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Higher KdV equations approximating long waves of two-dimensional water surface

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1. There are more than 100 years now since Korteweg and de Vries have discovered their nowadays very famous equation as the second nonlinear approxmation for long waves of water surface in the two-dimensional flows being characterized by the following physical structure:

> $\delta^2 = (h/\lambda)^2$ and $\varepsilon = \alpha/h$ are of the same order as infinitesimals when λ tends to infinity and α tends to zero, λ being the wave length and α the wave amplitude.

We have given effectively a mathematical justification for THIS KdV equation for analytic solutions in 1980's [1] .

Our discussion, however, has been a little bit obscure. In fact, the KdV equation gives a good approximation for long waves propagating in the direction of one of the characteristic of the linear wave equation being known as the first approximation since L.Lagrange, but for long waves propagating in the direction of the other characteristic is looser with the additional inhomogeneous terms consisting of the differential polynomials of these long waves.

2. Remember now our dimensionless equations for long waves in Lagrangian coordinates in [5]:

the equations of motion:

$$(2.1)\qquad \mathbf{X}_{tt} + (1/\delta)D_\xi(\delta\mathbf{Y}) + \delta^2\mathbf{X}_{tt}D_\xi\mathbf{X} + \delta^2(\delta\mathbf{Y})_{tt}D_\xi(\delta\mathbf{Y}) + D_\xi\mathbf{P} = 0,$$

$$(2.2)\qquad (\delta\mathbf{Y})_{tt} + (1/\delta)D_\eta(\delta\mathbf{Y}) + \delta^2\mathbf{X}_{tt}D_\eta\mathbf{X} + \delta^2(\delta\mathbf{Y})_{tt}D_\eta(\delta\mathbf{Y}) + D_\eta\mathbf{P} = 0;$$

and th equation of continuity:

$$(2.3)\qquad D_\xi\mathbf{X} + D_\eta\mathbf{Y} + \delta^2(D_\xi\mathbf{X}D_\eta\mathbf{Y} - D_\eta\mathbf{X}D_\xi\mathbf{Y}) = 1,$$

in

$$(2.4)\qquad \Omega_1 = \{(\xi,\eta): 0 < |\eta| < 1\},$$

where $(D_\xi, D_\eta) \equiv (\partial/\partial\xi, (1/\delta)\partial/\partial\eta)$.

Their solutions with initial data $\mathbf{X}(0,\xi,\eta)=\mathbf{X}^0(\xi,\eta)$, $\mathbf{X}_t(0,\xi,\eta)=\mathbf{X}^1(\xi,\eta)$, analytic, give us the Friedrichs expansion as follows:

$$(2.5)\qquad \mathbf{x}_{tt} + \delta^2\mathbf{x}_{tt}\mathbf{x}_\xi - \mathbf{x}_{\xi\xi} - (1/3)\delta^2\mathbf{x}_{\xi\xi\xi\xi} = O(\delta^4),$$

for $\mathbf{x}(t,\xi)=\mathbf{X}(t,\xi,1)$, in Banach spaces of analytic functions provided with the norms $E(DX^n)$ and $E(D^mX^n)$, $m \geqq 1$, for $n \geqq 1$, defined by:

Definition 2.1 $E(\phi)$ for any $\phi \in C^\infty(\Omega_1) \Leftrightarrow$

$$\Leftrightarrow \quad E(\phi) \equiv |||\phi||| + |||D_\xi\phi||| + |||D_\eta\phi||| + ||\phi(1)|| + ||D_\xi\phi(1)|| + ||(D_\eta\phi)(1)||,$$

where

$$|||\phi||| = |||\phi|||_0 + |||D\phi|||_0 + |||D^2\phi|||_0\ ;$$
$$||\phi(1)|| = ||\phi(1)||_0 + ||D\phi(1)||_0 + ||D^2\phi(1)||_0$$

with $D = D_\xi$ or D_η and

$$|||\phi|||_0{}^2 = \int_{-1}^{1}\int_{-\infty}^{\infty} |\phi(\xi,\eta)|^2\, d\xi d\eta;\quad ||\phi(1)||_0{}^2 = \int_{-\infty}^{\infty} |\phi(\xi,1)|^2\, d\xi;$$

And for $m \geqq 1$,

$$E(D^m\phi) \quad \text{for} \quad \phi \in C^\infty(\Omega_1) \Leftrightarrow$$

$$\Leftrightarrow \qquad E(D^m\phi) \equiv |||D^m\phi||| + |||D_\xi D^m\phi||| + |||D_\eta D^m\phi||| +$$
$$+ ||D^m\phi(1)|| + ||D_\xi D^m\phi(1)|| + ||(D_\eta D^m\phi)(1)||,$$

where

$$|||D^m\phi||| = \Sigma_{0\leqq\mu\leqq m}\ \theta^\mu |||D_\xi{}^{m-\mu} D_\eta{}^\mu \phi|||\ ;$$
$$||D^m\phi(1)|| = \Sigma_{0\leqq\mu\leqq m}\ \theta^\mu ||(D_\xi{}^{m-\mu} D_\eta{}^\mu)\phi(1)||,\ 0 \leqq \theta \leqq 1.$$

This gives us actually

$$\mathbf{x}_{tt} - \mathbf{x}_{\xi\xi} + \delta^2(\mathbf{x}_\xi \mathbf{x}_{\xi\xi} - (1/3)\mathbf{x}_{\xi\xi\xi\xi}) = O(\delta^4),$$

since we have

$$(2.6)\qquad \mathbf{x}_{\xi\xi}/(1+\delta^2\mathbf{x}_\xi) - (1-\delta^2\mathbf{x}_\xi)\,\mathbf{x}_{\xi\xi} = \delta^4\mathbf{x}_\xi{}^2\,\mathbf{x}_{\xi\xi}/(1+\delta^2\mathbf{x}_\xi) + O(\delta^4).$$

In particular, for initial data

$$(2.7)\qquad u(0,\xi) = \mathbf{x}_\xi(0,\xi) = \mathbf{X}_\xi(0,\xi,1),\ \ v(0,\xi) = \mathbf{x}_t(0,\xi) = \mathbf{X}_t(0,\xi,1)$$

satisfying

$$(2.8)\qquad E(\mathbf{x}_\xi(0,\xi) + \mathbf{x}_t(0,\xi)) = O(1),\quad E(\mathbf{x}_\xi(0,\xi) - \mathbf{x}_t(0,\xi)) = O(\delta^2),$$

surface waves defined by $f = (u+v)/2$, $g = (u-v)/2$ with $u = -\mathbf{x}_\xi$, $v = -\mathbf{x}_t$ satisfy

$$(2.9)\qquad f_t - f_\xi - \delta^2(ff_\xi + (1/3)f_{\xi\xi\xi}) = O(\delta^4),$$

and

$$(2.10)\qquad g_t + g_\xi + \delta^2(gg_\xi + (1/3)g_{\xi\xi\xi}) = -\delta^2(ff_\xi + (1/3)f_{\xi\xi\xi}) + O(\delta^4),$$

since we have

$$u_t - v_\xi = 0,\ v_t - u_\xi - \delta^2(uu_\xi + (1/3)u_{\xi\xi\xi}) = O(\delta^4).$$

We have had in consequence [5] a mathematical justification for Korteweg-de Vries equations

$$(2.11)\qquad \mathbf{F}_t - \mathbf{F}_\xi - \delta^2(\mathbf{F}\mathbf{F}_\xi + (1/3)\mathbf{F}_{\xi\xi\xi}) = 0\ ,$$

$$(2.12)\qquad \mathbf{G}_t + \mathbf{G}_\xi + \delta^2(\mathbf{G}\mathbf{G}_\xi + (1/3)\mathbf{G}_{\xi\xi\xi}) = -\delta^2(\mathbf{F}\mathbf{F}_\xi + (1/3)\mathbf{F}_{\xi\xi\xi})\ ,$$

with initial data $\mathbf{F}(0) = f(0)$ and $\mathbf{G}(0) = g(0)$, as the approximate equations for long waves of water surface. As we mentioned above, the approximations for long waves by solutions $\mathbf{G}$ for inhomogeneous KdV equation above are less accurate than the approximations by $\mathbf{F}$.

3. We have now discovered long waves of water surface approximated more accurately than $f = u+v$, $g = u-v$ with $u = \mathbf{x}_\xi$, $v = \mathbf{x}_t$, by solutions of the KdV equations propagating not only in the direction of f, but also in the direction of g. Let us, in fact, define now $\{\mathbf{u},\mathbf{v}\}$ by

$$(3.1) \qquad \mathbf{u} = (\mathbf{x}_\xi + (1/6)\delta^2\mathbf{x}_{\xi\xi\xi})/(1+(1/4)\delta^2\mathbf{x}_\xi\,, \ \mathbf{v} = \mathbf{x}_t\,,$$

then we see readily that $\mathbf{f} = -(\mathbf{u}+\mathbf{v})/2$, $\mathbf{g} = -(\mathbf{u}-\mathbf{v})/2$ satisfy

$$(3.2) \qquad \mathbf{f}_t - \mathbf{f}_\xi - (1/4)\delta^2(\mathbf{f}\mathbf{f}_\xi + (2/3)\mathbf{f}_{\xi\xi\xi}) = O(\delta^4),$$

and

$$(3.3) \qquad \mathbf{g}_t + \mathbf{g}_\xi + (1/4)\delta^2(\mathbf{g}\mathbf{g}_\xi + (2/3)\mathbf{g}_{\xi\xi\xi}) = O(\delta^4).$$

In fact, (3.1) assures us that there are no inhomogeneous terms of order $O(\delta^2)$ in (3.3) consisting of differential polynomials of f only as coefficients, which differs essentially (3.3) from (2.12).

We see then readily that

$$F_t - F_\xi - (1/4)\delta^2(FF_\xi + (2/3)F_{\xi\xi\xi}) = 0$$

and

$$G_t + G_\xi + (1/4)\delta^2(GG_\xi + (2/3)G_{\xi\xi\xi}) = 0\,,$$

with initial data

$$F(0)=\mathbf{f}(0)= \mathbf{x}_t(0)+(\mathbf{x}_\xi(0) + (1/6)\delta^2\mathbf{x}_{\xi\xi\xi}(0))/(1+(1/4)\delta^2\mathbf{x}_\xi(0) = O(1)$$

and

$$G(0)=\mathbf{g}(0)= \mathbf{x}_t(0)-(\mathbf{x}_\xi(0) + (1/6)\delta^2\mathbf{x}_{\xi\xi\xi}(0))/(1+(1/4)\delta^2\mathbf{x}_\xi(0) = O(\delta^2)$$

respectively, give approximations for (3.2) and (3.3). Thus $\mathbf{f}=-(\mathbf{u}+\mathbf{v})/2$ and $\mathbf{g}=-(\mathbf{u}-\mathbf{v})/2$ are approximated by F and G. We have in consequence approximations for $u = -\mathbf{x}_\xi$, $v = -\mathbf{x}_t$ by F and G via $\mathbf{u}, \mathbf{v}$.

Thus we have renewed and modified our theory in 1986 giving another mathematical justification for KdV equations as the second nonlinear approximate equations for long waves of water surface in two-dimensional flows, propagating in both directions of the characteristics of the linear wave equation as the first approximation.

If we have done this here in the Lagrangean coordinates system, it is to show that our theory is absolutely valid both in Euler's and Lagrange's point of view. And also, it is suggested by (2.5) that the KdV equation approximates long waves of rather complicated nonlinear structure. This observation faciliates our deeper analysis of long waves of water surface in two-dimensional flows.

Let us make here a remark, very important for the passage to the higher approximations, that $\mathbf{g}(t)$ itself is of order $O(\delta^4)$ for initial data $\mathbf{g}(0)$ of order $O(\delta^4)$ by the continuous dependence of solutions on inhomogeneous second terms.

4. In this paragraph, we return once more to the analysis in Euler coordinates system. The above analysis in **nos. 2.-3.** is of a general character. By a deepen study of our Friedrichs expansion in [2],[3],[4], we discovered, in fact, certain nonlinear transformations of long waves of the water surface approximated by solutions for the higher KdV equations, the first of which is

$$(4.1)\qquad f_t+f_x+(\delta^2/2)((3/2)f^2+(1/3)f_{xx})_x+ +(\delta^4/2)((5/2)f^3+(5/3)ff_{xx}+(5/6)f_x{}^2+(1/9)f_{xxxx})_x=0.$$

Firstly, we see that dimensionless equations governing long waves of water surface in two-dimensional flows are

$$(4.2)\qquad \delta^2\phi_{xx}+\phi_{yy}=0 \text{ in } \Omega=\{(x,y):\ -\infty< x<\infty,\ 0<1+\delta^2\gamma\},$$

$$(4.3)\qquad \phi_y=0,\ -\infty< x<\infty,\ y=0,$$

$$(4.4)\qquad \phi_t+(\delta^2/2)\phi_x{}^2+\gamma+(1/2)\phi_y{}^2=0,\quad -\infty< x<\infty,\ y=1+\delta^2\gamma,$$

$$(4.5)\qquad \gamma_t+\delta^2\gamma_x\phi_x-\delta^{-2}\phi_y=0,\quad -\infty< x<\infty,\ y=1+\delta^2\gamma,$$

with initial data $\phi(0,x,y)=\phi_0(x,y),\ \ \gamma(0,x)=\gamma_0(x)$.

For analytic solutions, we have the following Friedrichs expansion for (4.4) - (4.5), i.e. on the water surface we have:

$$\phi_t + \gamma + (\delta^2/2)\phi_x{}^2 - (\delta^4/2)\phi_{xx}{}^2 - \delta^6(\gamma\phi_{xx}{}^2 + (1/3)\phi_{xx}\phi_{xxxxx}) = O(\delta^8)$$

$$\gamma_t + \phi_{xx} + \delta^2(\gamma\phi_x)_x + (\delta^2/3)\phi_{xxxxx} + \delta^4\{(\gamma\phi_{xx})_{xx} + (2/15)\phi_{xxxxxxx}\} -$$

$$- \delta^6\{(\gamma^2\phi_{xx})_{xx} + (1/3)\gamma_{xxxx}\phi_{xx} + (4/3)\gamma_{xxx}\phi_{xxx} + (7/3)\gamma_{xx}\phi_{xxxx} +$$

$$+ 2\gamma_x\phi_{xxxxx} + (2/3)\gamma\phi_{xxxxxx} + (17/315)\phi_{xxxxxxxxx}\} = O(\delta^8).$$

Secondly, we renew the definition of **u,v** as follows:

$$\mathbf{u} = \{\gamma + G(\delta^2/2)u^2 - H(\delta^4/2)u_x{}^2\} \,/\, \{1 + A_1\delta^2 u + A_2\delta^4 u^2\},$$

$$\mathbf{v} = \{u + \delta^2 D(\gamma u) + (\delta^2/6)u_{xx} + \delta^4 E(\gamma u_x)_x + (2/15)Fu_{xxxx}\} \,/$$
$$/\{1 + A_1\delta^2 u + A_2\delta^4 u^2\},$$

with $u = \phi_x$ on the surface $y = 1+\delta^2\gamma$. Starting from these, we get our above mentioned conclusion by discussions of the same order as those in **3.**

We see in fact that the redefined $\mathbf{f} = (\mathbf{u}+\mathbf{v})/2$ satisfies

$$\mathbf{f}_t + \mathbf{f}_x + (\delta^2/2)((3/2)\mathbf{f}^2 + (1/3)\mathbf{f}_{xx})_x +$$
$$+ (\delta^4/2)((5/2)\mathbf{f}^3 + (5/3)\mathbf{f}\mathbf{f}_{xx} + (5/6)\mathbf{f}_x{}^2 + (1/9)\mathbf{f}_{xxxx})_x = O(\delta^6),$$

for suitable constants A_1, A_2,D,E,F,G and H which assure us the absence of coefficients consisting of differential polynomials of f only in terms of order $O(\delta^4)$ in the development of **g**.

5. We show now how this first higher KdV equation approximates the long waves of water surface. Remind first the KdV hierarchy $\{F_n\}$, n=1,2,3,... defined by

$$(5.1)\quad F_{n+1,x} = f_x F_n + 2fF_{n,x} + (1/3)F_{n,xxx}, \quad \text{for} \quad n \geqq 0 \quad \text{with} \quad F_0 = f,$$

for solutions (periodic or rapidly decreasing etc) f of KdV equation:

$$f_t + f_x + (1/2)\delta^2(3ff_x + (1/3)f_{xxx}) = 0,$$

namely, $f_t + f_x + F_{n,x} = 0$ is the n^{th} KdV equation.

We see now that our first higher KdV equation can be written by these KdV hierarchy as follows:

$$(5.2)\qquad f_t + f_x + (1/2)\delta^2 F_{1,x} + (1/2)\delta^4 F_{2,x} = 0.$$

These F_1 and F_2 satisfy the following linearized KdV equations:

$$(5.3)\qquad F_{1,t} + F_{1,x} + (1/2)\delta^2(3fF_{1,x} + (1/3)F_{1,xxx}) = 0,$$

and

(5.4) $$F_{2,t} + F_{2,x} + (1/2)\delta^2(3fF_{2,x} + (1/3)F_{2,xxx}) = 0 .$$

Applying the Cauchy-Kowalevski theorem in Nirenberg-Nishida version to this system (5.2) - (5.4) of KdV equations for analytic initial data, we get analytic solutions for (5.1) which approximate long waves **f.**

Remark One can write (5.2) as

$$f_t + f_x + (1/2)\delta^2 F_{1,x} + (1/2)\delta^4(f_x F_1 + 2fF_{1,x} + (1/3)F_{1,xxx}) = 0.$$

6. After that, our attempt has been to analyse these " liaisons secrètes" between a concrete problem of water surface in mathematical physics and the theory of completely integrable systems which seems totally stranger to the former in her character. Very briefly speaking, for N, an integer, we have a set of nonlinear transformations $\{\mathbf{u}_N,\mathbf{v}_N\}$ of $\{u,v\} = \{u=\phi_x , \gamma\}$ for which we have an asymptotic expansion such that

$$\mathbf{f}_t + \mathbf{f}_x + (\delta^2/2)\mathbf{F}_{1,x} + (\delta^4/2)\mathbf{F}_{2,x} + \dots + (\delta^{2N}/2)\mathbf{F}_{N,x} = O(\delta^{2(N+1)}),$$

f being long waves defined by $\mathbf{f}=\mathbf{f}_N = (\mathbf{u}_N+\mathbf{v}_N)/2$ and $\{\mathbf{F}_n\}_{n=1,2,3,\dots}$ being "pseudo-KdV hierarchy" defined as above by

$$\mathbf{F}_{n+1,x} = \mathbf{f}_x\mathbf{F}_n + 2\mathbf{f}\mathbf{F}_{n,x} + (1/3)\mathbf{F}_{n,xxx}, \text{ for } n\geqq 0 \text{ with } \mathbf{F}_0 = \mathbf{f}.$$

We are preparing an article on these " liaisons secrètes" to be published elsewhere. We gave a talk on it in the International Conference in Honour of Professor Jean Vaillant held in Karlskrona, Sweden, in june 2002.

References

[1] T.Kano -T. Nishida: A mathematical justification for Korteweg-de Vries equation and Boussinesq equation of water surface waves, Osaka J.Math., 23(1986), 389-413.

[2] T.Kano -T. Nishida: Sur les ondes de surface de l'eau avec une justification mathématique des équations des ondes en eau peu profonde, J.Math.Kyoto Univ., 19(1979), 335-370.

[3]T.Kano -T. Nishida: Water waves and Friedrichs expansion, Lect.Note.Num.Appl.Anal., Kinokuniya-North Holland, 6(1983), 39-57.

[4] T.Kano: Une théorie trois-dimensionnelle des ondes de surface de l'eau et le développement de Friedrichs, J.Math. Kyoto Univ., 26(1986), 101-155 et 157-175.

[5] T.Kano - S.Miki: Sur les ondes superficielles de l'eau et le développement de Friedrichs dans le système de coordonnées de Lagrange, in "Hyperbolic Differential Operators and Related Problems", éd. Ancona/Vaillant, Marcel Dekker, 2003, Chap.14, 199-232.