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SMOOTHNESS OF HIGHER ORDER TERMS IN A BACKSCATTERING TRANSFORMATION

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ABSTRACT. The consideration of backscattering data of Schrödinger operators $H_v = |D|^2 - v$ in $\mathbf{R}^n$, when $n \geq 3$ is odd, motivates the introduction of a nonlinear transformation $v \mapsto Bv$ from $L^q_{\mathrm{comp}}(\mathbf{R}^n)$ to $\mathcal{D}'(\mathbf{R}^n)$ when $q > n$. We define Bv by considering the wave group associated to the equation $(\partial_t^2 - \Delta_x - v(x))K(x,t) = 0$. Simple estimates show that Bv is entire analytic in v. When v is sufficiently small and real-valued, Bv is uniquely determined from the backscattering data. If $n = 3$ and ∇v has a small norm in L^1 it is known also that v is uniquely determined by Bv. We prove that the N:th order term $B_N v$ in the power series expansion of Bv is μ_N-times continuously differentiable for N large, where $\mu_N/N \to 1 - n/q$ as $N \to \infty$.

1. INTRODUCTION

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Denote by $\mathcal{C}^k([0,\infty);\mathcal{B}(\mathcal{H},\mathcal{K}))$ the space of mappings

$$[0,\infty) \ni t \mapsto A(t) \in \mathcal{B}(\mathcal{H},\mathcal{K})$$

which are k times continuously differentiable in the strong sense, i.e. $t \mapsto A(t)f \in \mathcal{K}$ is a C^k-mapping for every $f \in \mathcal{H}$. Let $\mathcal{H}_s$ be the standard Sobolev space of functions in $\mathbf{R}^n$ with all derivatives up to order s in $L^2(\mathbf{R}^n)$, so that $\mathcal{H}_0 = L^2(\mathbf{R}^n)$. When $v \in L^q(\mathbf{R}^n)$ and $q \geq n/2$ it follows from the Sobolev embedding theorem that the operator M_v, multiplication by v, is continuous from $\mathcal{H}_2$ to $\mathcal{H}_0$. The Schrödinger operator $H_v = -\Delta - M_v = H_0 - M_v$ is therefore a continuous linear operator between the same spaces.

Main assumptions: *It will be assumed throughout this paper that $n \geq 3$ is odd and that $n < q \leq \infty$.*

In Section 2 we shall present a simple proof of the following theorem.

Theorem 1. *Assume $v \in L^q(\mathbf{R}^n)$ (with q as above). Then there is a unique*

$$K_v \in \mathcal{C}^2([0,\infty);\mathcal{B}(\mathcal{H}_2,\mathcal{H}_0)) \cap \mathcal{C}^0([0,\infty);\mathcal{B}(\mathcal{H}_2,\mathcal{H}_2))$$

such that

$$K_v''(t)f + H_v K_v(t)f = 0, \tag{1}$$

and

$$K_v(0)f = 0, \quad K_v'(0)f = f \tag{2}$$

when $f \in \mathcal{H}_2$.

The family of operators $K_v(t)$, $t \geq 0$ will sometimes be referred to as the *wave group*. We are also going to use the following properties of K_v, where $K_v(x,y,t)$ denotes the distribution kernel of $K_v(t)$:

$$|x-y| \leq t \quad \text{in the support of } K_v(x,y,t) \text{ with equality when } v = 0, \tag{3}$$

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$$K_v \in \mathcal{C}^0([0,\infty);\mathcal{B}(\mathcal{H}_0,\mathcal{H}_1)), \tag{4}$$

and

$$K_v \in \mathcal{C}^1([0,\infty);\mathcal{B}(\mathcal{H}_0,\mathcal{H}_0)). \tag{5}$$

It follows from Sobolev's embedding theorem and (4) (with $v=0$) that $K_0(t)$ is continuous from L^2 to L^p when $2 \le p \le 2n/(n-2)$. Hence $M_vK_0 \in \mathcal{C}^0([0,\infty);\mathcal{B}(\mathcal{H}_0,\mathcal{H}_0))$ by Hölder's inequality when $v \in L^q$, and it follows then from (5) that $K_v'(t)M_vK_0(t)$ is a strongly continuous family of bounded operators on $L^2(\mathbf{R}^n)$.

Let $L^p_{\mathrm{comp}}(\Omega)$ be the space of functions in $L^p(\mathbf{R}^n)$ with compact support contained in Ω, where $\Omega \subset \mathbf{R}^n$ are open bounded sets. Assume that $v \in L^q_{\mathrm{comp}}(\mathbf{R}^n)$. It follows from property (3) that for every Ω there is a constant $T = T(\Omega, \mathrm{supp}(v))$ such that $M_vK_0(t)f = 0$ when $f \in L^2_{\mathrm{comp}}(\Omega)$ and $t \ge T$. Another application of property (3) shows that the union of the supports of the $K_v'(t)M_vK_0(t)f$ when t ranges from 0 to ∞ is contained in a compact set which depends on Ω and $\mathrm{supp}(v)$ only. It follows that the operator $G = G_v$ defined by

$$Gf = \int_0^\infty K_v'(t)M_vK_0(t)f\,dt \tag{6}$$

is a continuous linear operator on $L^2_{\mathrm{comp}}(\mathbf{R}^n)$. Since $v \in L^2$ the operator M_vG is continuous from L^2_{comp} to L^1, and hence also from $C_0^\infty(\mathbf{R}^n)$ to $\mathcal{E}'(\mathbf{R}^n)$. Let $(M_vG)(x,y)$ denote its distribution kernel. A linear change of variables in $\mathbf{R}^n \times \mathbf{R}^n$ allows us to consider the distribution $(M_vG)(y, 2x-y)$. Since this distribution is compactly supported in y, we may define its integral with respect to that variable, formally written as $\int v(y)G(y,2x-y)\,dy$. This procedure gives rise to a nonlinear mapping from $L^q_{\mathrm{comp}}(\mathbf{R}^n)$ to $\mathcal{D}'(\mathbf{R}^n)$, and we adopt the following definition:

Definition 2. The backscattering transform Bv of $v \in L^q_{\mathrm{comp}}(\mathbf{R}^n)$ is defined by

$$Bv(x) = v(x) - 2^n \int v(y)G(y,2x-y)\,dy,$$

where G is defined by (6).

Our terminology is motivated by the following. In the case when v is real-valued, compactly supported and satisfies some weak regularity conditions we have a scattering matrix corresponding to the two unitary groups e^{-itH_v} and e^{-itH_0}. Its anti-diagonal part is a function depending on the parameters (k,θ) where $k \in \mathbf{R}_+$ and $\theta \in S^{n-1}$. Viewing these as polar coordinates in frequency space and taking the inverse Fourier transform we get a distribution in $\mathbf{R}^n$. The real part of that distribution is after suitable normalization equal to the backscattering transform Bv defined above apart from a smooth term which is due to bound states that may occur when v becomes large. We refer to Lagergren [L] (in the case when $n=3$ and H_v has no bound states) and to a forthcoming paper by the author to a proof of these facts in arbitrary odd dimension (see also [M]). The advantage of this approach is that it gives a representation of backscattering data without reference to wave operators, and that there is no need to let the time parameter (in $K_v(t)$) tend to infinity when studying the local behaviour of the backscattering transform as long as the potentials are compactly supported. In other words, we take advantage of the finite speed of propagation in the wave equation, and in particular the validity of Huygen's principle in odd dimension. (For more extensive discussions on an approach to backscattering closely related to Lax-Phillips theory of scattering we refer to Uhlmann [U] and Wang [W].)

Inverse backscattering deals with the recovery of v from the backscattering data. (See [ER1] and [ER2].) In view of the previous discussions the recovery of v from Bv is closely related to the inverse backscattering problem. Since the leading part of Bv equals v one is tempted, at least when considering small potentials, to view the backscattering transformation as a nonlinear perturbation of the identity. The problem is then to find suitable spaces of functions to work within. In the case when $n=3$ it turns out (see [L]) that the completion of C_0^∞ in the norm

$\|\nabla v\|_{L^1}$ is a space for which $v \mapsto Bv$ is a homeomorphism in a neighbourhood of the origin. A natural candidate in the n-dimensional case, when $n > 3$ is odd, is the completion of C_0^∞ in the norm $\|\nabla^{n-2} v\|_{L^1}$.

A more modest version of the inverse backscattering problem would be to compare the singularities of Bv with those of v (see [J] and [OPS]). This paper will focus on some aspects of this question. As we shall see, Bv is an entire analytic function of v when viewed as an element of $\mathcal{D}'(\mathbf{R}^n)$. Thus

$$Bv = \sum_1^\infty B_N v$$

with convergence in $\mathcal{D}'(\mathbf{R}^n)$, where $B_N v$ is the part of Bv that is homogeneous of degree N in v. The main result of this paper, Theorem 8, says that the smoothness of $B_N v$ increases with N. In fact, we are going to prove that $B_N v \in C^{\mu_N}(\mathbf{R}^n)$ for N large where

$$\mu_N/N \to 1 - n/q \quad \text{as } N \to \infty. \tag{7}$$

Also, $\sum_{\mu_N \geq k} B_N v$ is convergent in $C^k(\mathbf{R}^n)$ for every k. This means that we may for every k write B as a sum of a map which is a polynomial in v and a map which is continuous from L^2_{comp} to C^k. A study of the finer regularity properties of Bv may therefore be reduced to the individual terms $B_N v$. Part of these results, which will be proved in the last section, may be summed up in the following theorem.

Theorem 3. *The backscattering transformation B may for any nonnegative integer k be written as a sum $B = B_{\text{pol}} + B_{\text{smooth}}$, where B_{pol} is a polynomial mapping and B_{smooth} is continuous from $L^q_{\text{comp}}(\mathbf{R}^n)$ to $C^k(\mathbf{R}^n)$.*

2. Proof of Theorem 1 and properties of the wave group

Proof of the uniqueness part of Theorem 1. We have to prove that $f(t) \equiv 0$ if

$$f \in C^2([0,\infty); \mathcal{H}_0) \cap C^0([0,\infty); \mathcal{H}_2), \quad f(0) = f'(0) = 0,$$

and

$$f''(t) + H_v f(t) = 0.$$

Set

$$G(t) = \|f'(t)\|^2 + ((I + H_0) f(t), f(t))$$

and

$$g_\varepsilon(t) = ((I + H_0)(I + \varepsilon H_0)^{-1} f(t), f(t))$$

when $0 \leq \varepsilon$. When $\varepsilon > 0$ we have

$$g'_\varepsilon(t) = 2\,\text{Re}((I + H_0)(I + \varepsilon H_0)^{-1} f(t), f'(t))$$

which converges in $L^1_{\text{loc}}(\mathbf{R}_+)$ to the continuous function

$$h(t) = 2\,\text{Re}((I + H_0) f(t), f'(t))$$

when $\varepsilon \to 0$. Since g_ε converges to g_0 in $L^1_{\text{loc}}(\mathbf{R}_+)$ when $\varepsilon \to 0$ it follows that g_0 is a C^1 function in $\mathbf{R}_+$ and that $g'_0 = h$. Hence $G \in C^1(\mathbf{R}_+)$ and

$$\begin{aligned} G'(t) &= 2\,\text{Re}(f''(t), f'(t)) + h(t) = 2\,\text{Re}(f''(t) + (I + H_0) f(t), f'(t)) \\ &= 2\,\text{Re}((1 + v) f(t), f'(t)). \end{aligned}$$

Since $v \in L^q$ and $(I + H_0)^{-1/2}$ is continuous from L^2 to L^p when $\frac{1}{2} - \frac{1}{n} \leq \frac{1}{p} \leq \frac{1}{2}$ we may estimate the norm in L^2 of vf by a constant times the norm in L^2 of $(I + H_0)^{1/2} f$. Hence, there is a constant C such that

$$G'(t) \leq C G(t), \quad t > 0,$$

and since $G(0) = 0$ we may conclude that G vanishes identically. □

We need some simple preparations in order to construct K_v. In the case when v is real one must have $K_v(t) = t\sigma(t^2 H_v)$, where σ is the unique entire analytic function which satisfies $\sigma(t^2) = (\sin t)/t$ when $t \in \mathbf{R}$. Since we allow v to be complex-valued, and since we are going to need rather precise information about K_v, we shall construct it by considering convolutions of operator valued functions on $\mathbf{R}_+$.

Convolutions of operator valued functions. Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and recall that $\mathcal{C}^k([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{K}))$ denotes the space of mappings

$$[0,\infty) \ni t \mapsto A(t) \in \mathcal{B}(\mathcal{H},\mathcal{K})$$

which are k-times continuously differentiable in the strong sense. We equip this space with the topology defined by the semi-norms

$$\|A\|_{T,f} = \sum_{0\le j\le k} \max_{0\le t\le T} \|A^{(j)}(t)f\|, \quad T \ge 0,\ f \in \mathcal{H}.$$

Under this topology $\mathcal{C}^k([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{K}))$ becomes a Fréchet space. We say that an element A in $\mathcal{C}^k([0,\infty); \mathcal{B}(\mathcal{H}))$ is *simple* if $A(t) = f(t)A_0$ where $A_0 \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is independent of t and $f \in \mathcal{C}^k([0,\infty))$. The finite linear combinations of simple elements form a dense subspace of $\mathcal{C}^k([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{K}))$, and if $A \in \mathcal{C}^k([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{K}))$, then the integral $\int_0^t A(s)\,ds$ is an element in $\mathcal{C}^{k+1}([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{K}))$ with derivative $A(t)$.

Assume that $A \in \mathcal{C}^0([0,\infty); \mathcal{B}(\mathcal{K},\mathcal{L}))$ and that $B \in \mathcal{C}^0([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{K}))$, where $\mathcal{H},\mathcal{K}$ and $\mathcal{L}$ are Hilbert spaces. Define

$$(A * B)(t) = \int_0^t A(t-s)B(s)\,ds = \int_0^t A(s)B(t-s)\,ds.$$

Then $A * B \in \mathcal{C}^0([0,\infty), \mathcal{B}(\mathcal{H},\mathcal{L}))$. The convolution is associative, i.e.

$$(A * B) * C = A * (B * C) \tag{8}$$

when A, B and C take values in appropriate spaces so that the convolutions are defined. For reasons of continuity and linearity it suffices to prove this when A, B and C are simple, and then it follows from the corresponding properties for convolution of scalar valued functions. We shall use the fact that if $A \in \mathcal{C}^1([0,\infty); \mathcal{B}(\mathcal{K},\mathcal{L}))$ and $B \in \mathcal{C}^0([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{K}))$, then $A * B \in \mathcal{C}^1([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{L}))$ and

$$(A * B)' = A' * B + A(0)B.$$

When $A \in \mathcal{C}^0$, $B \in \mathcal{C}^1$ we have instead $(A * B)' = A * B' + AB(0)$.

If f and g are locally integrable function on $[0,\infty)$ we define their convolution by

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds.$$

In this formula we may replace g by G where $G \in \mathcal{C}^0([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{K}))$. Then we get an element $f * G$ in the same space of operator valued functions. The obvious laws of associativity hold so that in particular $\mathcal{C}^0([0,\infty); \mathcal{B}(\mathcal{H},\mathcal{K}))$ becomes a module with respect to the convolution algebra of locally integrable functions on $[0,\infty)$.

Since it will be important for us also to consider fractional derivatives of operator valued functions we need one more definition. Set

$$\mathcal{B} = \mathcal{B}(\mathcal{H}_0,\mathcal{H}_0) = \mathcal{B}(L^2,L^2)$$

and define

$$X_0 = \mathcal{C}^0([0,\infty); \mathcal{B}).$$

In order to define X_a when $a > 0$ we introduce

$$\chi_a(t) = t^{a-1}/\Gamma(a), \quad t > 0.$$

Then $\chi_a * \chi_b = \chi_{a+b}$. If $A \in X_0$ we say that $A \in X_a$ if $A = \chi_a * B$, where $B \in X_0$. If $a = k$ is a positive integer this implies that $A \in C^k([0,\infty);\mathcal{B})$ and $B = A^{(k)}$. If $a = k + b$ where $0 < b < 1$ then $A^{(k)} = \chi_b * B$ and $B = C'$ where $C = \chi_{1-b} * A^{(k)} \in X_1$. It follows that B is uniquely determined by A and we write $B = A^{(a)}$. The following lemma is immediate from the definitions.

Lemma 4. *Assume $A \in X_a$ and $B \in X_b$ then $A * B \in X_{a+b}$ and*

$$(A * B)^{(a+b)} = A^{(a)} * B^{(b)}. \tag{9}$$

If $0 \le a \le b$ then $X_b \subset X_a$, and if $A \in X_b$ then $A^{(a)} = \chi_{b-a} * A^{(b)}$.

Mapping properties of K_0. It is easily verified that the conditions (1)–(5) are satisfied by

$$K_0(t) = (\sin t|D|)/|D| \quad \text{where } D = \partial/i \text{ and } |D| = H_0^{1/2}. \tag{10}$$

This is a convolution operator, and its distribution kernel $k_0(x,t)$ is supported in the wave cone $|x| = t$. We notice that $K_0(t)$ extends to a continuous operator on $\mathcal{S}'(\mathbf{R}^n)$. We have $K_0'(t) = \cos(t|D|)$, and $K_0 \in X_1$ since $K_0(0) = 0$. If $0 < a < 1$ then

$$K_0^{(a)} = \chi_{1-a} * K_0'. \tag{11}$$

From this follows that

$$K_0^{(a)}(t) = |D|^{a-1} h_a(t|D|), \tag{12}$$

where

$$h_a(t) = \int_0^t (t-s)^{-a} \cos s \, ds / \Gamma(1-a)$$

is a bounded function. Since $|D|^{-1}$ is convolution by a constant times $|x|^{1-n}$ it follows from formula (10) and the Hardy-Littlewood-Sobolev (HLS) inequality (see [H], Sec. 4.5) that $K_0(t)$ is continuous from L^p to L^2 and from L^2 to $L^{p'}$ when $\frac{1}{p} \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]$ and p' is the conjugated exponent, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. It follows from Hölder's inequality then that the operators

$$Y_-(t) = K_0(t) M_v, \quad Y_+(t) = M_v K_0(t) \tag{13}$$

are continuous in L^2, and from some simple estimates one deduces that $Y_\pm \in C^0([0,\infty);\mathcal{B}) = X_0$.

Define $\delta = \delta_q \in (0,1]$ by

$$\delta = 1 - \frac{n}{q}. \tag{14}$$

Lemma 5. *We have $Y_\pm \in X_\delta$, and there is a constant $C = C_{q,n}$, which depends on q and n only, such that*

$$\|Y_\pm^{(\delta)}(t)\| \le C \|v\|_{L^q}, \quad t \ge 0.$$

Proof. Since $|D|^{\delta-1}$ is convolution by a constant times $|x|^{1-\delta-n}$ it follows from (12) and the HLS-inequality that $K_0^{(\delta)}$ is continuous from L^r to L^2 and from L^2 to $L^{r'}$, where

$$\frac{1}{r} = \frac{1}{2} + \frac{1-\delta}{n} = \frac{1}{2} + \frac{1}{q}.$$

It follows then from Hölder's inequality that $M_v K_0^{(\delta)}(t)$ and $K_0^{(\delta)}(t) M_v$ are continuous operators in L^2, and as such they are strongly continuous in t. The operator norm may be estimated from above by $C\|v\|_{L^q}$. The lemma follows since

$$Y_- = \chi_\delta * (M_v K_0^{(\delta)}), \quad Y_+ = \chi_\delta * (K_0^{(\delta)} M_v).$$

The construction of K_v. Let $q \in (n, \infty]$ and $v \in L^q(\mathbf{R}^n)$ be as before. Define K_N inductively when $N \geq 1$ by

$$K_N = Y_- * K_{N-1}. \tag{15}$$

Since $Y_- \in X_\delta$ by Lemma 5, and since $K_0 \in X_1$, it follows by induction over N that

$$K_N \in X_{N\delta+1}, \quad N \geq 1. \tag{16}$$

An application of Lemma 4 and Lemma 5 shows that

$$\begin{aligned} \|K_N^{(1+N\delta)}\| &\leq \|Y_-^{(\delta)}\| * \|K_{N-1}^{(1+(N-1)\delta)}\| \\ \leq C\|v\|_{L^q}\chi_1 * \|K_{N-1}^{(1+(N-1)\delta)}\| &\leq C^2\|v\|_{L^q}^2\chi_1 * \chi_1 * \|K_{N-2}^{(1+(N-2)\delta)}\| \\ = C^2\|v\|_{L^q}^2\chi_2 * \|K_{N-2}^{(1+(N-2)\delta)}\| &\leq \cdots \leq C^N\|v\|_{L^q}^N\chi_N * \|K_0^{(1)}\| \\ &\leq C^N\|v\|_{L^q}^N\chi_N * \chi_1 = C^N\|v\|_{L^q}^N\chi_{N+1}. \end{aligned}$$

Since $K_N^{(a)} = \chi_{1+N\delta-a} * K_N^{(1+N\delta)}$, when $0 \leq a < 1 + N\delta$, it follows that $K_N \in X_a$ when $0 \leq a \leq 1 + N\delta$, and one has the estimate

$$\|K_N^{(a)}(t)\| \leq C^N t^{1+N(1+\delta)-a}\|v\|_{L^q}^N/\Gamma(2 + N(1+\delta) - a), \quad 0 \leq a \leq 1 + N\delta. \tag{17}$$

We now define

$$K_v = \sum_0^\infty K_N. \tag{18}$$

It follows from (17) with $a = 1$ that the sum converges in $C^1([0, \infty); \mathcal{B})$. Hence condition (5) is fulfilled and (3) holds since $|x - y| = t$ in the support of the distribution kernel $K_0(x, y, t)$.

Lemma 6. *We have $(K_v - K_0)(I + H_0)^{-1} \in X_2$.*

Proof. Set $P = M_v(I + H_0)^{-1}$. Then P is bounded on $L^2(\mathbf{R}^n)$ and

$$M_vK_0(I + H_0)^{-1} = PK_0 \in X_1$$

since $K_0 \in X_1$. Since

$$K_1(I + H_0)^{-1} = K_0 * (M_vK_0(I + H_0)^{-1}) = K_0 * (PK_0),$$

it follows from Lemma 4 that

$$K_1(I + H_0)^{-1} \in X_2. \tag{19}$$

Let us introduce

$$V_N = Y_- * \cdots * Y_-, \quad W_N = Y_+ * \cdots * Y_+, \tag{20}$$

where the number of factors equals N. Then

$$K_N = V_{N-1} * K_1 = K_1 * W_{N-1}, \quad N \geq 2. \tag{21}$$

It follows from Lemma 4 and Lemma 5 that $V_{N-1}, W_{N-1} \in X_{(N-1)\delta}$. Hence (19) and (21) imply that

$$K_N(I + H_0)^{-1} = V_{N-1} * (K_1(I + H_0)^{-1}) \in X_{(N-1)\delta+2}, \quad N \geq 2.$$

Arguments similar to those leading to (17) give the estimate

$$\|(K_N(I + H_0)^{-1})^{((N-1)\delta+2)}\| \leq C^N\|v\|_{L^q}^N\chi_N, \quad N \geq 1. \tag{22}$$

The lemma is an immediate consequence of these estimates, since (22) implies that $K_N(I + H_0)^{-1} = \chi_2 * Z_N$ when $N \geq 1$, where $\sum_1^\infty Z_N$ is convergent in $C^0([0, \infty), \mathcal{B})$. □

It follows from the previous lemma that

$$K_v \in C^2([0,\infty); \mathcal{B}(\mathcal{H}_2, \mathcal{H}_0))$$

and that (2) holds. We need also to verify (1) and that

$$K_v \in C^0([0,\infty); \mathcal{B}(\mathcal{H}_2, \mathcal{H}_2)), \tag{23}$$

or, equivalently, that

$$(I+H_0)K_v(I+H_0)^{-1} \in C^0([0,\infty); \mathcal{B}). \tag{24}$$

We notice that

$$K_{N-1}M_v = V_N \in X_{N\delta}, \quad M_v K_{N-1} = W_N \in X_{N\delta}, \quad N \geq 1,$$

since $Y_\pm \in X_\delta$. Hence we have

$$P_N \in X_{N\delta}, \quad \text{where } P_N = (W_N - V_N)(I+H_0)^{-1}. \tag{25}$$

Lemma 7. *Assume $N \geq 1$. Then*

$$K_N''(t) = V_N - K_N H_0, \quad \textit{(on } \mathcal{H}_2\textit{)} \tag{26}$$

and

$$(I+H_0)K_N(t)(I+H_0)^{-1} = K_N(t) + P_N(t). \tag{27}$$

Proof. The estimates (17) and (22) (and their polarized versions) show that both sides of (26) and (27), viewed as mappings from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{S}'(\mathbf{R}^n)$ depend continuously on $v \in L^q$. It suffices therefore to prove the lemma when $v \in C_0^\infty(\mathbf{R}^n)$. Consider first $K_1 = (K_0 M_v) * K_0$. Since $K_0 \in C^2([0,\infty); \mathcal{B}(\mathcal{H}_2, \mathcal{H}_0))$, $K_0(0) = 0$, $K_0'(0) = I$ and $K_0'' = -K_0 H_0$, it follows that

$$K_1'' = K_0 M_v - K_1 H_0 = V_1 - K_1 H_0.$$

If $N \geq 2$ we write $K_N = (K_{N-2}M_v) * K_1$ and get

$$\begin{aligned} K_N'' &= (K_{N-2}M_v) * K_1'' = (K_{N-2}M_v) * (K_0 M_v) - (K_{N-2}M_v) * (K_1 H_0) \\ &= K_{N-1}M_v - K_N H_0 = V_N - K_N H_0. \end{aligned}$$

This proves (26). Since K_N is its own transpose we also have

$$K_N'' = M_v K_{N-1} - H_0 K_N = W_N - H_0 K_N. \tag{28}$$

Hence

$$(H_0 + I)K_N = K_N(H_0 + I) + W_N - V_N$$

from which (27) follows. □

We notice that (1) follows from (28). The only remaining part in the proof of Theorem 1 is therefore the assertion (24). The series $\sum_1^\infty P_N$ converges in $C^0([0,\infty); \mathcal{B})$ and its sum $(M_v K_v - K_v M_v)(I+H_0)^{-1}$ is an element in X_δ. It follows from Lemma 7 therefore that

$$\begin{aligned} &(I+H_0)K_v(t)(I+H_0)^{-1} \\ &= (M_v K_v(t) - K_v(t) M_v)(I+H_0)^{-1} + K_v(t) \in C^0([0,\infty); \mathcal{B}). \end{aligned}$$

This completes the proof of Theorem 1.

We have already verified (3) and (5) and want to prove now that (4) holds. Since

$$K_N = K_1 * W_{N-1}, \quad N \geq 2$$

a summation over N gives

$$K_v = K_0 + K_1 + K_1 * W,$$

where $W = \sum_1^\infty W_N \in X_\delta$. It suffices therefore to observe that

$$K_1 = K_0 * Y_+ \in C^0([0,\infty); \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)),$$

since K_0 is in that space.

3. The backscattering transform

Let $v \in L^q_{\mathrm{comp}}(\mathbf{R}^n)$ where $q > n$. Define $G = G_v$ as in (6) and recall that the backscattering transformation B was introduced in Definition 2.

Define $B_1 v = v$ and

$$B_N v(x) = -2^n \int v(y) G_{N-1}(y, 2x - y)\, dy, \quad N > 1, \tag{29}$$

where

$$G_{N-1} = \int_0^\infty K'_{N-2}(t) M_v K_0(t)\, dt. \tag{30}$$

It is a simple consequence from these definitions and the estimates in the previous section that

$$Bv = \sum_1^\infty B_N v$$

with convergence in $\mathcal{D}'(\mathbf{R}^n)$, and also that Bv is entire analytic in v when viewed as an element of that space.

The main result of this paper is a proof for the fact that the smoothness of $B_N v$ increases with N. (We shall not discuss the smoothness of the lower order terms in the expansion of Bv.) It follows from the theorem below that for every nonnegative integer k there is a positive integer N_k such that $B_N \in C^k$ when $N \geq N_k$, and $\sum_{N \geq N_k} B_N$ is convergent in $C^k(\mathbf{R}^n)$. Moreover, $k/N_k \to \delta = 1 - n/q$ as $k \to \infty$.

Theorem 8. *Let n^* be the smallest integer $> n/4$ and set $\delta = 1 - n/q$, where $q > n$. Assume $2(n^* + k) < (N-2)\delta$. Then $\Delta^k B_N v \in L^2_{\mathrm{loc}}(\mathbf{R}^n)$ when $v \in L^q_{\mathrm{comp}}(\mathbf{R}^n)$. Moreover, if Ω_1 and Ω_2 are open bounded sets in $\mathbf{R}^n$ there is a constant $C = C_k$, depending on k, Ω_1, Ω_2 and q only such that*

$$\left(\int_{\Omega_1} |\Delta^k B_N v(x)|^2\, dx \right)^{1/2} \leq C_k^N \|v\|^N_{L^q} / N! \tag{31}$$

when $v \in L^q_{\mathrm{comp}}(\Omega_2)$.

We notice that Theorem 3 in the introduction is an immediate consequence of this theorem and its polarized version, which we leave to the reader to formulate.

Proof of the theorem. Let Ω_1 and Ω_2 be open bounded sets in $\mathbf{R}^n$ and let $v \in L^q_{\mathrm{comp}}(\Omega_2)$. If $f \in C_0^\infty(\mathbf{R}^n)$ then $F(t) = M_v K_0(t) f$ is a smooth function of t with values in $L^2_{\mathrm{comp}}(\mathbf{R}^n)$ and $F^{(2k)}(t) = M_v K_0(t) \Delta^k f$. It follows when $N \geq 2$ that

$$G_{N-1} \Delta^{k+n^*} f = \int_0^\infty K'_{N-2}(t) F^{(2n^*+2k)}(t)\, dt.$$

Since $2(n^* + k) < (N-2)\delta$ it follows from (17) that

$$K'_{N-2} \in C^{2k+2n^*}([0,\infty); \mathcal{B}),$$

and its derivatives up to order $2k + 2n^*$ vanish at the origin. Integrating by parts $2k + 2n^*$ times we get

$$G_{N-1} \Delta^{k+n^*} f = \int_0^\infty K_{N-2}^{(1+2n^*+2k)}(t) F(t)\, dt.$$

Set $Q_{N-1,k} = G_{N-1} \circ \Delta^k$ and define

$$G_{N-1,k} = \int_0^\infty K_{N-2}^{(1+2n^*+2k)}(t) M_v K_0(t)\, dt.$$

This is a continuous operator on $L^2_{\rm comp}$. Let E be a properly supported pseudo-differential operator of order $-2n^*$ which is a parametrix of Δ^{n^*}. Since $Q_{N-1,k} \circ \Delta^{n^*} = G_{N-1,k}$ we have

$$Q_{N-1,k} = G_{N-1,k} \circ E + Q_{N-1,k} \circ R$$

where R is an integral operator with a smooth and properly supported kernel (i.e. the projections $\operatorname{supp}(R) \ni (x,y) \to x$ and $\operatorname{supp}(R) \ni (x,y) \to y$ are proper). Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ and choose $\psi \in C_0^\infty(\mathbf{R}^n)$ such that $EM_\varphi = M_\psi EM_\varphi$. Then

$$Q_{N-1,k}M_\varphi = (G_{N-1,k}M_\psi)EM_\varphi + Q_{N-1,k}(RM_\varphi). \tag{32}$$

We notice that $G_{N-1,k}M_\psi$ is a continuous linear operator on $L^2(\mathbf{R}^n)$, and its distribution kernel is compactly supported. It follows from (17) that its norm in $\mathcal{B}$ can be estimated from above by $C_k^N\|v\|^N_{L^q}/N!$, where C_k depends on Ω_2, q and ψ only. Since EM_φ is a Hilbert-Schmidt operator we get the same kind of estimate for the Hilbert-Schmidt norm of $G_{N-1,k}M_\psi EM_\varphi$, if let C_k depend on φ also. Writing $Q_{N-1,k}RM_\varphi = G_{N-1}(\Delta^k RM_\varphi)$ we may also estimate the second term in the right-hand side of (32) in this way. Since $\varphi \in C_0^\infty$ was arbitrary it follows that $\Delta_y^k G_{N-1}(x,y) = Q_{N-1,k}(x,y)$ is in $L^2_{\rm loc}(\mathbf{R}^n \times \mathbf{R}^n)$ and we have the estimates

$$\Big(\iint_{\mathbf{R}^n\times\Omega_0} |\Delta_y^k G_{N-1}(x,y)|^2\,dx\,dy\Big)^{1/2} \le C_k^N\|v\|^{N-1}_{L^q}/N! \tag{33}$$

when v is supported in Ω_2, $\Omega_0 \subset \mathbf{R}^n$ is an open bounded set and $2(n^*+k) < (N-2)\delta$. Here C_k depends also on Ω_0, Ω_2 and q.

It is now a straight-forward procedure to deduce the conclusion of the theorem from the inequality above. In fact, if one chooses $\Omega_0 = 2\Omega_1 - \Omega_2$, then Caychy's inequality and the definition of B_N gives the estimate

$$\int_{\Omega_1} |B_N(x)|^2\,dx \le 2^n\|v\|^2_{L^2} \iint_{\mathbf{R}^n\times\Omega_0} |G_{N-1}(x,y)|^2\,dx\,dy,$$

and the estimates for $\Delta^k B_N(x)$ are obtained by replacing G_{N-1} in the right-hand side by $2^{2k}\Delta_y^k G_{N-1}(x,y)$ and then using (33). □

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