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Initial boundary value problem for the equations of ideal magneto-hydrodynamics in a half space

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1. Introduction

We consider the equations of ideal Magneto-Hydrodynamics (MHD) for the motion of an electrically conducting fluid, where 'ideal' means that the effect of viscosity and electrical resistivity is neglected. We study the initial boundary value problem in the half space. More precisely, we consider the equations of MHD

$$\rho_p(\partial_t + u\cdot\nabla)p + \rho\nabla\cdot u = 0, \tag{1.1a}$$

$$\rho(\partial_t + u\cdot\nabla)u + \nabla p + \mu H\times(\nabla\times H) = 0, \tag{1.1b}$$

$$\partial_t H - \nabla\times(u\times H) = 0, \tag{1.1c}$$

$$\nabla\cdot H = 0 \tag{1.1d}$$

in $[0,T]\times\Omega$ with the initial condition

$$(p,u,H)|_{t=0} = (p^0,u^0,H^0) \quad \text{in} \quad \Omega \tag{1.2}$$

and with the boundary condition

$$u\cdot\nu = 0, \quad H\times\nu = 0 \quad \text{on} \quad [0,T]\times\Gamma. \tag{1.3}$$

Here Ω is the half space $\{x\in\mathbf{R}^3;\ x_1>0\}$ with the boundary $\Gamma=\{x_1=0\}$; the pressure p (scalar), the velocity $u=(u_1,u_2,u_3)$, and the magnetic field $H=(H_1,H_2,H_3)$ are unknown functions of (t,x); the permeability μ is supposed to be a positive constant; the density $\rho=\rho(p)$ is also supposed to be a smooth known function of $p>0$ such that $\rho>0$ and $\rho_p\equiv\partial\rho/\partial p>0$ for $p>0$; we write $\partial_t=\partial/\partial t$, $\partial_i=\partial/\partial x_i\,(i=1,2,3)$, $\nabla=(\partial/\partial x_1,\partial/\partial x_2,\partial/\partial x_3)$ and use the conventional notation in the vector analysis; $\nu=(-1,0,0)$ denotes the unit outward normal to Ω. Thus our boundary condition (1.3) can be written as

$$u_1 = H_2 = H_3 = 0 \quad \text{on} \quad [0,T]\times\Gamma. \tag{1.4}$$

The initial value problem (1.1), (1.2) in the whole space has been solved by Kato [2]. Other initial boundary value problems for the equations (1.1) with boundary conditions different from (1.3) have been studied by Yanagisawa [19], Yanagisawa-Matsumura [21]. To explain the details, let us set

$$\Gamma_0 = \{x\in\Gamma;\ (H^0\cdot\nu)(x)=0\}, \quad \Gamma_1 = \{x\in\Gamma;\ (H^0\cdot\nu)(x)\neq 0\}.$$

They consider the case when Γ consists only of Γ_0 or Γ_1. In this case, their problems can be reduced into initial boundary value problems for quasi-linear symmetric hyperbolic systems with boundary characteristic of constant multiplicity.

A general theory for initial boundary value problems for symmetric hyperbolic systems has been developed by many authors. The case when the boundary is non-characteristic has been studied by Friedrichs [1], Lax-Phillips [5], Tartakoff [17], Rauch-Massey III [12] and so on (see also [13]). The case when the boundary is characteristic of constant multiplicity has been treated by Lax-Phillips [5], Tsuji [18], Majda-Osher [7], Rauch [11], Ohno-Shizuta-Yanagisawa [10], Secchi [14] and so on.

If Γ consists only of Γ_0 or Γ_1, then our problem (1.1)–(1.3) can be also reduced into an initial boundary value problem with boundary characteristic of constant multiplicity. So, in this case, we can find a solution (p, u, H).

Our main concern is the case when both Γ_0 and Γ_1 are not empty. In this case, the equations form a quasi-linear symmetric hyperbolic system with boundary characteristic of *non-constant* multiplicity. However, only few studies have so far been made at the case when the boundary is characteristic of non-constant multiplicity (see [8], [16]). The purpose of this paper is to show that the solution to our problem (1.1)–(1.3) has *full regularity*.

2. Main Theorem

We use the following notation for the function spaces: For $m \in \mathbf{Z}_+$, we define

$$X^m(T;\Omega) = \bigcap_{j=0}^{m} W^{j,\infty}(0,T;H^{m-j}(\Omega)).$$

Let $\bar{p}$ be a positive constant and set $\bar{V} = (\bar{p}, 0, 0)$. Our main theorem is as follows:

Theorem 2.1. *Let $m \geq 3$ be an integer. Suppose that the initial data $V^0 = (p^0, u^0, H^0)$ satisfies the following conditions:*

(i) $V^0 - \bar{V} \in H^m(\Omega)$;

(ii) V^0 *satisfies the compatibility conditions up to order* $m-1$;

(iii) Γ_1 *is dense in* Γ;

(iv) $\nabla \cdot H^0 = 0$ *in* Ω;

(v) $p^0(x) > 0$ *in* Ω.

Then there exists a $T_0 > 0$ such that the initial boundary value problem (1.1)–(1.3) *has a unique solution $V = (p, u, H)$ with $V - \bar{V} \in X^m(T_0;\Omega)$.*

Remark. Physically, we must impose the conditions (iv), (v). The conditions (i), (ii) are necessary to get a regular solution. Therefore only the condition (iii) is unreasonable. The initial boundary value problem (1.1)–(1.3) under the condition weaker than (iii) is an open problem.

3. Preliminaries

For the equations (1.1) we may assume $\mu = 1$ without loss of generality; otherwise it suffices to introduce new variables $\mu^{1/2}H$ instead of H. Moreover we can write the equations (1.1) in the following equivalent form:

$$\alpha(p)(\partial_t + u\cdot\nabla)p + \nabla\cdot u = 0, \tag{3.1a}$$

$$\rho(p)(\partial_t + u\cdot\nabla)u + \nabla p - (H\cdot\nabla)H + (1/2)\nabla|H|^2 = 0, \tag{3.1b}$$

$$(\partial_t + u\cdot\nabla)H - (H\cdot\nabla)u + H(\nabla\cdot u) = 0, \tag{3.1c}$$

$$\nabla\cdot H = 0 \tag{3.1d}$$

where $\alpha(p) = \rho_p(p)/\rho(p)$. The following lemma is needed later.

Lemma 3.1. *Let $m \geq 3$ be an integer. Suppose that the initial data $V^0 = (p^0, u^0, H^0)$ satisfies the following conditions:*

(i) $V^0 - \bar{V} \in H^m(\Omega)$;

(ii) V^0 *satisfies the compatibility conditions up to order* $m-1$;

(iii) Γ_1 *is dense in* Γ.

Then it holds that

$$\partial_1^k u_1^0 = \partial_1^k H_2^0 = \partial_1^k H_3^0 = 0 \quad \text{on} \quad \Gamma \quad (k \text{ is an even number}); \tag{3.2a}$$

$$\partial_1^k p_1^0 = \partial_1^k u_2^0 = \partial_1^k u_3^0 = \partial_1^k H_1^0 = 0 \quad \text{on} \quad \Gamma \quad (k \text{ is an odd number}). \tag{3.2b}$$

for $k = 0, 1, \dots, m-1$.

Proof. Given the system (3.1) and the initial data $V|_{t=0} = V^0$ in Ω, we define the function "$\partial_t^j V|_{t=0}$" in Ω by fomally applying ∂_t^{j-1} to the system, solving for $\partial_t^j V$ and evaluating at time $t = 0$. Furthermore let us take the 7×7 matrix M_i ($i \in \mathbf{Z}_+$) such that

$$M_i V = \begin{cases} (0, u_1, 0, 0, 0, H_2, H_3) & (i \text{ is an even number}), \\ (p, 0, u_2, u_3, H_1, 0, 0) & (i \text{ is an odd number}) \end{cases}$$

for $V = (p, u_1, u_2, u_3, H_1, H_2, H_3) \in \mathbf{R}^7$. It suffices to show that

$$M_i(\partial_1^i \text{“}\partial_t^j V|_{t=0}\text{”}) = 0 \quad \text{on} \quad \Gamma \quad \text{for} \quad 0 \leq i + j \leq k \quad (k = 0, 1, \dots, m-1). \tag{3.3}$$

Indeed, letting $i = k$ and $j = 0$, we conclude the proof.

Now we shall show the statement (3.3). We proceed by induction on k. From the boundary condition (1.4), the case $k = 0$ is trivial. Inductively assume that the statement is true up to $k-1$ and consider the case of k. It is enough to prove that

$$M_i(\partial_1^i \text{“}\partial_t^{k-i} V|_{t=0}\text{”}) = 0 \quad \text{on} \quad \Gamma \quad (i = 0, 1, \dots, k). \tag{3.4}$$

In order to prove the assertion (3.4), we proceed by induction on i. First we consider the case $i = 0$. The compatibility condition of order k implies that

$$\text{“}\partial_t^k u_1|_{t=0}\text{”} = \text{“}\partial_t^k H_2|_{t=0}\text{”} = \text{“}\partial_t^k H_3|_{t=0}\text{”} = 0 \quad \text{on} \quad \Gamma,$$

and hence the case $i = 0$ is clear. Inductively assuming that the assertion (3.4) is true up to $i-1$, we consider the case of i.

First suppose that i is an odd number. Applying $\partial_1^{i-1}\partial_t^{k-i}$ to the both sides of (3.1d), we obtain

$$\partial_1^i\partial_t^{k-i}H_1 + \partial_1^{i-1}\partial_2\partial_t^{k-i}H_2 + \partial_1^{i-1}\partial_3\partial_t^{k-i}H_3 = 0. \tag{3.5}$$

From the inductive hypothesis it follows that ∂_1^{i-1}"$\partial_t^{k-i}H_l|_{t=0}$" $= 0$ on Γ $(l = 2,3)$, which implies that

$$\partial_1^{i-1}\partial_l\text{“}\partial_t^{k-i}H_l|_{t=0}\text{”} = \partial_l(\partial_1^{i-1}\text{“}\partial_t^{k-i}H_l|_{t=0}\text{”}) = 0 \quad \text{on} \quad \Gamma \quad (l = 2,3),$$

and hence

$$(\text{the left-hand side of (3.5)})|_{t=0} = \partial_1^i\text{“}\partial_t^{k-i}H_1|_{t=0}\text{”} \quad \text{on} \quad \Gamma.$$

Thus it holds that ∂_1^i"$\partial_t^{k-i}H_1|_{t=0}$" $= 0$ on Γ.

Similarly, applying $\partial_1^{i-1}\partial_t^{k-i}$ to the both sides of the first component of (3.1b), we have

$$\begin{aligned}\partial_1^{i-1}\partial_t^{k-i}\{(\rho(p)\partial_t u_1 + u_1\partial_1 u_1 + u_2\partial_2 u_1 + u_3\partial_3 u_1) + \partial_1 p \\ -(H_2\partial_2 H_1 + H_3\partial_3 H_1) + (H_2\partial_1 H_2 + H_3\partial_1 H_3)\} = 0.\end{aligned}$$

Calculating the differentiations of the product, recalling the inductive hypothesis and observing ∂_1^i"$\partial_t^{k-i}H_1|_{t=0}$" $= 0$ on Γ, we obtain "$\partial_1^i\partial_t^{k-i}p|_{t=0}$" $= 0$ on Γ.

Moreover applying $\partial_1^{i-1}\partial_t^{k-i}$ to the both sides of the second and third components of (3.1c) and using the inductive hypothesis, we get

$$H_1^0(\text{“}\partial_1^i\partial_t^{k-i}u_l|_{t=0}\text{”}) = 0 \quad \text{on} \quad \Gamma \quad (l = 2,3).$$

Since H_1^0 is continuous on Γ and Γ_1 is dense in Γ, we have $H_1^0 \neq 0$ a.e. on Γ, and hence

$$\partial_1^i\text{“}\partial_t^{k-i}u_l|_{t=0}\text{”} = 0 \quad \text{on} \quad \Gamma \quad (l = 2,3).$$

Therefore if i is an odd number, then the assertion (3.4) is true.

Next suppose that i is an even number. Applying $\partial_1^{i-1}\partial_t^{k-i}$ to the both sides of (3.1a) and using the inductive hypothesis, we obtain ∂_1^i"$\partial_t^{k-i}u_1|_{t=0}$" $= 0$ on Γ. In the same way, applying $\partial_1^{i-1}\partial_t^{k-i}$ to the both side of the second and third components of (3.1b) and recalling the inductive hypothesis, we have

$$H_1^0(\partial_1^i\text{“}\partial_t^{k-i}H_l|_{t=0}\text{”}) = 0 \quad \text{on} \quad \Gamma \quad (l = 2,3).$$

As argued above, we obtain "$\partial_1^i\partial_t^{k-i}u_l|_{t=0}$" $= 0$ on Γ $(l = 2,3)$. Therefore if i is an even number, then the assertion (3.4) is true.

Thus the assertion (3.4) is true by induction on i, and hence the statement (3.3) is also true by induction on k. □

4. Proof of the Main Theorem

The equations (3.1) can be converted into the following equivalent form as a symmetric system:

$$\alpha(p)(\partial_t + u\cdot\nabla)p + \nabla\cdot u = 0, \tag{4.1a}$$

$$\rho(p)(\partial_t + u\cdot\nabla)u + \nabla p - (H\cdot\nabla)H + (1/2)\nabla|H|^2 = 0, \tag{4.1b}$$

$$(\partial_t + u\cdot\nabla)H - (H\cdot\nabla)u + H(\nabla\cdot u) = 0. \tag{4.1c}$$

The equivalence of (3.1) and (4.1), under the initial and boundary conditions (1.2) and (1.3), follows by observing that if the solution of (4.1) satisfies $\nabla \cdot H = 0$ in Ω at $t = 0$, then $\nabla \cdot H = 0$ in Ω is true for all $t > 0$. Thus for the proof of Theorem 2.1, we shall find a unique solution to the initial boundary value problem (4.1), (1.2), (1.4).

Proof of Theorem 2.1. The uniqueness of the solution to the initial boundary value problem (4.1), (1.2), (1.4) is easily checked. We consider the existence of the solution to this problem. For the proof, we introduce the extension $\tilde{V}^0(x) = (\tilde{p}^0, \tilde{u}^0, \tilde{H}^0)$ $(x \in \mathbf{R}^3)$ of the initial data $V^0(x) = (p^0, u^0, H^0)$ $(x \in \Omega)$ as follows: $\tilde{u}_1^0$, $\tilde{H}_2^0$, $\tilde{H}_3^0$ are odd functions and $\tilde{p}^0$, $\tilde{u}_2^0$, $\tilde{u}_3^0$, $\tilde{H}_1^0$ are even functions with respect to x_1. Then the assertion (3.2) yields that $\tilde{V}^0 \in H^m(\mathbf{R}^3)$.

Now we consider the initial value problem for the system (4.1) in whole space with the initial condition

$$V|_{t=0} = \tilde{V}^0 \quad \text{in} \quad \mathbf{R}^3. \tag{4.2}$$

Since the equations (4.1) is a symmetric hyperbolic system, this initial value problem (4.1), (4.2) has a unique solution $V = (p, u, H)$ with $V - \bar{V} \in X^m(T_0; \mathbf{R}^3)$ for some $T_0 > 0$ (see [3], [6] and so on). We shall show that V restricted to $[0, T_0] \times \Omega$ is a desired solution to our initial boundary value problem (4.1), (1.2), (1.4). For this purpose, it suffices to prove that V satisfies the condition (1.4).

For a function $f(t, x)$ $((t, x) \in [0, T_0] \times \mathbf{R}^3)$, we define the functions $Odd(f)(t, x)$ and $Even(f)(t, x)$ $((t, x) \in [0, T_0] \times \mathbf{R}^3)$ as

$$Odd(f)(t, x) = -f(t, -x_1, x_2, x_3), \quad Even(f)(t, x) = f(t, -x_1, x_2, x_3).$$

Using this notation, we set

$$\begin{gathered} \hat{u}_1 = Odd(u_1), \quad \hat{H}_2 = Odd(H_2), \quad \hat{H}_3 = Odd(H_3), \\ \hat{p} = Even(p), \quad \hat{u}_2 = Even(u_2), \quad \hat{u}_3 = Even(u_3), \quad \hat{H}_1 = Even(H_1) \end{gathered}$$

where $V = (p, u, H)$ is as above. By direct calculations, we can prove that $\hat{V} = (\hat{p}, \hat{u}, \hat{H})$ is also a solution to the initial value problem (4.1), (4.2). Thus the uniqueness of the solution to the initial value problem (4.1), (4.2) implies that $V = \hat{V}$. This yields that u_1, H_2, H_3 are odd functions, and hence V satisfies the condition (1.4). Therefore V restricted to $[0, T_0] \times \Omega$ is a desired solution to our initial boundary value problem (4.1), (1.2), (1.4). $\square$

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