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An introduction to semi-Lagrangian schemes for second order Hamilton-Jacobi equations

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The paper presents the theory of semi-Lagrangian (SL) schemes for Hamilton-Jacobi equations. The main emphasis is on second order problems starting from the linear advection-diffusion equation and continuing with the MC flow equation. Several results will be illustrated as well as some connections to other techniques and hints for future extensions and developments. Finally, some numerical tests will be presented.

1 Introduction

The goal of this paper is to give a short introduction to the theory of semi-Lagrangian schemes for Hamilton-Jacobi equations with a special emphasis on second order problems. Let us recall that semi–Lagrangian schemes are interesting extentions of the Courant–Isaacson–Rees [17] method for conservation laws which allow for large time–steps still guaranteeing high accuracy. The general idea of such schemes is to reconstruct the solution by integrating (numerically) the equation along characteristics starting from every grid point for just a single time step. The solution is computed coupling a numerical method for ODEs (to compute the upwind points with respect to the grid nodes) with an interpolation formula (to recover the value of the solution at such points, which are not grid points). In this respect SL shemes differ from particle methods and Monte Carlo methods where there is no grid and the paths of the particles must be tracked up to the boundary of the domain of computation. Although SL schemes are typically used for first order problems, they can also be applied to second order equations giving some advantages when the advection term is dominant or when the second order operator is degenerate. It should be noted that in Finite Differences(FD) or Finite Elements (FE) techniques it is quite common to split the differential operator in order to treat by semi–Lagrangian techniques only the first order part. This splitting results either in very severe time–step bounds or in the additional computational effort of solving an implicit scheme for the second order term. Moreover, the approximation of second order terms by a more traditional approaches (FD or FE) can make rather difficult to handle the degeneracies which are typical of the Hamilton-Jacobi equations arising in stochastic control problems, front propagations and image processing.

We will present here a different approach where second order *and* first order terms are treated in the same way. In fact the schemes are based on a weak notion of characteristics which relies on the stochastic representation formula of Feynman–Kac (see e.g. [46]). In this framework, characteristics for the second order terms are stochastic processes, f.e. for the Laplacian the characteristics are the trajectories of a Wiener process. There are several advantages in this approach: it allows for larger time steps, the schemes are stable also in presence of degeneracies, they can be implemented on structured and unstructured grids and they may achieve higher consistency rates (although a precise proof of this property is still missing for second order problems). Since the goal of this paper is to survey the underlying philosophy and some results, we will concentrate on model problems.

We will start in Section 2 from first order equations (linear and nonlinear) where this approach has shown to be very effective and the interpretation of characteristic lines is standard. We will continue our presentation dealing with a classical linear problem arising in many applications to fluiddynamics, the advection-diffusion problem. Finally, we will conclude with the second order degenerate parabolic equations related to the study of the Mean Curvature (MC) flow. The general framework for our results is the theory of viscosity solutions, the interested reader can find the main results, several applications and a long list of references in the monographs [3], [2], [37] and in the survey paper [18].

In order to set these results into perspective we should give some background on other techniques that have been adopted to solve the obove problems. As we said, the typical problem which has been attacked with semi-Lagrangian scheme is the advection problem since the method of characteristics gives the exact solution. The approximation via the semi-Lagrangian approach gives a discrete representation formula for the solution that converges to the continuous representation formula when the discretization steps Δt and Δx go to zero. Naturally, a CFL stability condition is required to have convergence but iterestingly this condition does not imply for SL schemes that Δt must have the same size of Δx (see Section 2). An analysis of SL schemes for the trasport equation can be found in [29]. The interested reader can also find in [57] and [61] many informations and examples on SL methods for linear and nonlinear advection dominated problems arising in atmospheric models. It should be emphasized that the meteorological community has greatly contributed to the development of SL schemes since the possibility to integrate models with large time steps is crucial for weather prediction where the simulation of complex models should cover at least 24 hours. Section 2 will be devoted to the construction of SL schemes for first order equations. We refer to [28], [33], [30], [35] for the analysis of SL-schemes for first order nonlinear Hamilton-Jacobi equations (see also the references therein for other discretizations). Finite differences schemes have been analyzed f.e. in [19].

The advection-diffusion problem treated in Section 3 has been extensively studied by finite differences and finite elements methods so that it is almost impossible to give a complete list of references. We limit ourselves to some contribution which are closer to our approach. It is well known that classical finite elements fail for advection dominated problems (see e.g. [54], [56]) since stability is lost. A great effort has been done to stabilize Galerkin methods finally obtaining theoretical results and accurate simulations (see e.g. [41], [38], [8] and [9]). In [22] and [55] a coupling between the method of characteristics and the finite element method has been proposed. The methods based on this coupling allow for rather large time steps so that they can be compared to SL-schemes. The analysis of SL-schemes for the advection-diffusion problem is based on the results in [10]. The approximation of more general nonlinear second order Hamilton-Jacobi equations arising in stochastic control problems can be found in [46] (see also the references therein), [40], [1], [11], [47]. An analysis of the rate of convergence has recently appeared in [45], [44] and [5].

The MC flow problem has obtained a great attention because of its importance in many applications which ranges from phase transition to image processing (nonlinear filtering). It is a prototype for degenerate second order Hamilton–Jacobi equations. Several results have been obained via viscosity solutions starting from [16] and [26]. The interested reader can find a long list of references in the survey papers [60], [25] and in the lecture notes [39]. Among the many papers related to the numerical approximation of MCM we limit ourselves to few of them based on the level set formulation (see e.g. [50], [7] for other approaches). In the pioneering paper [53] a finite difference approximation of the mean curvature is used there to replace the velocity c of a front propagating in the normal direction according to the first order Hamilton-Jacobi equation

$$\begin{cases} v_t(x,t) + c(x)|Dv(x,t)| = 0 \\ v(x,0) = v_0(x). \end{cases} \tag{1}$$

The interested reader can find in the monographs [59] and [52] several developments and applications of this approach. Another class of finite differences approximation schemes has been proposed in [48]. Those schemes are based on the description of MCM via the (short time) solution of the heat equation with discontinuous initial data. An analysis of this technique and convergence results can be found in [24], [4]. Other finite differences schemes have been proposed in [20] where a convergence result is also proved. An error analysis for these schemes has appeared in [21] (see also [23]). More recently, a semi-Lagrangian scheme strictly connected to the approach described in [53] has been implemented in [62]. The analysis of the SL-scheme presented in Section 4 can be found in [31], [27], [12] and it is still going on.

2 Basics on semi-Lagrangian schemes for first order problems

Let us start with the simplest linear problem, the advection equation

$$v_t + av_x = 0 \quad \text{in } \boldsymbol{R} \times (0, +\infty) \tag{2}$$

with the initial condition

$$v(x,0) = v_0(x) \quad \text{in } \boldsymbol{R} \tag{3}$$

The exact solution of problem (2), (3), obtained via the method of characteristics, is

$$v(x,t) = v_0(x-at), \qquad \text{for any } (x,t) \in \mathbb{R} \times [0,+\infty) \,. \tag{4}$$

The above representation formula means that the solution at the point (x,t) coincides with the value of the initial condition at the foot of the characteristic passing through (x,t). That information can be used also for numerical purposes, trying to mimic the method of characteristics on a grid. To simplify the presentation, let us take a uniform space step Δx and define its nodes by $x_i = i\Delta x$, $i \in \mathbb{Z}$. Considering the representation formula on a single time interval of width Δt for every node of the grid, we get

$$v(x_i, \Delta t) = v_0(x_i - a\Delta t) \,, \qquad \text{for any } i \in \mathbb{Z}. \tag{5}$$

Now keep the time step Δt constant and define $t_n = n\Delta t$, $n \in \mathbb{N}$. We can iterate the same argument on every time step obtaining

$$v(x_i, n\Delta t) = v(x_i - a\Delta t, (n-1)\Delta t) \,, \qquad \text{for any } i \in \mathbb{Z},\ n \in \mathbb{N}. \tag{6}$$

It is interesting to note that the representation formula (6), coincides with the SL approximation scheme derived by applying the forward Euler scheme to (2). In fact, using the standard notation for numerical approximations

$$v_i^n \equiv v(x_i, t_n), \qquad \text{for } i \in \mathbb{Z} \text{ and } n \in \mathbb{N} \tag{7}$$

we can write the SL scheme for (2) as

$$\frac{v_i^{(n+1)} - v_i^n}{\Delta t} - \frac{v^n(x_i - a\Delta t) - v_i^n}{\Delta t} = 0, \qquad \text{for } i \in \mathbb{Z},\ n \in \mathbb{N} \tag{8}$$

which is equivalent to (6). There are three important remarks to be made. The first is that the advection term av_x has been approximated as a directional derivative in the backward direction of characteristics $(-a)$ so that the up-wind correction is built in the scheme. The second is that to implement the scheme we need a local reconstruction of the approximate vale of v^n at the foot of the characteristic $x_i - a\Delta t$ which is not a grid point (except for the special choice $\Delta x = a\Delta t$). This problem can be easily solved by interpolation over the (known) values v_i^n. When the drift is constant the foot of the characteristic can be computed exactly so that the only numerical error involved in this computation is the interpolation error which can be reduced by means of high order interpolation formulas. The last remark is that, for this problem, stability is guaranteed for any Δt since

$$||v^n||_\infty \leq ||v_0||_\infty \tag{9}$$

For a precise analyis of this approach to linear advection problems we refer to [29].

It is also interesting to note that the same approach can be adopted for nonlinear problems such as

$$\begin{cases} v_t + H(\nabla v) = 0 \\ v(x,0) = v_0(x) \end{cases} \tag{10}$$

which also have a representation formula for the solution, the so-called Hopf-Lax-Oleinik formula. A crucial role in the representation formula for (10) is played by the Legendre-Fenchel conjugate of convex analysis which we recall here for reader's convenience.

Definition 2.1 *Let $H : \mathbf{R}^n \to \mathbf{R}$ be a continuous and convex function such that*

$$\frac{H(p)}{|p|} \to +\infty \qquad \text{for } |p| \to +\infty\,. \tag{11}$$

The Legendre-Fenchel conjugate of H is the continuous and convex function, H^*, defined by

$$H^*(p) \equiv \sup_{q \in \mathbf{R}^n} \{p \cdot q - H(q)\}\,. \tag{12}$$

It is worth to note that (11) guarantees that $H^*(p)$ is always properly defined and $(H^*(p))^* = H(p)$ for any $p \in \mathbf{R}^n$. The Legendre-Fenchel conjugate is crucial to establish a link between the general Cauchy problem (10) and a control problem. Infact, if the hamiltonian H satisfies the assumptions required in Definition 2.1, we can write the equation in (10) as

$$v_t + \sup_{a \in \mathbf{R}^n} \{a \cdot \nabla v - H^*(a)\} = 0\,. \tag{13}$$

which is the Bellman equation for a finite horizon control problem with the controls varying in $A \equiv \mathbf{R}^n$ (see [2] and [3] for details). The Hopf-Lax-Oleinik representation formula for the solution of this equation is

$$v(x,t) = \inf_{y \in \mathbf{R}^n} \left[v_0(y) + tH^*\left(\frac{x-y}{t}\right)\right]\,. \tag{14}$$

Extensions of the above representation formula to more general hamiltonians $H(x, \nabla v)$ can be found in [13] and [14].

Let us examine now the typical SL-scheme in two dimensions. Let us define the lattice $L(\Delta x, \Delta y, \Delta t)$ by

$$L \equiv \{(x_i, y_j, t_n) : x_i = i\Delta x, y_j = j\Delta y \text{ and } t_n = n\Delta t, \text{for } i,j \in \mathbb{Z} \text{ and } n \in \mathbf{N}\} \tag{15}$$

where $(x_i, y_j, t_n) \in \mathbf{R}^2 \times \mathbf{R}_+$, Δx and Δy are the space steps and Δt is the time step. In order to obtain the SL-scheme let us consider the following approximation

$$-a \cdot \nabla v(x_i, y_j, t_n) = \frac{v(x_i - a_1\Delta t, y_j - a_2\Delta t, t_n) - v(x_i, y_j, t_n)}{\Delta t} + O(\Delta t) \tag{16}$$

We will use the standard notation $v^n_{i,j}$ for an approximation of $v(x_i, y_j, t_n)$, $i, j \in \mathbb{Z}$ and $n \in \mathbf{N}$ and $v^n : \mathbf{R}^2 \to \mathbf{R}$ for its reconstruction, i.e. its extension to any triple (x, y, t_n). Replacing in (13) the term v_t by forward finite differences and the directional derivative by (16), we get

$$\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta t} = \min_{a \in R^2} \left[\frac{v^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t) - v^n(x_i, y_j)}{\Delta t} + H^*(a) \right] \quad (17)$$

which leads to the time explicit scheme

$$v_{i,j}^{n+1} = \min_{a \in R^2} \left[v^n(x_i - a_1 \Delta t, y_j - a_2 \Delta t) + \Delta t H^*(a) \right] \quad (18)$$

Let us set $a = (x-y)/t$ in (14). It is clear from (18) that the SL-schemes has the same structure of the representation formula of the exact solution written for $v_0 = v^n$ and $t = \Delta t$. Naturally, in order to compute the solution one has to compute first the value of v^n on the right-hand side by an interpolation procedure based on the values on the nodes of the lattice L. The last step is to determine $H^*(a)$ so that we can finally compute the minimum for $a \in R^2$. Although this step can be rather expensive (or unfeasible) for a general Hamiltonian, there are many interesting cases which can be solved explicitly. We refer the interested reader to [30] for some applications to isotropic front propagation problems and for the properties of SL schemes for (1). A convergence result for first order SL schemes can be found in [33] whereas some hints for the implementation and the optimal choice of the discretization steps are discussed in [32]. More recently, a convergence result for high order accurate semi-Lagrangian schemes in one dimension has been proved in [35].

3 Approximation of the linear advection-diffusion equation

Let us consider the stationary advection-diffusion problem in one dimension,

$$\begin{cases} -\varepsilon u'' + \beta(x) u' = f(x), & x \in I = (0,1) \\ u(0) = u(1) = 0 \end{cases} \quad (19)$$

As we have done in the Section 2, we can write the first order term $\beta(x)u'$ as the directional derivative of the solution u in the direction of the vectorfield β.

$$\beta(x) \cdot \nabla u = \left. \frac{du(X(t))}{dt} \right|_{t=0} \quad (20)$$

where $X(t)$ is the solution of the Cauchy problem

$$\begin{cases} X'(t) = \beta(X(t)) \\ X(0) = x \end{cases} \quad (21)$$

which defines the characteristic lines of (19).

For $\varepsilon = 0$, this is enough to start building the time discrete scheme (integration along the characteristics). For $\varepsilon \neq 0$, the interpretation in terms of characteristics becomes more complicated because they must be understood in a weak sense. In that case (see e.g. [46]) the characteristics are the solution of the Cauchy problem for the system of stochastic differential equations

$$\begin{cases} \mathrm{d}X(t) = \beta(X(t))\mathrm{d}t + \mathrm{d}w \\ X(0) = x, \end{cases} \tag{22}$$

. The time discretization can be obtained then by applying a one-step method for the integration of (22) chosen in the large collection of methods presented in [43]. The simplest choice is the Euler scheme which will be used here to simplify the presentation. Let us start defining the second order operator

$$L[u](x) = \varepsilon u''(x) - \beta(x)u'(x). \tag{23}$$

and let δ be a positive parameter (virtual time step) which we will use in the definition of the discrete operator $L_\delta[u]$,

$$L_\delta[u](x) := \frac{1}{\delta}\left[\frac{1}{2}u\left(x - \beta\delta + \sqrt{2\delta\varepsilon}\right) + \frac{1}{2}u\left(x - \beta\delta - \sqrt{2\delta\varepsilon}\right) - u(x)\right]. \tag{24}$$

Proposition 3.1 *(Consistency)*
Let the solution $u \in C^2(I)$, then $L_\delta[u]$ converges pointwise to $L[u]$ for δ tending to 0.

Proof.
We will give tha proof for $\beta(x) \equiv \beta$. The proof is similar when $\beta(\cdot)$ is a regular function.

Let us set

$$z = x - \beta\delta \tag{25}$$

$$z^+ = z + \sqrt{2\delta\varepsilon} \tag{26}$$

$$z^- = z - \sqrt{2\delta\varepsilon} \tag{27}$$

By Taylor expansion, we get

$$u(z^+) = u(x - \beta\delta + \sqrt{2\delta\varepsilon}) = u(x) + (-\beta\delta + \sqrt{2\delta\varepsilon})u'(x) +$$

$$+(-\beta\delta + \sqrt{2\delta\varepsilon})^2\frac{u''(x)}{2} + O((-\beta\delta + \sqrt{2\delta\varepsilon})^3) \tag{28}$$

and

$$u(z^-) = u(x - \beta\delta - \sqrt{2\delta\varepsilon}) = u(x) + (-\beta\delta - \sqrt{2\delta\varepsilon})u'(x) +$$

$$+(-\beta\delta - \sqrt{2\delta\varepsilon})^2\frac{u''(x)}{2} + O((-\beta\delta - \sqrt{2\delta\varepsilon})^3) \tag{29}$$

Replacing (29) and (28) in (24) we conclude that

$$L_\delta[u](x) = \varepsilon u''(x) - \beta u'(x) + O(\sqrt{\delta}) = L[u] + O(\sqrt{\delta}).$$

□

We get then the following consistent discretization of (19):

$$u_\delta(x) = \delta f(x) + \frac{u_\delta(x - \beta\delta + \sqrt{2\delta\varepsilon}) + u_\delta(x - \beta\delta - \sqrt{2\delta\varepsilon})}{2}. \tag{30}$$

Note that for $\varepsilon = 0$, we obtain the SL scheme for the advection equation already examined in Section 2.
Let us introduce the space discretization for (30). Let us divide $I = (0,1)$ in sub-intervals of width $k = \Delta x$. We introduce the following representation

$$u_\delta^k = \sum_{j \in Q} u_j \phi_j(x) \tag{31}$$

where u_δ^k denotes the space discretization of $u_\delta(x)$, $Q = \{1, ..., N\}$ is the set of indices for the internal nodes x_j and $\{\phi_j(x)\}_{j\in Q}$ is a basis of bounded functions. The boundary Dirichlet condition is imposed on the nodes x_0 and x_{N+1}.

$$\phi_j(x_j) = 1,\ \phi_j(x_m) = 0\ (m \neq j) \ \text{ and } \ \| \phi_j \|_\infty = 1. \tag{32}$$

Replacing (31) in (30) we obtain the fully discrete scheme corresponding to (19)

$$u_j = \delta f_j + \frac{1}{2} \sum_{m\in\sigma(j)} u_m \phi_m(z_j^+) + \frac{1}{2} \sum_{m\in\sigma(j)} u_m \phi_m(z_j^-) \tag{33}$$

where $z_j^+ = x_j - \beta(x_j)\delta + \sqrt{2\delta\varepsilon}$, $z_j^- = x_j - \beta(x_j)\delta - \sqrt{2\delta\varepsilon}$ and $\sigma(j)$ denotes the stencil of the method.
Note that at this stage we can set $\sigma(j) \equiv Q$, for any j, and sum over all the indices belonging to Q although only those corresponding to the vertices of the cells containing $z_j^\pm$ are contributing to the sum (because of (32)). We will introduce a notation for the real stencil later on in this paragraph.

Our problem is equivalent to the solution of the linear system of algebraic equations in the unknown u_j:

$$U = F + \Phi U \qquad \text{i.e.} \qquad AU = F \tag{34}$$

where

$$U \equiv (u_1, ...u_N)^T, \quad F \equiv (\delta f_1, ..., \delta f_N)^T, \quad A \equiv I - \Phi$$

and

$$\Phi \equiv \{\phi_{ij}\}, \quad \phi_{ij} \equiv \frac{\phi_j(z_i^+) + \phi_j(z_i^-)}{2}.$$

System (34) has a unique solution if and only if $A = (I - \Phi)$ is a non singular matrix. This is equivalent to demand that the scheme (33) satisfies the discrete maximum principle.

Now we want to introduce a subset of indices which will represent the stencil of our method. For any $j \in \{q, \ldots, N-q\}$ we define the stencil

$$\overline{S}(j) = \{j-q, ..., j-1, j, j+1, ..., j+q\} \tag{35}$$

and the set of indices

$$S(j) = \{j-q, ..., j-1, j+1, ..., j+q\} \tag{36}$$

Let the discrete operator connected to the elliptic operator $L[u]$ be defined as

$$L_h[u](x_j) = \alpha_j u_j - \sum_{p \in S(j)} \alpha_p u_p = f_j, \quad j = 1, ..., N. \tag{37}$$

We want to solve the discrete problem

$$L_h[u](x_j) = f_j, \quad j = 1, ..., N \tag{38}$$

coupled with a Dirichlet boundary condition. The following general result holds true.

Lemma 3.2 *(Discrete Maximum Principle)*
Let A be the matrix of the system (34) and let the following assumptions be satisfied:
(i) A has nonnegative elements, i.e. for any $j = 1, ..., N$

$$\alpha_j > 0, \quad \alpha_p \geq 0, \quad for p \in S(j); \tag{39}$$

(ii) A is diagonally dominant,

$$\sum_{p \in S(j)} |\alpha_p| \leq \alpha_j, \quad for\ any\ j = 1, ..., N \tag{40}$$

where the strict inequality holds for at least one index j.
Moreover, let

$$L_h[u](x_j) \leq 0, \quad for\ any\ j = 1, ..., N. \tag{41}$$

Then the value u_j at the internal nodes x_j, $j \in Q$, satisfies

$$u_j \leq \max\{u_0, u_{N+1}\}, \quad \forall\, j = 1, ..., N. \tag{42}$$

Sketch of the proof
Let us assume that $u_j \geq 0$ for all $j = 0, ..., N+1$ (this is not restrictive since we can always add to u_j, $j = 0, ..., N+1$, a suitable positive constant).
Let $1 < j < N$, the condition (41) implies

$$\alpha_j u_j \leq \sum_{p \in S(q)} \alpha_p u_p.$$

From (39) and (40), we have

$$\alpha_j u_j \leq \left(\max_{p \in S(j)} u_p \right) \sum_{p \in S(j)} |\alpha_p| \leq \left(\max_{p \in S(j)} u_p \right) \alpha_j,$$

then

$$u_j \leq \max_{p \in S(j)} u_p,$$

i.e. u_j in the interior of the region is less than or equal to the maximum of its "neighbours". This easily leads to the conclusion. □

The question is: which kind of SL schemes satisfy the Discrete Maximum Principle? A sufficient condition for Lemma 3.2 to hold for the SL-scheme described by (33), (34) is

$$\sigma(j) = \overline{S}(j). \tag{43}$$

which means that the stencil of (33) is centered at the node x_j, $\forall j = 1, ..., N$. Let us write the scheme (33) as:

$$u_j = \delta f_j + \sum_{m \in Q} c_m u_m \tag{44}$$

where

$$c_m = \frac{\phi_m(z_j^-) + \phi_m(z_j^+)}{2}. \tag{45}$$

By (32) we have

$$| c_m | \leq 1, \quad \forall m \in Q.$$

Let define the value $\widehat{\delta}$ such that, for any $\delta \leq \widehat{\delta}$, the condition (43) is verified. Then, for any $\delta \leq \widehat{\delta}$ we can write (33) as

$$L_\delta^h u = \delta f_j, \quad j = 1, ..., N \tag{46}$$

where

$$L_\delta^h u = \alpha_j u_j - \sum_{p \in S(j)} \alpha_p u_p \tag{47}$$

and $\alpha_p = c_p$, $\alpha_j = 1 - c_j$. The following result holds (see [10] for the proof).

Proposition 3.3
Let the coefficients c_m defined in (45) be non negative. Then, the scheme (46) verifies the discrete maximum principle.

Note that we can write the scheme (33) as (46) only for particular choices of the δ, i.e. for the values of δ such that $\delta \leq \widehat{\delta}$.

In fact, to guarantee that the stencil is centered at the node x_j the two points z_j^- and z_j^+ should be close to x_j and this of course depend on β and δ.

An analysis of the SL-scheme and of the recipes for the choice of δ when the local reconstruction is obtained by $\boldsymbol{P}_1$ finite elements can be found in [10].

Note that in our SL-scheme the diffusion term is discretized at the foot of the characteristic instead that at the node x_j. This means that *first* we move up-wind according to the advection term and *then* we discretize the diffusive term. This choice, opposite to the finite difference approximation, allows to deal with the degeneracies of the second order operator. The scheme is also well suited to treat the two limit cases $\varepsilon = 0$ (pure advection) and $\beta = 0$ (pure diffusion). In the first case (30) correspond to the upwind scheme

$$u_\delta(x) = \delta f(x) + u_\delta(x - \beta\delta). \tag{48}$$

In the second case, (30) gives

$$u_\delta(x) = \delta f(x) + \frac{u_\delta(x + \sqrt{2\delta\varepsilon}) + u_\delta(x - \sqrt{2\delta\varepsilon})}{2}. \tag{49}$$

In $\boldsymbol{R}^2$ version of the above SL scheme can be written as:

$$\begin{aligned} u_\delta(x,y) = \frac{1}{4}\Big[& u_\delta(x - a_1\delta - \sqrt{2\delta\varepsilon}, y - a_2\delta) + \\ & + u_\delta(x - a_1\delta + \sqrt{2\delta\varepsilon}, y - a_2\delta) + u_\delta(x - a_1\delta, y - a_2\delta - \sqrt{2\delta\varepsilon}) + \\ & + u_\delta(x - a_1\delta, y - a_2\delta + \sqrt{2\delta\varepsilon})\Big] + \delta f(x,y) \end{aligned} \tag{50}$$

We approximate the solution u at (x,y) with the mean of the values $u(z_1)$, $u(z_2)$, $u(z_3)$ and $u(z_4)$, where:

$$\begin{aligned} z_1 &= (x - a_1\delta - \sqrt{2\delta\varepsilon}, y - a_2\delta) \\ z_2 &= (x - a_1\delta, y - a_2\delta - \sqrt{2\delta\varepsilon}) \\ z_3 &= (x - a_1\delta + \sqrt{2\delta\varepsilon}, y - a_2\delta) \\ z_4 &= (x - a_1\delta, y - a_2\delta + \sqrt{2\delta\varepsilon}). \end{aligned}$$

In order to obtain a fully discrete scheme we construct a (structured or unstructured) grid and we "project" on the grid by local interpolation.

Note that δ is a free parameter which makes the scheme very flexible. In [10] it has been proved that tuning δ and choosing a piecewise linear local interpolation one can reconstruct other well known stabilized FE methods, such as the SUPG method or the Bubble method (see [41], [38], [8] and [9] for more informations about stabilized FE methods for this problem). A comparison of the numerical results on this test problem will be given in the last section.

4 Approximation of MC flow

Let us consider the second order evolutive Hamilton-Jacobi equation which arises in the level set formulation of mean curvature motion, that is:

$$\begin{cases} v_t(x,t) = \operatorname{div}\left(\frac{Dv(x,t)}{|Dv(x,t)|}\right)|Dv(x,t)|, & \text{in } \boldsymbol{R}^2 \times (0,T) \\ v(x,0) = v_0(x), \qquad \text{in } \boldsymbol{R}^2. \end{cases} \tag{51}$$

where v_0 is a representation function for the front Γ_0 at time 0 (i.e. a continuous function which vanishes on Γ_0 and changes sign crossing Γ_0). A large amount of papers have studied this problem from the analytical point of view, the interested reader can find in the lectures [25], [60] and [39] an introduction to the subject and a rather complete list of references. It is worth to note that problem (51) can develop singularities in finite time so that its solutions should be understood in the viscosity sense (see [16], [26]).

The scheme we propose here can be regarded as the discrete version of a representation formula for the viscosity solution of (51). In [63] Soner and Touzi have proved a representation formula for the solution of a large class of geometric second order Hamilton–Jacobi equations, including (51) (see also [64], [65]). The formula is based on a stochastic control interpretation of the MCM which leads to look at it as a target problem for a degenerate diffusion dynamics where the target is the initial configuration of the front. Although their representation formula is suitable for more general situations, we will focus for simplicity on the mean curvature evolution of a curve in $\mathbf{R}^2$. In this special case, Soner-Touzi representation formula has the form

$$v(x,t) = E\{v_0(y(x,t,t))\} \tag{52}$$

where $E(\cdot)$ is the probabilistic expectation, $y(x,t,t)$ is the solution of the Stochastic Initial Value Problem

$$\begin{cases} dy(x,t,s) = \sqrt{2}P(y,t,s)dW(s) \\ y(x,t,0) = x \end{cases} \tag{53}$$

and P is defined by

$$P(y,t,s) = \frac{1}{|Dv(y,t-s)|^2}\begin{pmatrix} v_{x_2}^2(y,t-s) & -v_{x_1}(y,t-s)v_{x_2}(y,t-s) \\ -v_{x_1}(y,t-s)v_{x_2}(y,t-s) & v_{x_2}^2(y,t-s) \end{pmatrix} \tag{54}$$

In this construction, the Wiener process appearing in (53) is 2–dimensional, but the projection matrix P has rank one. Neglecting for simplicity the arguments in (53), (54), we also have:

$$\begin{aligned} \sqrt{2}PdW &= \frac{\sqrt{2}}{|Dv|^2}\begin{pmatrix} v_{x_2}^2 & -v_{x_1}v_{x_2} \\ -v_{x_1}v_{x_2} & v_{x_2}^2 \end{pmatrix}\begin{pmatrix} dW_1 \\ dW_2 \end{pmatrix} \\ &= \frac{\sqrt{2}}{|Dv|}\begin{pmatrix} v_{x_2} \\ -v_{x_1} \end{pmatrix}\left(\frac{v_{x_2}}{|Dv|}dW_1 - \frac{v_{x_1}}{|Dv|}dW_2\right) \\ &= \sigma d\widehat{W} \end{aligned}$$

where

$$d\widehat{W} = \left(\frac{v_{x_2}}{|Dv|}dW_1 - \frac{v_{x_1}}{|Dv|}dW_2\right) \tag{55}$$

is still the differential of a standard Wiener process, and

$$\sigma(y,t,s) = \frac{\sqrt{2}}{|Dv(y,t-s)|}\begin{pmatrix} v_{x_2}(y,t-s) \\ -v_{x_1}(y,t-s) \end{pmatrix}. \tag{56}$$

Replacing (53) by

$$\begin{cases} dy(x,t,s) = \sigma(y,t,s)d\widehat{W}(s) \\ y(x,t,0) = x \end{cases} \tag{57}$$

it is then possibile to reformulate the representation formula in order to reduce the dimension of the Brownian process, replacing (53) by

$$\begin{cases} dy(x,t,s) = \sigma(y,t,s)d\widehat{W}(s) \\ y(x,t,0) = x \end{cases} \tag{58}$$

Also for the MC flow equation, the semi-Lagrangian scheme parallels the representation formula written on a single time step. In fact, writing the Soner–Touzi representation formula between t and $t+\Delta t$ we get

$$v(x,t+\Delta t) = E\{v(y(x,t,\Delta t),t)\} \tag{59}$$

with y defined by (58).

Following [43], we discretize (59), (58) according to the theory of numerical schemes for stochastic ODEs. A simple choice is to discretize equation (58) by a stochastic Euler scheme and to compute the expectation in (59) considering for $\Delta\widehat{W}$ only two determinations, namely $\Delta\widehat{W} = \pm\sqrt{\Delta t}$, each one with probability $1/2$. Setting up a space grid of step Δx, we obtain the scheme (written at the node x_j and at the $(n+1)$–th time step):

$$v_j^{n+1} = \frac{1}{2}\left(I[V^n](x_j + \sigma_j^n\sqrt{\Delta t}) + I[V^n](x_j - \sigma_j^n\sqrt{\Delta t})\right) \tag{60}$$

where σ_j^n is defined by

$$\sigma_j^n = \frac{\sqrt{2}}{|D_j^n|}\begin{pmatrix} D_{2,j}^n \\ -D_{1,j}^n \end{pmatrix} \tag{61}$$

$D_{1,j}^n$, $D_{2,j}^n$ and D_j^n are suitable numerical approximations of $v_{x_1}(x_j,t_n)$, $v_{x_2}(x_j,t_n)$ and $Dv(x_j,t_n)$. In (60), (61) we have replaced the expectation by a weighted average, the computation of the function $v(x,t_n)$ by a numerical interpolation $I[V^n](x)$, the solution of (58) by its Euler approximation and the derivatives in σ by finite differences.

It is very easy to show that the scheme (60) satisfies a discrete maximum principle as long as the reconstruction $I[V](x)$ is monotone (e.g. piecewise linear or bilinear). In fact, the right–hand side of (60) consists of a convex combination of values of $I[V^n]$ so that if

$$\min_j v_j^n \le I[V^n](x) \le \max_j v_j^n,$$

the same bounds also apply to v_j^{n+1}.

Let us examine the consistency of the scheme. Assume now that the vector V^n of the node values is obtained from the exact solution by setting $v_j^n = v(x_j, t_n)$. As usual, the solution v will be assumed to be smooth enough to carry out all the differentiations required, and we will assume in addition that $|Dv| \geq c > 0$. The consistency assumptions on the elementary building blocks of the scheme are:

$$\|I[V^n] - v(t_n)\|_\infty \leq C\Delta x^r, \quad \forall n \in N \tag{62}$$

$$|D_{1,j}^n - v_{x_1}(x_j, t_n)| \leq C\Delta x^q, \quad \forall n \in N \tag{63}$$

$$|D_{2,j}^n - v_{x_2}(x_j, t_n)| \leq C\Delta x^q, \quad \forall n \in N \tag{64}$$

where C is a positive constant.

Let us recall that the stochastic Euler scheme is first–order in terms of weak convergence (see [43]). For our purposes, this means that, for any smooth function g:

$$\frac{1}{2}g(x + \sigma(x,t,0)\sqrt{\Delta t}) + \frac{1}{2}g(x - \sigma(x,t,0)\sqrt{\Delta t}) = E\{g(y(x,t,\Delta t))\} + O(\Delta t^2). \tag{65}$$

Theorem 4.1 *Assume v is a smooth solution of (51), such that $|Dv| \geq c > 0$. Assume moreover that (62), (63), (64), (65) hold. Then, the local truncation error of the scheme (60)–(61) has the form*

$$L_{\Delta x, \Delta t}(x_j, t_n) = O\left(\Delta t + \frac{\Delta x^r}{\Delta t} + \frac{\Delta x^q}{\Delta t^{1/2}}\right). \tag{66}$$

Proof

Let us start observing that

$$\begin{aligned}
I[V^n](x_j \pm \sigma_j^n\sqrt{\Delta t}) &= I[V^n](x_j \pm \sigma_j^n\sqrt{\Delta t}) - v(x_j \pm \sigma_j^n\sqrt{\Delta t}, t_n) + \\
&\quad + v(x_j \pm \sigma_j^n\sqrt{\Delta t}, t_n) - v(x_j \pm \sigma(x_j, t_n, 0)\sqrt{\Delta t}, t_n) + \\
&\quad + v(x_j \pm \sigma(x_j, t_n, 0)\sqrt{\Delta t}, t_n) = \\
&= O(\Delta x^r) + O(\Delta x^q \Delta t^{1/2}) + v(x_j \pm \sigma(x_j, t_n, 0)\sqrt{\Delta t}, t_n)
\end{aligned}$$

where we have estimated the right–hand side row by row, used the fact that v is smooth and $\sigma_j^n - \sigma(x_j, t_n, 0) = O(\Delta x^q)$, along with (62), (63), (64). Therefore we get

$$\begin{aligned}
&\frac{1}{2}\left(I[V^n](x_j + \sigma_j^n\sqrt{\Delta t}) + I[V^n](x_j - \sigma_j^n\sqrt{\Delta t})\right) = \\
&\quad = \frac{1}{2}\left[v(x_j + \sigma(x_j, t_n, 0)\sqrt{\Delta t}, t_n) + v(x_j - \sigma(x_j, t_n, 0)\sqrt{\Delta t}, t_n)\right] + \\
&\quad + O(\Delta x^r) + O(\Delta x^q \Delta t^{1/2}) = \\
&\quad = E\{v(y(x,t,\Delta t), t)\} + O(\Delta t^2) + O(\Delta x^r) + O(\Delta x^q \Delta t^{1/2})
\end{aligned} \tag{67}$$

in which also (65) has been used.

By (67) and (59), we can finally get our estimate of the local truncation error,

$$L_{\Delta x,\Delta t}(x_j,t_n) = \frac{1}{\Delta t}\left(v(x_j,t_{n+1}) - v_j^{n+1}\right). \tag{68}$$

□

The convergence for the SL scheme can be obtained by applying the abstract convergence theorem by Barles-Souganidis [6]. In fact, it has been proved that the scheme is monotone, stable and consistent for monotone interpolations and short time steps, cfr. [12]. Naturally, these restrictions should be removed or at least weakened also because there is some numerical evidence that this can be actually done. This is the goal of our future work for a different proof which should cover large time steps and high–order approximation schemes which have been tested in [27].

5 Numerical tests

In this section, we provide some numerical experiments on the two model problems examined in Section 3 and 4.

5.1 The advection diffusion problem

Let us consider the following boundary value problem: problem:

$$\begin{cases} -\varepsilon u'' + \beta u' = f \quad in \ (0,1) \\ \\ u(0) = u(1) = 0 \end{cases} \tag{69}$$

where we fix $\varepsilon = 0.01$ and $\beta = 1$ and $f(x) = 1$.
The exact solution is:

$$u(x) = -\frac{1}{\beta(e^{\frac{\beta}{\varepsilon}} - 1)}(e^{\frac{\beta x}{\varepsilon}} - 1) + \frac{x}{\beta}.$$

In Figures 1 and 2 we represent the approximate solution using respectively the standard Galerkin method and a stabilized FE method (SUPG) with $k = \Delta x = 0.05$. We can observe spurious oscillations using G (Figure 1) and smooth approximate solutions using SUPG (see Figure 2).
For $k = 0.05$, we have choosen for SL method the δ parameter in order to obtain the same results as SUPG (Figure 3)). The approximation by the SL scheme remains stable also decreasing Δx, Figure 4 shows the solution for $k = 0.025$.

5.2 Shrinking of simple curves

In the following tests the space grid is orthogonal and uniform, with 50 nodes per side of the computational domain. The approximation of the gradient has

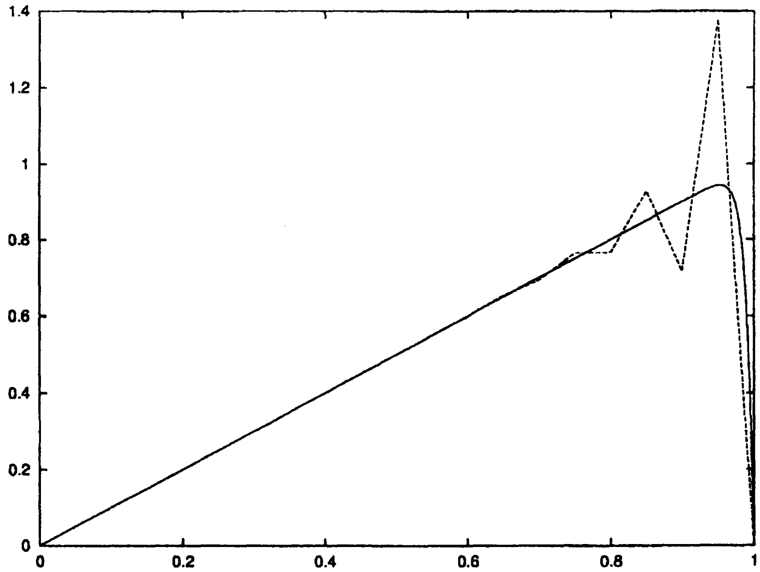

Figure 1: Galerkin approximation, $\Delta x = 0.05$

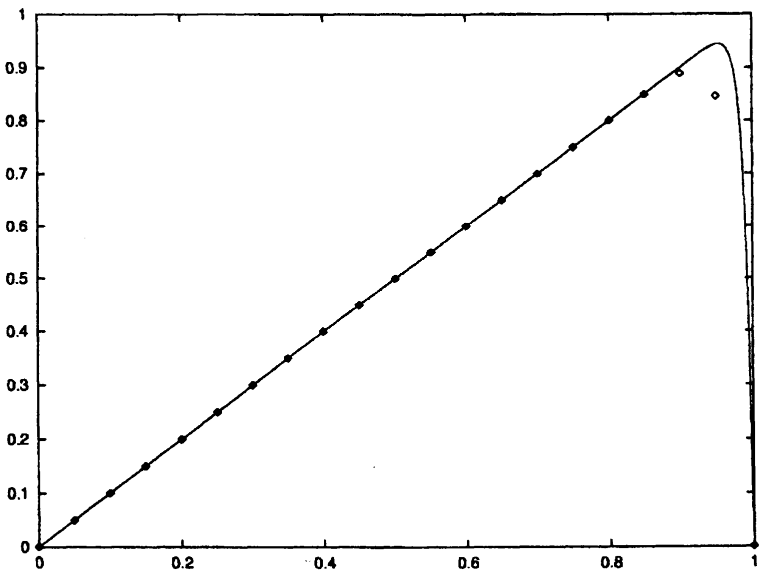

Figure 2: SUPG approximation, $\Delta x = 0.05$

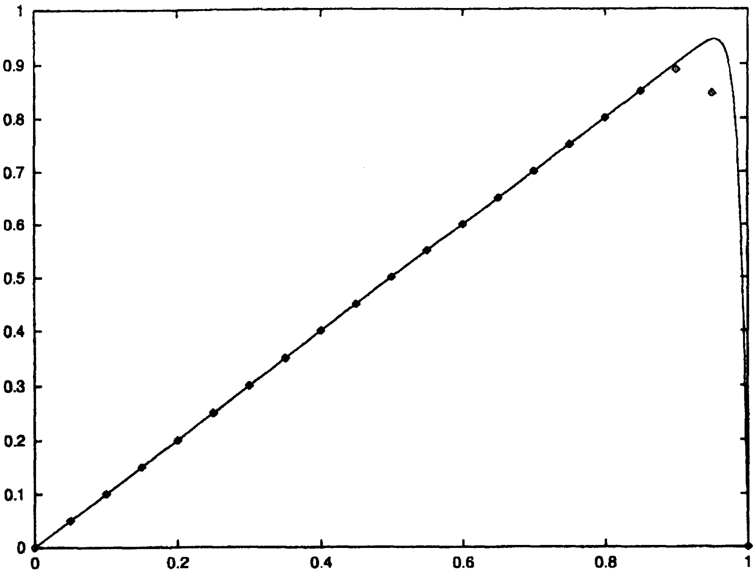

Figure 3: SL approximation, $\Delta x = 0.05$

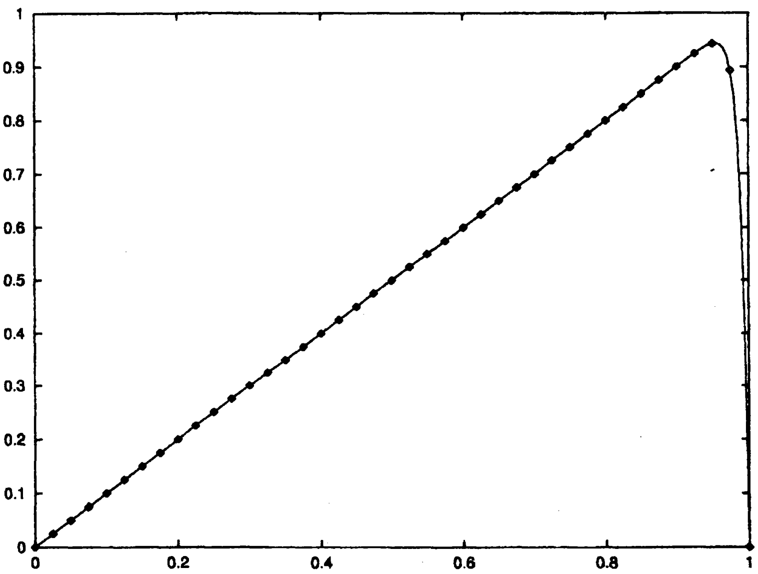

Figure 4: SL approximation, $\Delta x = 0.025$

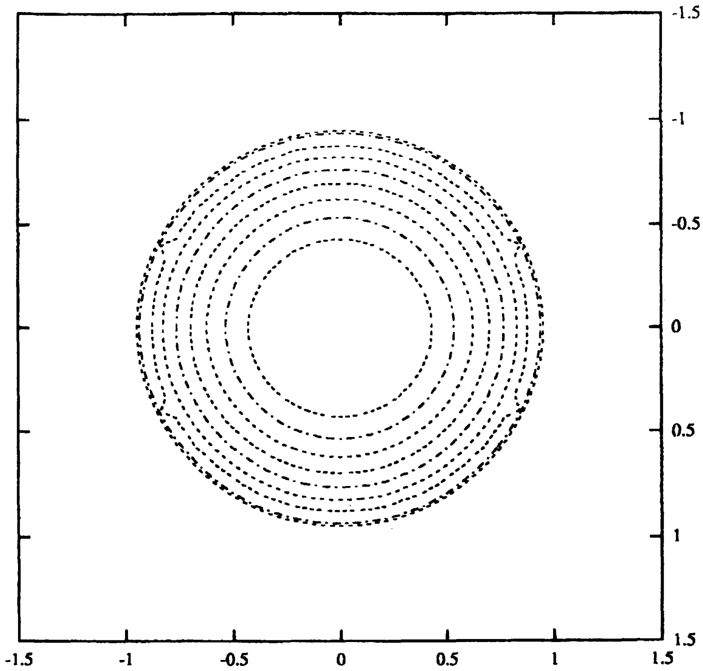

Figure 5: A shrinking circle

been performed by means of centered differences. In all tests we have used a third order reconstructions (Lagrange type polynomials or ENO type reconstructions). Figure 5 shows the shrinking of a circular front evolving by mean curvature, a test example for which the exact solution is known (see e.g. [39]). The level curve of the numerical solution is plotted every 5 iterations up to the extinction.

Figure 6 illustrates a well–known feature of motion by mean curvature: the evolution of any closed simple curve tends towards a shrinking circle which eventually collapses (see [42]). The initial front is a square rotated by 30 degrees and the level curves are plotted every 4 iterations in the first part of the evolution. Note that in this test, the initial front is not aligned with the grid.

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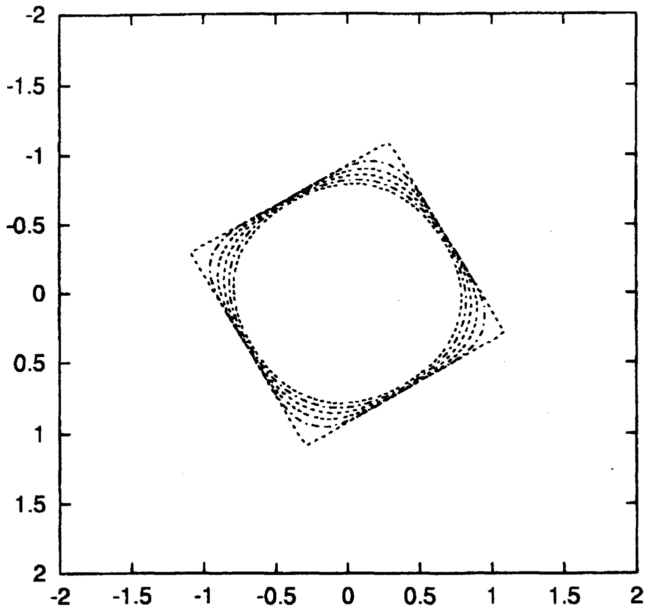

Figure 6: A shrinking square

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