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Note on the small-scale structure of the phase boundaries

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Let $\Omega \in \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. The following energy functional

$$E_\varepsilon(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon},$$

where W is a double well potential with strict minima at ± 1, $\varepsilon > 0$ is a small parameter and $u : \Omega \to \mathbf{R}$ has attracted many researchers in recent years and, to much extent, has been quite well understood mathematically. Under the fixed integral constraint $\int_\Omega u = m$, the critical point of E_ε satisfies the second order semilinear equation

$$-\varepsilon\Delta u + \frac{W'(u)}{\varepsilon} = \lambda, \tag{1}$$

where λ is the Lagrange multiplier. With a minor structure condition on W such as the growth at infinity, the standard minimization in the Sobolev space shows the existence of minimizers for all $\varepsilon > 0$. The family of energy minimizers $\{u_\varepsilon\}_{\varepsilon>0}$ has a subsequence which converges to a bounded variation function u which takes only ± 1 for almost all point on Ω and which has the least area interface among all the competing functions having the same integral value ([3, 4]). The value of $E_\varepsilon(u_\varepsilon)$ converges to $\sigma\mathcal{H}^{n-1}(\partial\{u = 1\} \cap \Omega)$, where $\mathcal{H}^{n-1}$ is the (n-1)-dimensional Hausdorff measure and $\sigma = \int_{-1}^{1} \sqrt{2W(s)}\, ds$. The picture is that the minimizer u_ε takes values very close to ± 1 except for the transition region with thickness of order ε. If we let M_ε be the level set $\{u_\varepsilon = 0\}$, and $\phi : \mathbf{R} \to \mathbf{R}$ be the unique solution to the 2nd order ordinary differential equation $\phi'' = W'(\phi)$ with $\phi(0) = 0$, $\phi(\infty) = 1$ and $\phi(-\infty) = -1$, then one expects that, in a suitable sense, u_ε should look very close to $\phi(d(x)/\varepsilon)$. Here, $d(x)$ is the appropriately signed distance function to the hypersurface M_ε. This is how the comparison function is constructed in [3, 4] in fact. The approach using the monotonicity formula for general critical points of the energy is considered in our previous work [2], which also shows that the profile of the transition region is what one expects. What these arguments show rigorously is that the sequence of Radon measures defined naturally from the energy $\mu_\varepsilon(B) = \int_B \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W}{\varepsilon}$ converges to a Radon measure μ whose support concentrates on some hypersurface and the (n-1)-dimensional density of μ is integral multiple of the constant σ, at least for almost all points on the support of μ. If $\{u_\varepsilon\}$ are global or local minimizers, the limit u has the interface minimizing the (n-1)-dimensional area globally or locally, respectively, so $\partial\{u = 1\} \cap \Omega$ is a real analytic hypersurface outside the closed singular set of codimension at least 7 by the standard results from geometric measure theory. If $\{u_\varepsilon\}$ is only assumed to be a sequence of critical points, then the support of μ (which includes $\partial\{u = 1\} \cap \Omega$ but may not coincide) is again real analytic on a dense open set of the support of μ.

If we assume that W is smooth, then the standard elliptic regularity theory shows that u_ε is also smooth, so the level set of u_ε is a smooth hypersurface for almost all value. This does not necessarily imply that the convergence of the level sets is smooth as ε approaches 0. In [2], we proved the convergence in the Hausdorff distance topology at least. At the point where the limit interface is smooth and density is σ, one may wonder if the convergence of the interface is even better than the Hausdorff distance convergence. Here we describe what we know so far in this regard. For dimension $n = 2$, on the support of μ where the density is σ, we show that the level sets for u_ε, which are smooth curves, converge to the limit curve in C^1 graph topology. That is, the C^1 norms of the level curves represented as graphs over a line are uniformly bounded as $\varepsilon \to 0$. For higher dimensions, we have been unable to prove or disprove the statement for better convergence so far. We believe that one needs to have a better understanding of the rescaled problem $\Delta u = W'(u)$, where it is defined on the domain of the order of $1/\varepsilon$ so it is asymptotically approaching to the entire function on $\mathbf{R}^n$.

Suppose that W is C^3 and has only three critical points. Assume also that $W''(\pm 1) > 0$. Suppose each of $\{u_\varepsilon\}_{1>\varepsilon>0}$ satisfies (1) and that the energy, the Lagrange multiplier and the supremum norm of u_ε are uniformly bounded with respect to ε by a constant C. With this setting, the results from [2] shows that the sequence of measures $\{\mu_\varepsilon\}$ as defined above has a subsequence which converges to a Radon measure μ with the (n-1)-rectifiable support. Suppose that the density at $0 \in \operatorname{supp}\mu$ is σ and that the approximate tangent plane exists. This implies from [2] and Allard's regularity theory [1] that the support of μ is a smooth manifold in a small neighborhood. In the following, we restrict n to be equal to 2. We then claim that there exist a line L through 0 and some constants $\beta > 0$ and $\tilde{C}$ such that, for all sufficiently small $\varepsilon > 0$, the level set $\{u_\varepsilon = 0\} \cap B_{r/2}$ can be represented as a graph over L as $\{(x, f_\varepsilon(x)) \,|\, x \in (a_\varepsilon, b_\varepsilon) \subset L\}$ and $||f_\varepsilon||_{C^{1,\beta}((a_\varepsilon,b_\varepsilon))} \le \tilde{C}$. Thus, the level set converges subsequencially at least in C^1 topology to the limit curve as $\varepsilon \to 0$. The proof uses the analogies with Allard's regularity results in many ways, in that one approximates the level sets by a family of lipschitz functions and utilizes the reverse Hölder-type estimates. What one shows is the following decay estimates. Define

$$G(T,r) = \frac{1}{r^{n+1}} \int_{B_r} |Tx|^2 \varepsilon |\nabla u_\varepsilon|^2,$$

where T is a projection to a (n-1)-dimensional plane identified with the set T. Note that $G(T,r)$ roughly corresponds to the L^2 norm of the graph of the level set normalized appropriately. We then suppose $G(T,r) > (\varepsilon/r)^{2-\delta}$ and $\{u_\varepsilon = 0\} \cap B_{r/4}$ is not empty. One can then prove that there exist $0 < \theta < 1/2$ and $\tilde{T}$ such that $G(\tilde{T}, \theta r) \le \theta G(T,r)$. For example, the Reverse Hölder inequality in this setting corresponds to

$$\frac{1}{r^{n-1}} \int_{B_{r/2}} \left| T\left(\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right)\right|^2 \varepsilon |\nabla u_\varepsilon|^2 + \left(\frac{W}{\varepsilon} - \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\right)^+ \le c(G(T,r) + r\varepsilon^{1/2}).$$

This estimate can be obtained from the equation (1) with a suitable multiplication of function and two integration by parts as well as a uniform L^∞ estimate on the positive part of the function $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{W}{\varepsilon}$. Once the decay estimate is established, then one can iterate the estimates for smaller and smaller scale until the assumption on $G(T,x)$ does not hold for appropriately rescaled setting. If it does not hold, then in case the dimension $n = 2$, one can prove that the level set has a uniform $C^{1,\beta}$ estimate with respect to a suitably chosen line. Note that if $G(T,x) \approx (\varepsilon/r)^2$, this means that the deviation of the level set from the plane T is roughly of order ε/r in L^2 norm, and one cannot expect that $G(\cdot)$ will decay further in a smaller scale unless some suitable quantity is subtracted from $G(T,r)$.

So if $G(T,r) >> (\varepsilon/r)^2$, then we can see the bending of the interface and we can show the decay in a smaller scale, while we do not know what one can say furthermore in case $G(T,r) \approx (\varepsilon/r)^2$. It would be interesting to know precisely what to be said when the interface is almost flat in the L^2 norm in this sense. The rescaled problem $\Delta u = W'(u)$ on the entire $\mathbf{R}^n$ has been studied by C. Gui, L. Ambrosio, X. Cabre and others in connection with the so called De Giorgi's conjecture. They show certain rigidity properties and various Harnack-type estimates and it should be interesting to investigate the relationships between the asymptotic problems arising from (1) and the entire solution.

References

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