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# ON PRIME-INDEPENDENT MULTIPLICATIVE FUNCTIONS

WENGUANG ZHAI(JINAN)

## 1. Introduction

An arithmetical function $f:\mathbb{N}\rightarrow\mathbb{C}$ with $f(\mathbb{N})\neq\{0\}$ is called a *multiplicative* function if $f(mn)=f(m)f(n)$ holds for any $m,n\in\mathbb{N}$ with $(m,n)=1$. Obviously, if $f$ is multiplicative, then $f(1)=1$ and the values of $f(n)(n\geq 2)$ depend on $f(p^{\alpha})(p\in\mathbb{P},\alpha\in\mathbb{N})$. We say a multiplicative function $f(n)$ is *prime-independent multiplicative* function if for any $p\in\mathbb{P}$ and $\alpha\in\mathbb{N}$, the value of $f(p^{\alpha})$ does not depend on $p$ but only on $\alpha$.

We can define the prime-independent multiplicative function in another way. Suppose $g:\mathbb{N}\rightarrow\mathbb{C}$ is any map such that $g(\mathbb{N})\neq\{0\}$. Define

$$f(n)=\begin{cases}1, & \text{if } n=1,\\ \prod_{p^{\alpha}\| n}g(\alpha), & \text{if } n>1.\end{cases}\tag{1.1}$$

Then $f(n)$ is a prime-independent multiplicative function and we say it is generated from $g$. Throughout this paper , we use this definition.

There are many well-known prime-independent multiplicative functions.

**Example 1.1.** Let $a(n)$ denote the number of non-isomorphic abelian groups with $n$ elements. It is well-known that $a(n)$ is multiplicative and $a(p^{\alpha})=P(\alpha)$ for any $p\in\mathbb{P}$, $\alpha\in\mathbb{N}$, where $P(\alpha)$ is the number of unrestricted partitions of $\alpha$. Thus $a(n)$ is prime-independent multiplicative.

**Example 1.2.** The Dirichlet divisor function $d(n)$ is prime-independent multiplicative since $d(p^{\alpha})=\alpha+1$ for any $p\in\mathbb{P}$ and $\alpha\in\mathbb{N}$.

**Example 1.3.** Suppose $n>1$ is an integer and write $n=p_{1}^{\alpha_{1}}\cdots p_{s}^{\alpha_{s}}$. We say an integer $u$ is an exponential divisor of $n$ if

$$u=p_{1}^{\beta_{1}}\cdots p_{s}^{\beta_{s}}|n\Rightarrow\beta_{j}|\alpha_{j}(j=1,\cdots,s).$$

Let $d^{(e)}(n)$ denote the number of exponential divisors of $n$ for $n>1$ and $d^{(e)}(1)=1$. Then $d^{(e)}(n)$ is prime-independent multiplicative since $d^{(e)}(p^{\alpha})=d(\alpha)$ for any $p\in\mathbb{P}$ and $\alpha\in\mathbb{N}$.

The aim of this paper is to study the local density property of **integer-valued** prime-independent multiplicative functions.

**Definition.** If $l\geq 1$ is a fixed integer and $\{a_{n}\}$ ia any subset of $\mathbb{N}$. We say $\{a_{n}\}$ possesses the local density $d_{l}$ if the limit

$$d_{l}=\lim_{x\rightarrow\infty}x^{-1}\sum_{n\leq x,a_{n}=l}1\tag{1.2}$$

We suppose now that $g : \mathbb{N} \to \mathbb{N}$ is any map such that $g(\mathbb{N}) \neq \{1\}$ and $f$ is generated from $g$. For any fixed $l \geq 1$, define

$$F_{l,f} := \{n \in \mathbb{N} : f(n) = l\}, F_{l,f}(x) = \#\{n \in \mathbb{N} : f(n) = l, n \leq x\}.$$

When $f(n) = a(n)$, The asymptotic behaviour of $F_{l,a}(x)$ was studied by Kendall and Rankin[10], Ivić[6], Krätzel[12,13], Krätzel and Wolke[14].

The asymptotic behaviour of $F_{l,f}(x)$ for general prime-independent multiplicative functions was studied by Ivić[7]. Define

$$F_{l,f}(x) = d_{l,f}x + R(x), \tag{1.3}$$

where $d_{l,f} \geq 0$ is a constant depending only on $g$ and $l$.

When $g(1) = 1$, Ivić proved that uniformly in $l$

$$R(x) \ll x^{1/3} \log^2 x, \text{if} \quad g(2) = 1, \tag{1.4}$$

$$R(x) \ll x^{1/2} e^{-A\gamma(x)} \tag{1.5}$$

if there is a prime $p$ such that $p|g(2)$ but $p \not| l$, where $A > 0$ is a positive constant and

$$\gamma(x) := (\log x)^{3/5} (\log\log x)^{-1/5},$$

and

$$R(x) \ll x^{1/2} (\log\log x)^{c-1} \log^{-1} x \tag{1.6}$$

where $c = \min\{B \geq 1 : p|g(2), p^B \parallel l\}$.

When $g(1) > 1$, Ivić proved that if $l = p^c l'$, $p \not| l'$, where $p$ is a prime divisor of $g(1)$, then

$$F_{l,f}(x) \ll x (\log\log x)^{c-1} \log^{-1} x. \tag{1.7}$$

In this paper we shall further improve Ivić's results .

## 2. The case $g(1) = 1$

### 2.1. On $F_{l,f}(x)$.

In this section we consider the case $g(1) = 1$. First introduce some definitions connected with $g$. Since $g(\mathbb{N}) \neq \{1\}$, there exists an integer $k \geq 2$ such that $g(1) = \cdots = g(k-1) = 1$, but $g(k) > 1$. We define $r_0$ to be the smallest $j > k$ with $g(j) = 1$ if $1 \in \{g(n) : n > k\}$ otherwise we define $r_0 = \infty$.

Let $\mathcal{Q}_k(x)$ denote the number of $k-$free numbers not exceeding $x$. If the Riemann Hypothesis (RH) is true, then for some constant $0 < \theta_k < 1/k$ the asymptotic formula

$$\mathcal{Q}_k(x) = \frac{x}{\zeta(k)} + O(x^{\theta_k}) \tag{2.1}$$

holds. For example, we can take

$$\theta_2 = 17/54 + \varepsilon, \theta_k = 7/(8k+6) + \varepsilon (3 \leq k \leq 5), \theta_6 = 67/514 + \varepsilon, \text{etc.}$$

See Jia[9] , Graham and Pintz[2].

**Remark.** We always suppose that $1/\theta_k \notin \mathbb{N}$.

Now we state the results of $F_{l,f}(x)$. For $l = 1$, we have

**Theorem 2.1.** The asymptotic formula

$$F_{1,f}(x) = d_{1,f}x + O(x^{1/k}e^{-A\gamma(x)}) \tag{2.2}$$

holds.

If RH is true , then

$$F_{1,f}(x) = d_{1,f}x + d_{2,f}x^{1/r_0} + O(x^{1/(r_0+1-r_0\theta_k)}) \tag{2.3}$$

for $r_0 < 1/\theta_k$ and

$$F_{1,f}(x) = d_{1,f}x + O(x^{\theta_k}), \tag{2.4}$$

for $r_0 > 1/\theta_k$.

**Remark.** If $r_0 = \infty$, then $F_{1,f}$ is the set of all $k-$free numbers.

Now we suppose $l \geq 2$. Let $r$ denote the smallest $j$ such that $g(j) > 1$ and $g(j)|l$. Suppose $l = g^c(r)l', c > 0, g(r) \not| l'$. Obviously , $r \geq k$.

**Theorem 2.2.** If $r = k$, then for any fixed integer $N \geq 1$ we have

$$\begin{aligned} F_{l,f}(x) = d_{l,f}x + x^{1/k}\sum_{j=1}^{N} Q_j(\log\log x)\log^{-j-1} x \\ +O\left(x^{1/k}(\log x)^{-N-2}(\log\log x)^{c-1}\right), \end{aligned} \tag{2.5}$$

where $Q_j(t)$ is a polynomial in $t$ of degree not exceeding $c-1$.

If $r > k$, then

$$F_{l,f}(x) = d_{l,f}x + O(x^{1/k}e^{-A\gamma(x)}). \tag{2.6}$$

If RH is true , then (2.6) can be further improved.

**Theorem 2.3.** Suppose RH is true and $k < r < r_0$.

If $r < 1/\theta_k$, then for any fixed integer $N \geq 1$, we have

$$\begin{aligned} F_{l,f}(x) = d_{l,f}x + x^{1/r}\sum_{j=1}^{N} Q_j(\log\log x)\log^{-j} x \\ +O\left(x^{1/r}(\log x)^{-N-1}(\log\log x)^{c-1}\right), \end{aligned} \tag{2.7}$$

where $Q_j(t)$ is a polynomial in $t$ of degree not exceeding $c-1$.

If $r > 1/\theta_k$, then

$$F_{l,f}(x) = d_{l,f}x + O(x^{\theta_k}). \tag{2.8}$$

**Theorem 2.4.** Suppose RH is true and $r > r_0$.

If $r_0 < 1/\theta_k$, then

$$F_{l,f}(x) = d_{l,f}x + d^*_{l,f}x^{1/r_0} + O(x^{1/(r_0+1-r_0\theta_k)}). \tag{2.9}$$

If $r_0 > 1/\theta_k$, then

$$F_{l,f}(x) = d_{l,f}x + O(x^{\theta_k}). \tag{2.10}$$

Taking $f(n) = d^{(e)}(n)$, we get the following Corollary 2.1.

**Corollary 2.1.**

If $l = 2^c l', c > 0, 2 \not| l'$, then for any fixed integer $N \geq 1$ we have

$$F_{l,d^{(e)}}(x) = d_{l,d^{(e)}}x + x^{1/2}\sum_{j=1}^{N} Q_j(\log\log x)\log^{-j-1} x \tag{2.11}$$

$$+O\left(x^{1/2}(\log x)^{-N-2}(\log\log x)^{c-1}\right),$$

where $Q_j(t)$ is a polynomial in $t$ of degree not exceeding $c-1$.

If $2 \not| l$, we have

$$F_{l,d^{(e)}}(x) = d_{l,d^{(e)}}x + O(x^{1/2}e^{-A\gamma(x)}). \tag{2.12}$$

If $2 \not| l$ and RH is true, then we have

$$F_{l,d^{(e)}}(x) = d_{l,d^{(e)}}x + O(x^{\theta_2}). \tag{2.13}$$

**Corollary 2.2.** Suppose $g(1) = g(3) = 1$, $g(2) > 1$.

(1) If $g^c(2) \parallel l$ for some $c \geq 1$, then for any fixed integer $N \geq 1$ we have

$$F_{l,f}(x) = d_{l,f}x + x^{1/2}\sum_{j=1}^{N} Q_j(\log\log x)\log^{-j-1} x \tag{2.14}$$

$$+O\left(x^{1/2}(\log x)^{-N-2}(\log\log x)^{c-1}\right),$$

where $Q_j(t)$ is a polynomial in $t$ of degree not exceeding $c-1$.

(2) If $g(2) \not| l$, then

$$F_{l,f}(x) = d_{l,f}x + O(x^{1/2}e^{-A\gamma(x)}). \tag{2.15}$$

If $g(2) \not| l$ and RH is true, then

$$F_{l,f}(x) = d_{l,f}x + d^*_{l,f}x^{1/3} + O(x^{18/55+\varepsilon}). \tag{2.16}$$

2.2. **On $F_{l,f}(x+y) - F_{l,f}(x)$.**

It is also interesting to study the above problem in the short interval $(x, x+y]$ with $y = o(x)$. In the case of $a(n)$, it was first proved by Ivić[8] that

$$F_{l,a}(x+y) - F_{l,a}(x) = d_{l,a}y + o(y) \tag{2.17}$$

holds for $y \geq x^{581/1744}\log x$ uniformly for $l \geq 1$. Krätzel[11] proved that (2.17) is true for $y \geq x^{11/42+\varepsilon}$ and even for $y \geq x^{2/9+\varepsilon}$ if $l \equiv \pm 1(mod\ 6)$. Li[15] proved that (2.17) is true for $y \geq x^{1/5+\varepsilon}$ uniformly for $l \geq 1$.

In the general case , we have the following Theorem.

**Theorem 2.5.** The asymptotic formula

$$F_{l,f}(x+y) - F_{l,f}(x) = d_{l,f}y + o(y) \tag{2.18}$$

holds for $y \geq x^{1/(2k+1)+\varepsilon}$.

## 3. Preliminary definitions for the case $g(1) > 1$

In this section , we shall make some further definitions for the use of the case $g(1) > 1$. Suppose $\mathcal{T}$ is a subset of $\mathbb{N}$ such that

$$k_0 = \min_{n \in \mathcal{T}} n \geq 2, \qquad \mathcal{T} \neq k_0\mathbb{N}.$$

It is easy to prove the following Lemma 3.1.

**Lemma 3.1** There exist integers $k_0 = a_0 < a_1 < \cdots < a_t < b$, $d \geq 1$ such that $\{a_0, a_1, \cdots, a_t\} \subset \mathcal{T}$ and for $|u| < 1$ we have

$$1 + \sum_{n \in \mathcal{T}} u^n = \frac{(1-u^b)^d}{\prod_{j=0}^t (1-u^{a_j})} \times (1 + O(|u|^{b+1})). \tag{3.1}$$

**Remark.** $d$ is always 1 or 2.

**Definition 3.1.** Define

$$S(\mathcal{T}) := \{a_0, a_1, \cdots, a_t\}, \quad k(\mathcal{T}) := b, \quad E(\mathcal{T}) := d.$$

Now we define the *primitive generating* subset of $\mathcal{T}$. We suppose $\mathcal{T} \setminus k_0\mathbb{N} \neq \emptyset$. Write $\mathcal{T} = \mathcal{T}_0 \bigcup \mathcal{T}_1$, where

$$\mathcal{T}_0 = \{n \in \mathcal{T} : k_0 | n\}, \mathcal{T}_1 = \{n \in \mathcal{T} : k_0 \not| n\}.$$

Then $\mathcal{T}_1 \neq \emptyset$.

A subset $\mathcal{T}_P$ of $\mathcal{T}$ is said the *primitive generating* subset if it satisfies the following conditions:

(1). Every element of $\mathcal{T}_1$ can be written as a linear combination of elements of $\mathcal{T}_P$ with non-negative integral coefficients;

(2). For any $\mathcal{A} \subset \mathcal{T}_P, \mathcal{A} \neq \mathcal{T}_P$, there exists an $n \in \mathcal{T}_1$ such that $n$ can't be written as a linear combination of elements of $\mathcal{A}$ with non-negative integral coefficients.

If $k_0 | \mathcal{T}$, then define $\mathcal{T}_P = \{k_0\}$.

We also need estimates of multi-dimensional divisor functions. Suppose $S = \{u_1, u_2, \cdots, u_t\}$ is a finite subset of $\mathbb{N}$ with $u_1 < u_2 < \cdots < u_t$. The multi-demensional divisor function $d(n; S)$ is defined by

$$d(n; S) = d(u_1, u_2, \cdots, u_t; n) := \sum_{n = n_1^{u_1} \cdots n_t^{u_t}} 1.$$

Write

$$D(x; S) = \sum_{n \leq x} d(n; S) = \sum_{j \in S} c_j x^{1/j} + \Delta(x; S).$$

Now we define $\delta(S)$ to be a real number such that $0 \le \delta(S) < 1/u_1$ and the estimate

$$\Delta(x;S) \ll x^{\delta(S)}$$

holds, and define $\delta^*(S)$ to be a real number such that $0 \le \delta^*(S) < 1/u_1$ and the estimate

$$\sum_{x<n\le x+y} d(n;S) = C_{u_1} y x^{1/u_1-1}(1+o(1)) + O(x^{\delta^*(S)})$$

holds. Obviously, $\delta^*(S) \le \delta(S)$.

It is important to get good values for $\delta(S)$, which can be done by the theory of exponential sums and the theory of the Riemann Zeta-function. The value of $\delta^*(S)$ is connected with many short interval problems. For example, the value of $\delta^*(\{1,k\})(k \ge 2)$ is connected with the distribution of $k$−free numbers in short intervals, the value of $\delta^*(\{2,3\})$ is connected with the distribution of square-full numbers in short intervals, the value of $\delta^*(\{3,4,5\})$ is connected with the distribution of cube-full numbers in short intervals,etc.

For more details about multi-demensional divisor problems, see Krätzel[11].

## 4. The case $g(1) > 1$

### 4.1. On $F_{l,f}(x)$.

Now we consider the case $g(1) > 1$. Let $k_0$ denote the smallest element in the set $\mathcal{G}_0 = \{n \in \mathbb{N} : g(n) = 1\}$ if it is not empty; otherwise define $k_0 = \infty$. Let $r$ denote the smallest $j$ such that $g(j) > 1$ and $g(j)|l$. Suppose $l = g^c(r)l', c > 0, g(r) \nmid l'$.

**Theorem 4.1.** If $r < k_0$, then for any $N \ge 1$ we have

(4.1)

$$F_{l,f}(x) = x^{1/r}\sum_{j=1}^{N} Q_j(\log\log x)\log^{-j} x + O\left(x^{1/r}(\log x)^{-N-1}(\log\log x)^{c-1}\right),$$

where $Q_j(u)$ is a polynomial in $u$ of degree not exceeding $c-1$.

Now suppose $l = 1$ or $l \ge 2$ with $r > k_0$. We have the following Theorem 4.2.

**Theorem 4.2.**

(I) Suppose $l = 1$ or $l \ge 2$ with $r > k(\mathcal{G}_0)$.

If $\delta(S(\mathcal{G}_0))k(\mathcal{G}_0) > 1$, then we have

$$F_{l,f}(x) = \sum_{j\in S(\mathcal{G}_0)} c_{l,j}x^{1/j} + O(x^{\delta(S(\mathcal{G}_0))}). \tag{4.2}$$

If $\delta(S(\mathcal{G}_0))k(\mathcal{G}_0) < 1$, then we have

$$F_{l,f}(x) = \sum_{j\in S(\mathcal{G}_0)} c_{l,j}x^{1/j} + O(x^{1/k(\mathcal{G}_0)}e^{-A\gamma(x)}). \tag{4.3}$$

(II) Suppose $k_0 < r \le k(\mathcal{G}_0)$ and let $S_1$ denote the set of elements in $S(\mathcal{G}_0)$ less

If $\delta(S_1)r > 1$, then

$$F_{l,f}(x) = \sum_{j\in S_1} c_{l,j}x^{1/j} + O(x^{\delta(S_1)}). \tag{4.4}$$

If $\delta(S_1)r < 1$, then for any fixed integer $N \geq 1$ we have

$$F_{l,f}(x) = \sum_{j\in S_1} c_{l,j}x^{1/j} + x^{1/r}\sum_{j=1}^{N} Q_j(\log\log x)\log^{-j-s_0} x \tag{4.5}$$

$$+O\left(x^{1/r}(\log x)^{-N-1-s_0}(\log\log x)^{c-1}\right),$$

where $Q_j(u)$ is a polynomial in $u$ of degree not exceeding $c-1$, and

$$s_0 = \begin{cases} 0, & \text{if } r < k(\mathcal{G}_0), \\ 1, & \text{if } r = k(\mathcal{G}_0), E(\mathcal{G}_0) = 1, \\ 2, & \text{if } r = k(\mathcal{G}_0), E(\mathcal{G}_0) = 2. \end{cases}$$

For the function $f(n) = d(n)$, we have the following Corollary 4.1.

**Corollary 4.1.** Suppose $l \geq 2$. Let $p$ denote the smallest prime divisor of $l$ and write $l = p^c l', c > 0, p \nmid l'$. Then for any fixed integer $N \geq 1$, we have

(4.6)

$$F_{l,d}(x) = x^{\frac{1}{p-1}}\sum_{j=1}^{N} Q_j(\log\log x)\log^{-j} x + O\left(x^{\frac{1}{p-1}}(\log x)^{-N-1}(\log\log x)^{c-1}\right),$$

where $Q_j(u)$ is a polynomial in $u$ of degree not exceeding $c-1$.

**Corollary 4.2.** Suppose $r_0 \geq 3$ is a fixed integer such that

$$\{2, 3, \cdots, r_0\} \subset \mathcal{G}_0, g(1) > 1, g(r_0+1) > 1.$$

If $r = 1$, then for any fixed integer $N \geq 1$, we have

(4.7)

$$F_{l,f}(x) = x\sum_{j=1}^{N} Q_j(\log\log x)\log^{-j} x + O\left(x(\log x)^{-N-1}(\log\log x)^{c-1}\right),$$

where $Q_j(u)$ is a polynomial in $u$ of degree not exceeding $c-1$.

If $r = r_0 + 1 (r_0 = 3, 4, 5)$, then for any fixed integer $N \geq 1$, we have

$$F_{l,f}(x) = c_{l,f}^{(2)}x^{1/2} + c_{l,f}^{(3)}x^{1/3} \tag{4.8}$$

$$+x^{\frac{1}{r_0+1}}\sum_{j=1}^{N} Q_{j,l,f}(\log\log x)\log^{-j-t_0} x + O\left(x^{\frac{1}{r_0+1}}(\log x)^{-N-1-t_0}(\log\log x)^{c-1}\right),$$

where $Q_{j,l,f}(u)$ is a polynomial in $u$ of degree not exceeding $c-1$, and

$$t_0 = \begin{cases} 1, & \text{if } r_0 = 3, 4, \\ 2, & \text{if } r_0 = 5. \end{cases}$$

If $r > r_0 + 1 (r_0 = 3, 4, 5)$ or $r \geq r_0 + 1 (r_0 \geq 6)$, then

$$F_{l,f}(x) = c_{l,f}^{(2)} x^{1/2} + c_{l,f}^{(3)} x^{1/3} + O(x^{1/t_1} e^{-A\gamma(x)}), \tag{4.9}$$

where

$$t_1 = \begin{cases} 4, & \text{if } r_0 = 3, \\ 5, & \text{if } r_0 = 4, \\ 6, & \text{if } r_0 \geq 5. \end{cases}$$

4.2. $F_{l,f}(x+y) - F_{l,f}(x)$.

Now we study the short inteval results . Note that if $F_{l,f} \neq \emptyset$, then $l$ is factorizable on $g(\mathbb{N} \setminus \mathcal{G}_0)$. If $l$ has a factorization $l = g^{c_1}(r_1) \cdots g^{c_e}(r_e)$, then let $\mathcal{G} = \{r_1, \cdots, r_e\}$. And we define

$$\mathcal{G}^* = \{\mathcal{G} | l \text{ has a factorization on } g(\mathcal{G})\}.$$

Let $\mathcal{G}_1, \cdots, \mathcal{G}_h$ denote all elements of $\mathcal{G}^*$ and define

$$\mathcal{T}_j = \mathcal{G}_0 \bigcup \mathcal{G}_j, (j = 1, \cdots, h).$$

If $k_0\mathbb{N}$ is not a subset of $\mathcal{G}_0$, then we define $k_1 \geq 2$ to be an integer such that

$$\{k_0, 2k_0, \cdots, (k_1 - 1)k_0\} \subset \mathcal{G}_0, k_1 k_0 \notin \mathcal{G}_0;$$

If $k_0\mathbb{N} \subset \mathcal{G}_0$, then define $k_1 = \infty$.

For $l = 1$, we have the following Theorem 4.3.

**Theorem 4.3.** Suppose $k_0 < \infty$. If

$$y \geq \begin{cases} x^{\frac{k_0-1}{k_0} + \frac{1}{(2k_1+1)k_0} + \varepsilon}, & \text{if } k_0 | \mathcal{G}_0, \\ x^{\frac{k_0-1}{k_0} + \max\left(\frac{1}{(2k_1+1)k_0},\ \delta^*(\mathcal{G}_{0P})\right) + \varepsilon}, & \text{if } k_0 \not| \mathcal{G}_0, \end{cases}$$

then

$$F_{1,f}(x+y) - F_{1,f}(x) = c_0 y x^{1/k_0 - 1}(1 + o(1)), \tag{4.10}$$

where $c_0 > 0$ is some positive constant.

Now suppose $l \geq 2$ and we have the following Theorem 4.4.

**Theorem 4.4.**

If $r = 1$, then for $y \geq x^{7/12+\varepsilon}$ we have

(4.11)

$$F_{l,f}(x+y) - F_{l,f}(x) = c_0 y (\log\log x)^{c-1} \log^{-1} x (1 + o(1)),$$

where $c_0$ is a constant.

If $1 < r < k_0$, then for

$$y \geq x^{\frac{r-1}{r} + \max\left(\frac{7}{12r}, \delta^*(\mathcal{T}_{1P}), \cdots, \delta^*(\mathcal{T}_{hP})\right) + \varepsilon}$$

we have

(4.12)

$$F_{l,f}(x+y) - F_{l,f}(x) = c_0 y x^{\frac{1}{r} - 1} (\log\log x)^{c-1} \log^{-1} x (1 + o(1)).$$

If $r > k_0$, then for

$$y \geq x^{\frac{k_0-1}{k_0}+\max\left(\frac{1}{(2k_1+1)k_0},\delta^*(\mathcal{T}_{1P}),\cdots,\delta^*(\mathcal{T}_{hP})\right)+\varepsilon}$$

we have

(4.13) $$F_{l,f}(x+y) - F_{l,f}(x) = c_{k_0,f}yx^{1/k_0-1}(1+o(1)).$$

Taking $f(n) = d(n)$, we get

**Corollary 4.3.**

(1) Suppose $2|l$, then the asymptotic formula

(4.14) $$F_{l,d}(x+y) - F_{l,d}(x) = c_0y(\log\log x)^{c-1}\log^{-1}x(1+o(1))$$

holds for $y \geq x^{7/12+\varepsilon}$.

(2) Suppose $l = p_1^{c_1}\cdots p_e^{c_e}$ with $2 < p_1 < \cdots < p_e$. Let $r = p_1 - 1$, then the asymptotic formula

(4.15)
$$F_{l,d}(x+y) - F_{l,d}(x) = c_0yx^{\frac{1}{r}-1}(\log\log x)^{c-1}\log^{-1}x(1+o(1))$$

holds for $y \geq x^{\frac{r-1}{r}+\max(\frac{7}{12r},\delta(l))+\varepsilon}$, where

$$\delta(l) = \max_{\substack{l=n_1\cdots n_d\\ d\geq 2, n_j>1}} \delta^*(\{n_1-1,\cdots,n_d-1\}_P).$$

**Corollary 4.4.** Suppose $r_0 \geq 3$ is a fixed integer and $\{2,3,\cdots,r_0\} \subset \mathcal{G}_0$. If $r = 1$, then for $y \geq x^{7/12+\varepsilon}$ the asymptotic formula

(4.16) $$F_{l,f}(x+y) - F_{l,f}(x) = c_0y(\log\log x)^{c-1}\log^{-1}x(1+o(1))$$

holds.

If $r \neq 1$, then

(4.17) $$F_{l,f}(x+y) - F_{l,f}(x) = c_0yx^{-1/2}(1+o(1))$$

holds for $y \geq x^{5/8+\varepsilon}$.

**Corollary 4.5.** Suppose $r_0 \geq 4$ is a fixed integer such that

$$\{3,4,\cdots,r_0\} \subset \mathcal{G}_0, g(1) > 1, g(2) > 1, g(r_0+1) > 1.$$

If $r = 1, 2$, then for $y \geq x^{\frac{r-1}{r}+\frac{7}{12r}+\varepsilon}$, we have

(4.18)
$$F_{l,f}(x+y) - F_{l,f}(x) = c_0yx^{\frac{1}{r}-1}(\log\log x)^{c-1}\log^{-1}x(1+o(1)).$$

Suppose $r_0 = 4$. If $l = 1$ or $r > 4$, then for $y \geq x^{2/3+1/11+\varepsilon}$, we have

(4.19) $$F_{l,f}(x+y) - F_{l,f}(x) = c_0yx^{-2/3}(1+o(1)).$$

Suppose $r_0 \geq 5$. If $l = 1$ or $r > r_0$, then for $y \geq x^{2/3+19/159+\varepsilon}$, we have

(4.20) $$F_{l,f}(x+y) - F_{l,f}(x) = c_0yx^{-2/3}(1+o(1)).$$

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Department of Mathematics,
Shandong Normal University,
Jinan, 250014, Shandong,
P.R.China
E-mail:zhaiwg@hotmail.com