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HEIGHT INEQUALITY FOR CURVES OVER FUNCTION FIELDS

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0. Introduction

The geometric case of the height inequality (cf. [V3]) was discussed at the conference. By the geometric case, we mean that the global field in the question is a function field of one variable over complex number field $\mathbb{C}$, instead of a number field which is a finite extension of $\mathbb{Q}$. Hence in our geometric case, problem is algebro-geometric nature. Since we consider geometry over $\mathbb{C}$, our problem is also complex analytic nature.

Our method belongs to the second view point. We use techniques of classical function theory such as Ahlfors' theory of covering surfaces, area-length method to prove the height inequality for curves in the geometric case, which is the main result of our discussion.

1. Notations

Let B be a smooth, projective, connected curve over $\mathbb{C}$. Let k be the function field of B. Let $S \subset B$ be a finite set of points which will be fixed throughout. Let X be a smooth, projective, geometricaly connected variety over k and $D \subset X$ be an effective divisor. Let L be a line bundle on X.

Following P. Vojta [V3], we define the functions

$$h_{L,k}(P),\ N_{k,S}(D,P),\ N^{(1)}_{k,S}(D,P),\ m_{k,S}(D,P),\ d_k(P)$$

as follows.

First, take a model of X over B, i.e., smooth variety $\mathfrak{X}$ projective over B such that the generic fiber is X. Then by taking the normalization of the Zariski closure of $P \in \mathfrak{X}(\overline{k}) = X(\overline{k})$, we can associate the following commutative diagram.

$$\begin{array}{ccc} B' & \xrightarrow{f_P} & \mathfrak{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & = & B \end{array}$$

Here B' is the curve whose function field is isomorphic to $k(P)$.

Let $\mathfrak{D} \subset \mathfrak{X}$ and $\mathfrak{L}$ be an extension of $D \subset X$ and L to $\mathfrak{X}$, respectively. Put

$$h_{\mathfrak{L},k}(P) = \frac{1}{\deg p} \deg f_P^* \mathfrak{L},$$

$$N_{k,S}(\mathfrak{D}, P) = \frac{1}{\deg p} \sum_{x \in B' \setminus p^{-1}(S)} \operatorname{ord}_x f_P^* \mathfrak{D} \quad (P \in X(\overline{k}) \setminus D),$$

$$N_{k,S}^{(1)}(\mathfrak{D}, P) = \frac{1}{\deg p} \sum_{x \in B' \setminus p^{-1}(S)} \min(1, \operatorname{ord}_x f_P^* \mathfrak{D}) \quad (P \in X(\overline{k}) \setminus D)$$

and

$$m_{k,S}(\mathfrak{D}, P) = \frac{1}{\deg p} \sum_{x \in p^{-1}(S)} \operatorname{ord}_x f_P^* \mathfrak{D} \quad (P \in X(\overline{k}) \setminus D).$$

If we replace the models $\mathfrak{X}$, $\mathfrak{D}$ and $\mathfrak{L}$ to other models $\mathfrak{X}'$, $\mathfrak{D}'$ and $\mathfrak{L}'$, we have

$$h_{\mathfrak{L},k}(P) = h_{\mathfrak{L}',k}(P) + O(1), \ N_{k,S}(\mathfrak{D}, P) = N_{k,S}(\mathfrak{D}', P) + O(1),$$

$$N_{k,S}^{(1)}(\mathfrak{D}, P) = N_{k,S}^{(1)}(\mathfrak{D}', P) + O(1), \ m_{k,S}(\mathfrak{D}, P) = m_{k,S}(\mathfrak{D}', P) + O(1),$$

where $O(1)$ are bounded terms independent of $P \in X(\overline{k})$. Hence we write as

$$h_{L,k}(P) = h_{\mathfrak{L},k}(P) + O(1), \ N_{k,S}(D, P) = N_{k,S}(\mathfrak{D}, P) + O(1),$$

$$N_{k,S}^{(1)}(D, P) = N_{k,S}^{(1)}(\mathfrak{D}, P) + O(1), \ m_{k,S}(D, P) = m_{k,S}(\mathfrak{D}, P) + O(1).$$

Finally, put

$$d_k(P) = \frac{1}{\deg p} \deg(\operatorname{ram} p),$$

where $\operatorname{ram} p \subset B'$ is the ramification divisor of p.

2. Main conjecture

Ofcourse, we have equality

$$h_{L(D),k}(P) = N_{k,S}(D, P) + m_{k,S}(D, P) + O(1), \tag{2.1}$$

where $L(D)$ is the line bundle associated to D. Our problem is that *What happens if we replace the right hand side of* (2.1) *by the term* $N_{k,S}^{(1)}(D, P)$?

In this case, we can't hope any equality. Instead, we hope the inequality like

$$h_{K_X(D),k} \leq N_{k,S}^{(1)}(D, P) + d_k(P) + (\text{small error term}), \tag{2.2}$$

where K_X is the canonical line bundle on X.

Heuristic proof of (2.2):

1. We only consider k rational points $P \in X(k)$ for simplicity. Let $\mathcal{M}$ be the connected component of the moduli space of sections of $\pi : \mathfrak{X} \to B$ containing the section $f_P : B \to \mathfrak{X}$.
2. For integers $k \geq 0$, put
$$\mathcal{M}_k = \{f' \in \mathcal{M} : \deg f'^*\mathfrak{D} - \#\operatorname{supp}(f'^*\mathfrak{D}) \geq k\}.$$
Then $\mathcal{M}_k \subset \mathcal{M}$ is a Zariski closed subset and form a sequence
$$\mathcal{M} = \mathcal{M}_0 \supset \mathcal{M}_1 \supset \mathcal{M}_2 \supset \cdots.$$
3. For a generic $f' \in \mathcal{M}$, $f'(B)$ and $\mathfrak{D}$ would intersect transverly. Hence we hope
$$\deg f'^*\mathfrak{D} = \#\operatorname{supp}(f'^*\mathfrak{D}),$$
which implies $\mathcal{M}_1 \subsetneqq \mathcal{M}_0 = \mathcal{M}$ and $\operatorname{codim}(\mathcal{M}_1, \mathcal{M}_0) \geq 1$.
4. More generaly, we hope $\operatorname{codim}(\mathcal{M}_{k+1}, \mathcal{M}_k) \geq 1$ for $k \geq 0$.
5. Hence, for $k = \dim \mathcal{M} + \varepsilon$, we hope "$\mathcal{M}_k = \emptyset$", which implies
$$\deg f_P^*\mathfrak{D} - \dim \mathcal{M} \leq \#\operatorname{supp}(f_P^*\mathfrak{D}) + \varepsilon.$$
6. By the equality "$\dim \mathcal{M} \doteqdot -h_{K_X,k}(P)$", which seems to be true, and the fact $\#S < \infty$ we get
$$h_{K_X(D),k}(P) \leq N_{k,S}^{(1)}(D, P) + \varepsilon + O(1)$$
as desired.

Unfortunately, the above inequality (2.2) is not correct in general, and it seems very difficult to justify the above argument.

The precise conjecture is

Conjecture ([V3]). *Let X be a smooth projective variety over k, let D be a normal crossings divisor on X, let L be a big line bundle on X, let $r \in \mathbb{Z}_{>0}$ and let $\varepsilon > 0$. Then there exists a proper Zariski closed subset $Z = Z(k, S, X, D, L, r, \varepsilon) \subsetneqq X$ such that*
$$h_{K_X(D),k}(P) \leq N_{k,S}^{(1)}(D, P) + d_k(P) + \varepsilon h_{L,k}(P) + O_\varepsilon(1)$$
for all $P \in X(\overline{k}) \backslash Z$ with $[k(P) : k] < r$.

Remark 2.3. *(1) Using Arakelov geometry, the number field case of the above conjecture can be formulated in the same manner (see* [V3]*).*

(2) When X is a curve, Z is a union of points. Hence $P \in Z$ satisfies $h_{K_X(D),k}(P) < O_\varepsilon(1)$, which means that we don't need Z in this case.

3. Main result

We can prove the one dimensional case of above conjecture.

Theorem . *Let X be a smooth projective curve over k, let D be a reduced divisor on X, let L be a big line bundle on X and let $\varepsilon > 0$. Then we have*

$$h_{K_X(D),k}(P) \leq N^{(1)}_{k,S}(D,P) + d_k(P) + \varepsilon h_{L,k}(P) + O_\varepsilon(1) \tag{3.1}$$

for all $P \in X(\overline{k})\backslash D$.

Remark 3.2. *(1) In our case, we don't need r in above conjecture.*

(2) When (X, D) has splitting, i.e., there exists (X_0, D_0) on $\mathbb{C}$ such that $(X, D) = (X_0 \otimes_{\mathbb{C}} k, D_0 \otimes_{\mathbb{C}} k)$, our theorem is an easy consequense of Hurwitz's formula.

The following corollary directly follows from our theorem (cf. [V1], [V2]).

Corollary . *Let X be a smooth projective curve over k and let $\varepsilon > 0$. Then we have*

$$h_{K_X,k}(P) \leq (1+\varepsilon) d_k(P) + O_\varepsilon(1)$$

for all $P \in X(\overline{k})$.

4. About proof

Our proof is based on Ahlfors' theory of covering surfaces [A], which is an important theory in classical complex analysis. Roughly speaking, main result of Ahlfors' theory is kind of Hurwitz's formula for non-proper covering of surfaces.

First, we reduce the general case of our theorem to the special case that $X = \mathbb{P}^1_k$ and $D = (P_1) + \cdots + (P_q)$ where P_i are distinct k-rational points of $\mathbb{P}^1_k$. This reduction step is algebraic; using a ramified cover and the ramification formula.

Then this special case is equivalent to the following; *Let $a_1, \cdots, a_q$ be distinct rational functions on B, let $\varepsilon > 0$. Then there is a positive constant $C(\varepsilon) > 0$ such that for all covering $\pi : Y \to B$ and rational function f on Y such that $f \neq a_i \circ \pi$, we have*

$$\begin{aligned}(q-2-\varepsilon)\deg f \leq \sum_{i=1}^{q} \#\{z \in Y;\ a_i \circ \pi(z) = f(z)\} \\ + \deg(\operatorname{ram}\pi) + C(\varepsilon)\deg\pi.\end{aligned} \tag{4.1}$$

To prove (4.1), we first divide B by sufficiently small, finite Jordan domains Δ_λ such that

$$B = \bigcup_{\lambda:\text{finite}} \overline{\Delta_\lambda}, \quad \Delta_\lambda \cap \Delta_{\lambda'} = \emptyset \text{ for } \lambda \neq \lambda'.$$

If each Δ_λ is small enough, then the move of rational functions a_i on Δ_λ are very small, hence the situation is close to the constant case. As already mentioned above, if rational functions a_i are constant, then (4.1) can be proved by Hurwitz's formula. In our case, since Δ_λ is non-compact, we use Ahlfors' theory instead of Hurwitz's formula to prove localized inequality of (4.1) on Δ_λ. Then we sum all these localized inequality over λ to obtain (4.1). In this part, we also need so-called area-length method, which is an important technique in complex analysis.

Our inequality (4.1) is an algebraic analogue of a long standing conjecture, called defect relation for small functions, in one dimensional value distribution theory. And above proof is a modification of an argument in [Y1].

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