.

In this note we consider the case that a wave of the δ-function type hits the boundary and is reflected totally. Then the reflected waves consist of not only the δ-function type but also the logarithmic type. Therefore, we employ the logarithmic type.

Inserting the function (0.1) into $(\partial_t^2 - L(x, \partial_x))u(t, x)$, we can write it in the following way.

$$\begin{aligned} (\partial_t^2 - L(x, \partial_x))u(t, x) = \rho_0''(t - \varphi)\{I - L_0(x, \partial_x\varphi)\}v^0(t, x) \\ + \rho_0'(t - \varphi)(\partial_t - H)v^0(t, x) \\ + \rho_0(t - \varphi)Lv^0(t, x) + \cdots , \end{aligned} \tag{2.4}$$

where $H = \sum_{p,q=1}^n (a_{pq} + a_{qp})(\partial_{x_p}\varphi)\partial_{x_q} - \sum_{p=1}^n b_p(\partial_{x_p}\varphi) + (L\varphi - c)$. We choose the phase function φ and the amplitude functions $v^j(t, x)$ inductively so that each of terms multiplied by $\rho_0'', \rho_0', \ldots$. vanishes (except the last remainder term). Let us explain the procedure of this choosing.

To eliminate the term in (2.4) containing $\rho_0''(t - \varphi(x))$, we require

$$\det\, (I - L_0(x, \partial_x\varphi)) = 0, \qquad v^0 \in \operatorname{Ker}\, (I - L_0(x, \partial_x\varphi)), \tag{2.5}$$

where $\mathrm{Ker}(I - L_0(x, \partial_x\varphi)) = \{v \in \mathbb{C}^n; \, (I - L_0(x, \partial_x\varphi))v = 0\}$. The former in (2.5) implies that $\lambda_l(x, \partial_x\varphi) = 1$ for some l. This is solved by the Hamilton-Jacobi method, as is well known. For a value imposed on the $(n-1)$-dimensional surface there exist two kinds of the solutions of this equation which are called the outgoing and incoming ones corresponding to the roots z_+^j and z_-^j stated in §1 (in detail, see the paper of Soga [7]). Thus the phase function $\varphi(x)$ is determined.

The amplitude functions $v^j(x)$ are constructed as follows. We denote by $P_l = P_l(x, \partial_x\varphi)$ the projection to the eigenspace of $\lambda_l(x, \partial_x\varphi)$. We decompose v^j into $v^j = P_l v^j + (I - P_l)v^j$ and determine the following parts inductively:

$$(I - P_l)v^0, \quad P_l v^0, \quad (I - P_l)v^1, \quad P_l v^1, \ldots .$$

Firstly, in view of (2.5) we set $(I - P_l)v^0 = 0$, and rewrite (2.4) as follows:

$$\begin{aligned}
&(\partial_t^2 - L(x, \partial_x))u(t, x) \\
&\quad = \rho_0''(t - \varphi)\{I - L_0(x, \partial_x\varphi)\}v^0(t, x) + \rho_0'(t - \varphi)P_l^*(\partial_t - H)P_l v^0(t, x) \\
&\qquad + \rho_0(t - \varphi)\{(I - P_l^*)(I - L)(I - P_l)v^1 + (I - P_l^*)(\partial_t - H)P_l v^0\} \\
&\qquad\quad + \rho_1(t - \varphi)\{P_l^*(\partial_t - H)P_l v^1 + \cdots\} + \cdots .
\end{aligned}$$

Taking appropriate bases in $P_l\mathbb{R}^n$, we can reduce the equation for $P_l^*(\partial_t - H)P_l v^0$ to a symmetric hyperbolic system, and choose $P_l v^0$ so that $P_l^*(\partial_t - H)P_l v^0 = 0$. Next, by the Cramer theorem we can solve the linear algebraic equation $(I - P_l^*)(I - L)(I - P_l)v^1 = -(I - P_l^*)(\partial_t - H)P_l v^0$. And we can determine $(I - P_l)v^1$ so that the term containing $\rho_0(t - \varphi)$ vanishes. Repeating these processes, we can determine the further terms of $(I - P_l)v^{j-1}$ and $P_l v^j$ inductively.

§3. Waves of Total Reflection

In this section we add the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ to the equation (1.1), and consider the case that a wave of the single mode hits the boundary $\partial\Omega$ and is reflected. We take the total reflection into consideration. In general, the reflected waves contain the modes different from the incident one, i.e., mode-conversion happens. We assume that the incident wave $u_-(t, x)$ is of δ-function type and has an asymptotic expansion of the form

$$u_-(t, x) = \delta(t - \varphi_-(x))v_-^0(t, x) + h(t - \varphi_-(x))v_-^1(t, x) + \cdots ,$$

where φ_- satisfies $\lambda_k(x, \partial_x\varphi_-) = 1$ for some k, and the wave front goes toward the boundary, which means that $(\partial_\xi\lambda_k(x, \partial_x\varphi_-)|_{\partial\Omega}) \cdot \nu < 0$ for the unit outer normal vector ν on $\partial\Omega \cap U$, where U is a small neighborhood of $\cup_{t\in\mathbb{R}, j\geq 0}\mathrm{supp}[v_-^j(t,\cdot)]$.

We suppose that the reflected waves have the following asymptotic expansion with the phase functions φ_+^l $(l = 1, \ldots, d)$ determined by the equation $\lambda_l(x, \partial_x\varphi_+^l) = 1$ and $(\partial_\xi\lambda_l(x, \partial_x\varphi_+^l)|_{\partial\Omega}) \cdot \nu > 0$:

$$(3.1) \qquad u_+(t,x) = \sum_{l=1}^{d}\{\rho_0(t - \varphi_+^l(x))v_+^{l0}(t,x) + \rho_1(t - \varphi_+^l(x))v_+^{l1}(t,x) + \cdots\}.$$

Adding the boundary condition due to the original condition ($u_-|_{\partial\Omega} + u_+|_{\partial\Omega} = 0$), we solve the equations for the phase functions φ_+^l and the amplitude functions v_+^{lj} (as is described in the previous section). These processes can be accomplished if the angle between $\partial_x\varphi_-|_{\partial\Omega}$ and $-\nu$ is close to 0 on $\partial\Omega \cap U$ (in detail, see Soga [5]). When the angle is far from 0, however, the glancing phenomenon or the total reflection happens. And in (3.1) we need to employ different forms.

Let us construct the asymptotic expansion in the case where total reflection happens. In this case, the non-real roots $z_\pm^{k'+1}, \ldots, z_\pm^{d'}$ appear in the equation

$$\det\,(I - L_0(x', \partial_x'\varphi_- + z\nu)) = 0 \quad (x' \in \partial\Omega),$$

where $\partial_x'\varphi_- = \partial_x\varphi_- - (\partial_x\varphi_- \cdot \nu)\nu$ ($\mathrm{Im}\, z_+^l = -\mathrm{Im}\ z_-^l > 0$). Let the other roots $z_\pm^1, \ldots, z_\pm^{k'}$ be all real. The part of the evanescent waves is connected with the non-real roots $z_\pm^{k'+1}, \ldots, z_\pm^{d'}$, and is added to (3.1). For the non-real roots $z_\pm^{k'+1}, \ldots, z_\pm^{d'}$ we assume that for any η near $\partial_x'\varphi_-$

(A.5) multiplicity of $z_\pm^l$ is constant and coincides with $\dim \mathrm{Ker}(I - L_0(x', \eta + z_\pm^l\nu))$.

This assumption is satisfied in the isotropic case.

For the evanescent waves also we need to construct the (complex-valued) phase functions $\tilde{\varphi}_\pm^l(x)$ satisfying $\tilde{\varphi}_\pm^l|_{\partial\Omega} = \varphi_-|_{\partial\Omega}$, $\partial_\nu\tilde{\varphi}_\pm^l|_{\partial\Omega} = z_\pm^l(x', \partial_x'\varphi_-)$ and $\det(I - L_0(x, \partial_x\tilde{\varphi}_\pm^l)) = 0$. But we cannot do so exactly, and only can solve the equation modulo $\{\text{dist }(x,\partial\Omega)\}^N$ for any $N > 0$, which is useful enough. Namely, for $l = k'+1,\dots,d'$ we get the complex-valued functions $\tilde{\varphi}^l_\pm(x)$ satisfying

$$(3.2) \qquad \begin{aligned} &\tilde{\varphi}^l_\pm(x'+r\nu) \sim \varphi_-(x') + z^l_\pm(x',\partial'_x\varphi_-)r + \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{as} \quad r = \text{dist }(x,\partial\Omega) \to 0\,, \\ &\det\,(I - L_0(x,\partial_x\tilde{\varphi}^l_\pm)) = 0 \quad \text{modulo} \quad r^N, \end{aligned}$$

where x' is a point on $\partial\Omega$ with $r = \text{dist}(x',x)$. The construction of these functions $\tilde{\varphi}^l_\pm$ is described in §3 of Soga [7].

For the complex variable $z = s + ir$ $(s,r \in \mathbb{R})$ we set

$$\begin{aligned} &\tilde{\rho}_0(z) = (2\pi i)^{-1}z^{-1}, \qquad \tilde{\rho}_1(z) = (2\pi i)^{-1}\log z, \\ &\tilde{\rho}_j(z) = \frac{1}{2\pi i\,(j-2)!}\int_0^z (z-\zeta)^{j-2}\log\zeta\, d\zeta \;\; (j = 2,3,\dots), \end{aligned}$$

where we take a branch (of $\log z$) connected with $\arg z = 0$ when Im $z = 0$ and Re $z > 0$, and the integral on a path linking 0 to z. Adding the terms $\tilde{\rho}_j(\tilde{\varphi}^l_\pm(x) - t)\tilde{v}^{lj}_\pm(t,x)$ to the sum (3.1), we introduce an expansion of the form

$$(3.3)\; \begin{aligned} u_+(t,x) &= \sum_{l=1}^{k'}\sum_{j=0}^{N}\rho_j(t-\varphi^l_+(x))v^{lj}_+(t,x) \\ &+ \sum_{l=k'+1}^{d'}\sum_{j=0}^{N}\{\tilde{\rho}_j(\tilde{\varphi}^l_+(x)-t)\tilde{v}^{lj}_+(t,x) + \tilde{\rho}_j(\tilde{\varphi}^l_-(x)-t)\tilde{v}^{lj}_-(t,x)\}. \end{aligned}$$

Then this can be the asymptotic expansion of the reflected wave:

Theorem 3.1. *Let $\partial_x\varphi_-$ be non-glaning on $U\cap\partial\Omega$, and assume that the conditions* (A.1)$\sim$(A.5) *are satisfied. Then we can construct the asymptotic expansion of $u_+(t,x)$ of the form* (3.3) *so that it satisfies the required boundary condition, i.e., $u_-|_{\partial\Omega} + u_+|_{\partial\Omega} = 0$.*

In the case where the expansions are of the oscillatory type $\sum_{j=0}^\infty e^{i\sigma\varphi(x)}$ $v^j(x)(i\sigma)^{-j}$, we have obtained a theorem corresponding to Theorem 3.1 (cf. Theorem 3.1 in Soga [7]). In this case the evanescent waves decay exponentially in distance from $\partial\Omega$ while they do not do so in the above Theorem 3.1.

Proof of Theorem 3.1 is based on the following Lemma and Proposition.

Lemma 3.2. *$\tilde{\rho}_j(s \pm ir)$ are continuous functions of $r \geq 0$ with the value in the Sobolev space $H^{j-1}_{loc}(\mathbb{R}^1_s)$, and satisfy*

$$\tilde{\rho_0}(s \pm i0) = (4\pi i)^{-1}\text{V.P.}\ \frac{1}{s} \pm 2^{-1}\delta(s),$$
$$\tilde{\rho_1}(s \pm i0) = (2\pi i)^{-1}\log|s| \pm 2^{-1}h(-s),$$
$$\tilde{\rho}_j(s \pm i0) = \frac{1}{2\pi i}\int_0^s \frac{(s-\tilde{s})^{j-2}}{(j-2)!}\log|\tilde{s}|\,d\tilde{s} \pm \frac{1}{2(j-1)!}s^{j-1}h(-s)\ (j \geq 2).$$

Proposition 3.3. *Let the assumptions in Theorem 3.1 be satisfied, and let $\varphi^l_+(x)$ and $\tilde{\varphi}^l_+(x)$ be the phase functions determined earlier. Then we have*

$$\sum_{l=1}^{k'}\text{Ker}\ [I - L_0(x', \partial_x\varphi^l_+(x'))] + \sum_{l=k'+1}^{d'}\text{Ker}\ [I - L_0(x', \partial_x\tilde{\varphi}^l_+(x'))]$$
$$= \mathbb{C}^n \qquad \text{for } x' \in \partial\Omega \cap U.$$

This proposition is obtained by Soga [7] (cf. Theorem 2.2 in [7]). We shall describe an outline of its proof in the next section. The method of the proof is developed in Kawashita-Ralston-Soga [2], and is a little different from the one in Soga [7].

Proof of Theorem 3.1. Noting that $\tilde{\rho}'_{j+1}(z) = \tilde{\rho}_j(z)$ $(j \geq 0)$, we insert the expansion $u = \sum \tilde{\rho}_j(\tilde{\varphi}^l_\pm - t)\tilde{v}^{lj}_\pm$ into $(\partial_t^2 - L)u$. Then, in order to eliminate each of the terms with $\tilde{\rho}''_j(\tilde{\varphi}^l_\pm - t)$ $(j \geq 0)$, we have the equations for $\tilde{v}^{lj}_\pm$ in the same way as in §2. But we cannot apply the same methods to these equations since the coefficients are complex-valued. In the same way as for $\tilde{\varphi}^l_\pm$ in (3.3), we solve them modulo r^N for any $N > 0$ considering the boundary condition for $\tilde{\rho}_j(\tilde{\varphi}^l_\pm(x) - t)\tilde{v}^l_\pm|_{\partial\Omega}$.

From (3.2) and Lemma 3.2 we see that

$$\tilde{\rho}_j(\tilde{\varphi}^l_+ - t)|_{\partial\Omega} - \tilde{\rho}_j(\tilde{\varphi}^l_- - t)|_{\partial\Omega} = \rho_j(t - \varphi_-)|_{\partial\Omega}.$$

Furthermore, the projections $P^l_\pm$ to $\text{Ker}(I - L_0(x, \partial_x\tilde{\varphi}^l_\pm))$ satisfy $P^l_+\mathbb{C}^n = P^l_-\mathbb{C}^n$ on $\partial\Omega$. Therefore, setting $P^l_+\tilde{v}^l_+|_{\partial\Omega} = -P^l_-\tilde{v}^l_-|_{\partial\Omega}$, we have

$$\sum_{l=1}^{k'} \rho_j(t-\varphi_+^l)P_l v_+^{lj}|_{\partial\Omega} + \sum_{l=k'+1}^{d'} \{\tilde\rho_j(\tilde\varphi_+^l - t)P_+^l \tilde v_+^{lj} + \tilde\rho_j(\tilde\varphi_-^l - t)P_-^l \tilde v_-^{lj}\}|_{\partial\Omega}$$

$$= \rho_j(t-\varphi_-)\{\sum_{l=1}^{k'} P_l v_+^{lj}|_{\partial\Omega} + \sum_{l=k'+1}^{d'} P_+^l \tilde v_+^{lj}|_{\partial\Omega}\}.$$

If we can make $\{\sum_{l=1}^{k'} P_l v_+^{lj}|_{\partial\Omega} + \sum_{l=k'+1}^{d'} P_+^l \tilde v_+^{lj}|_{\partial\Omega}\}$ coincide with any data, we obtain the expansion equal to $-u_-|_{\partial\Omega}$ modulo r^N. This coincidence follows from Proposition 3.3.

Thus we can determine φ_+^l, v_+^{lj} and $\tilde\varphi_\pm^l$, $\tilde v_\pm^{lj}$ inductively so that the expansion (3.3) satisfies the equation and the boundary condition modulo r^N. $\rho_j(t-\varphi_+^l)$ $(1 \le l \le k')$ and $\tilde\rho_j(\tilde\varphi_\pm^l - t)$ $(k'+1 \le l \le d')$ are C^{j-2} on $\overline{\Omega} \cap U$. Furthermore, $r^N\,\tilde\rho_j(\tilde\varphi_\pm^l - t)$ $(k'+1 \le l \le d')$ is C^{j+N-2} on $\overline{\Omega}\cap U$. This implies that difference between the true solution and the expansion (3.3) up to $j \le N$, is C^{j+N-2} on $\overline{\Omega} \cap U$. Hence the theorem is proved.

§4. Analysis of the Symbol $L(x,\xi)$

In this section we prove Proposition 3.3 in §3 by means of the method in Kawashita-Ralston-Soga [2], whose idea is due to Kostyuchenko-Shkalikov [3]. This proposition has been shown of the more abstract form in [2] (cf. Theorem 2.1 and Corollary 2.3 in [2]). The proof is based on complex analysis for the symbol $L_0(x,\xi)$. Let us give a brief explanation of the proof.

It suffices to discuss the requirement locally (in $U \cup \Omega$). And by appropriate local coordinates we transform $U \cup \Omega$ into a neighborhood of the origin in the half-space $\mathbb{R}^n_+ = \{x = (x', x_n);\ x_n > 0\}$. Hereafter we fix x arbitrarily and omit the letter x in the notation $L_0(x,\xi)$, etc., i.e. $L_0(\xi) = L_0(x,\xi)$, etc. In the symbol $L_0(\xi', \xi_n)$ $(\xi = (\xi', \xi_n))$ we vary ξ_n in the complex plane $\mathbb{C}$ and use the letter z instead of ξ_n.

The matrix $(I - L_0(\xi', z))^{-1}$ becomes a meromorphic function on $\mathbb{C}$ and has poles only at $z = z_\pm^l(\xi')$ $(l = 1, \dots, d')$. Furthermore we have

Lemma 4.1. *Let the assumptions* (A.1)$\sim$(A.5) *be satisfied and let ξ' be non-glancing. Then all the poles of $(I - L_0(\xi', z))^{-1}$ are simple. Furthermore, if the pole $\tilde z$ is real, $(I - L_0(\xi', z))^{-1}$ is of the following form near*

$$(I - L_0(\xi', z))^{-1} = -\frac{\{\partial_{\xi_n}\lambda_l(\xi', \tilde{z})\}^{-1}}{z - \tilde{z}} P_l(\xi', \tilde{z}) + R(z),$$

where $R(z)$ is analytic near $\tilde{z}$ and $P_l(\xi', \tilde{z})$ is the projection to the eigenspace of the eigenvalue $\lambda_l(\xi', \tilde{z}) = 1$.

Let us note that $\partial_{\xi_n}\lambda_l(\xi', \tilde{z}) > 0$ (resp. < 0) is corresponding to the outgoingness (resp. the incomingness) and that $\pm\partial_{\xi_n}\lambda_l(\xi', z^l_{\pm}(\xi')) > 0$. We can verify this lemma as follows.

It follows from (A.1) and (A.5) that rank of the cofactor cof$[I - L_0(\xi', \tilde{z})]$ is equal to $n -$ (multiplicity of $\tilde{z}$). This implies that

$$\partial_z^i \mathrm{cof}\,[I - L_0(\xi', z)]|_{z=\tilde{z}} = 0 \qquad \text{for} \quad i = 0, \dots, \alpha - 2,$$

if the multiplicity α of $\tilde{z}$ is larger than 2 (i.e., $\alpha \geq 2$). (cf. Lemma 2.3 and Remark 2.4 in Soga [7]). Noting that $(I - L_0(\xi', z))^{-1} = (\det\,[I - L_0(\xi', z)])^{-1}\mathrm{cof}\,[I - L_0(\xi', z)]$, we can see that the pole $\tilde{z}$ is simple. The simplicity is shown also in Kawashita-Ralston-Soga [2] (see Proposition 2.5 in [2]).

Next let $\tilde{z}$ be real. $L(\xi', z)$ is a real symmetric matrix for real z. Therefore, we have $L(\xi', z)P_j(\xi', z) = \lambda_j(\xi', z)P_j(\xi', z)$ and $I = \sum_{j=1}^{d} P_j(\xi', z)$ for real z. This yields that

$$(I - L_0(\xi', z))\sum_{j=1}^{d}(1 - \lambda_j(\xi', z))^{-1}P_j(\xi', z) = I \quad \text{for real } z\,(\neq \tilde{z})\,\text{near } \tilde{z},$$

which implies that when z varies in real values near $\tilde{z}$, $(I - L_0(\xi', z))^{-1}$ is expressed in the form $(1 - \lambda_l(\xi', z))^{-1}P^l(\xi', \tilde{z}) + R(z)$ for a C^∞ function $R(z)$. This proves the latter in Lemma 4.1.

To prove Proposition 3.3, we introduce the function

$$f_v(z) = ((I - L_0(\xi', z))^{-1}v, v), \tag{4.1}$$

where $v \in \mathbb{C}^n$ and $(\cdot\,,\,\cdot)$ is the inner product of $\mathbb{C}^n$. Then we have

Lemma 4.2. *If the pole τ of $(I - L_0(\xi', z))^{-1}$ is simple and v is orthogonal to* Ker$[I - L_0(\xi', \overline{\tau})]$, *then $f_v(z)$ becomes analytic at τ.*

This is verified in Kawashita-Ralston-Soga [2] (cf. Remark 3.3 in [2]).

PROOF of Proposition 3.3. We have only to show that v is equal to 0 if v is orthogonal to $\sum_{l=1}^{k'} \mathrm{Ker}\,[I - L_0(x', \partial_x \varphi_+^l(x'))] + \sum_{l=k'+1}^{d'} \mathrm{Ker}\,[I - L_0(x', \partial_x \tilde{\varphi}_+^l(x'))]$. For this v we consider the function (4.1), and integrate $z^l f_v(z)$ $(l = 0, 1)$ on a large circle $c_r = \{z : |z| = r\}$. Then, by Lemma 4.2, we obtain

$$(4.2)\quad \frac{1}{2\pi i}\int_{c_r} z^l f(z)dz = \sum_{j=1}^{k'} \frac{1}{2\pi i}\int_{c_-^j} z^l f(z)dz + \sum_{j=k'+1}^{d'} \frac{1}{2\pi i}\int_{c_+^j} z^l f(z)dz,$$

where $c_\pm^j$ is a small circle surrounding the pole $z_\pm^j(\xi')$. Noting $\overline{f_v(z)} = f_v(\overline{z})$ and $\overline{z_-^j(\xi')} = z_+^j(\xi')$ $(j = k'+1, \dots, d')$, we see that $z_+^j(\xi')$ is not a pole of $f_v(z)$ if $z_-^j(\xi')$ is not so. Therefore, the integrals $\int_{c_+^j} z^l f(z)dz$ in (4.2) are equal to 0 for $j = k'+1, \dots, d'$. Hence, by means of the form of $(I - L_0(\xi', z))^{-1}$ stated in Lemma 4.1, we obtain

$$\frac{1}{2\pi i}\int_{c_r} z^l f(z)dz = \sum_{j=1}^{k'} -\{\partial_{\xi_n}\lambda_j(\xi', z_-^j(\xi'))\}^{-1} z_-^j(\xi')^l (P_j(\xi', z_-^j(\xi'))v, v)$$

for $l = 0, 1$.

Calculating the residue of $z^l(I - L_0(\xi', z))^{-1}$ at $z = \infty$, for large $r > 0$ we have

$$\begin{aligned} \frac{1}{2\pi i}\int_{c_r} z^l f(z)dz &= 0 && \text{when } l = 0, \\ &= -(a_{nn}^{-1}v, v) && \text{when } l = 1, \end{aligned}$$

where a_{nn} is the coefficient of L_0. Note that a_{nn} is positive definite (from (A.2)). Thus we obtain

$$\begin{aligned} (4.3)\quad &\sum_{j=1}^{k'} \{\partial_{\xi_n}\lambda_j(\xi', z_-^j(\xi'))\}^{-1} z_-^j(\xi')^l (P_j(\xi', z_-^j(\xi'))v, v) \\ &\qquad = l(a_{nn}^{-1}v, v) \qquad \text{for } l = 0, 1. \end{aligned}$$

Since $\partial_{\xi_n}\lambda_j(\xi', z_-^j(\xi')) < 0$ for all $j = 1, \dots, k'$, we have $(P_j(\xi', z_-^j(\xi'))v, v) = 0$ $(j = 1, \dots, k')$ from (4.3) with $l = 0$. Therefore, by (4.3) with $l = 1$, we have $(a_{nn}^{-1}v, v) = 0$. Hence, we obtain $v = 0$, which proves Proposition

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