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Some regularity results for degenerate elliptic second-order partial differential operators.

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1 Introduction.

In this paper, we study the regularity properties of solutions of two types of degenerate elliptic problems. The first problem concerns with Lipschitz continuities and semi-concavities of solutions of a class of fully nonlinear degenerate elliptic second-order partial differential equations. (Collaboration with I. Capuzzo-Dolcetta.) The second problem concerns with a uniform gradient estimate for solutions of a class of second-order partial differential inequalities. We give typical examples which represent each problems.

Example 1.1. *(Lipschitz continuities, semi-concavities for a fully nonlinear degenerate elliptic PDE.) Let u be a solution of*

$$\lambda u-\Delta_{x'}u+|\nabla_{x''}u|-f(x)=0 \qquad in \quad x=(x',x'')\in\Omega\subset\mathbf{R}^N, \tag{1}$$

where $\Delta_{x'}u=\sum_{i=1}^m\frac{\partial^2u}{\partial x_i^2}$, $|\nabla_{x''}u|=\sqrt{\sum_{i=m+1}^N(\frac{\partial u}{\partial x_i})^2}$, $\lambda\geq 0$ a constant, $N=m+n$ $(m,n>0)$, Ω an open domain in $\mathbf{R}^N$, and $f(x)$ a bounded Lipschitz continuous function in Ω. Then, provided that u is bounded in Ω (i.e. $|u|_{L^\infty(\Omega)}\leq\exists M$, which is true when $\lambda>0$ and Ω is bounded), the following regularity properties hold for u. The directional Holder (including Lipschitz) *continuities in the first m variables : for any $\theta \in (0,1]$, there exists a constant $C > 0$ depending on θ and M such that*

$$|u(x',x'') - u(y',x'')| \leq C|x'-y'|^{\theta} \qquad \forall x', y' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n, \\ \text{such that} \quad (x',x''), \quad (y',x'') \in \quad \Omega. \qquad (2)$$

Moreover, if $\Omega = \mathbf{R}^N$ and $\lambda > 0$, the directional semi-concavities in the first m variables : there exists a constant $C > 0$ depending on $M > 0$ such that

$$|u(x'+h',x'') + u(x'-h',x'') - 2u(x',x'')| \leq C|h'|^2 \\ \forall x', h' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n. \qquad (3)$$

The "full" Holder (including Lipschitz) continuities in the whole variables : for any $\theta \in (0,1]$, there exists a constant $C > 0$ depending on θ and M such that

$$|u(x) - u(y)| \leq C|x-y|^{\theta} \qquad \forall x, y \in \mathbf{R}^N. \qquad (4)$$

Example 1.2. *(Interior gradient estimate for a system of second-order partial differential inequalities.) Consider any functions $u(x_1,x_2,x_3) \in C^2(\Omega)$ which satisfy the following inequalities in $(x_1,x_2,x_3) \in \Omega$.*

$$\begin{aligned} -\frac{\partial^2 u}{\partial x_1 \partial x_3} &\leq C_0, \\ -\frac{\partial^2 u}{\partial x_2^2} &\leq C_0, \\ -\frac{\partial}{\partial x_3}\Big(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial u}{\partial x_3}\Big) &\leq C_0, \end{aligned}$$

where $C_0 > 0$ is a constant. Then, if $\operatorname{supp} u \subset\subset \Omega$, there exists a constant $C > 0$ which depends on the matrix A and C_0 such that

$$|\nabla u| < C.$$

We treat in below general class of operators including Examples 1.1, 1.2. First, for the Lipschitz continuity and semiconcavity for degenerate elliptic operators, we consider the class which satisfies the assumptions (A1)-(A4) stated in below. Let $F(x,u,p,R)$ be a real-valued continuous function defined in $\Gamma = \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N \times \mathbf{S}^N$, where $N = m + n$ $(m,n > 0)$, Ω an open domain in $\mathbf{R}^N$, and $\mathbf{S}^N$ the set of $N \times N$ real valued symmetric matrices. We assume the following conditions for F.

(A1) There exists a constant $\nu > 0$, $0 < \rho < 2$ and C_0 such that

$$F(x,u,p,A) \leq F(x,u,p,B) - \nu Tr(A' - B') + C_0(|p|^\rho + 1) \tag{5}$$

$$\forall x \in \overline{\Omega}, \quad \forall u \in \mathbf{R}, \quad \forall p \in \mathbf{R}^N, \quad \forall A, B \in \mathbf{S}^N, \quad \text{such that}$$

$$A' \geq B' (A', B' \in \mathbf{S}^m), \quad A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B' & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

(A2) There exists a constant $C_1 \geq 0$ such that

$$|F(x,u,p,A) - F(y,u,p,A)| \leq C_1 + w(|x-y|)|x-y|^\tau |p'|^{2+\tau} + \mu(|x-y|)|A'|$$

$$\forall x, y \in \overline{\Omega}, \quad \forall u \in \mathbf{R}, \quad \forall p = (p', p'') \in \mathbf{R}^m \times \mathbf{R}^n,$$

$$\forall A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbf{S}^N \quad (A' \in \mathbf{S}^m),$$

where $0 \leq \tau \leq 1$, $w(\cdot)$, $\mu(\cdot) : [0, \infty) \to \mathbf{R}^+ \cap \{0\}$, such that

$$\lim_{\sigma \downarrow 0} w(\sigma) = 0, \quad \lim_{\sigma \downarrow 0} \mu(\sigma) = 0, \quad \int_{+0} \frac{\mu(\sigma)}{\sigma} d\sigma < \infty.$$

(A3) F is the Hamilton-Jacobi-Bellman operator, i.e.

$$F(x,u,\nabla u, \nabla^2 u) = \sup_{\alpha \in A} \{ -\sum_{ij=1}^N a_{ij}^\alpha(x) \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^N b_i^\alpha(x) \frac{\partial u}{\partial x_i} + c^\alpha(x) u - f^\alpha(x) \} x \in \Omega, \tag{6}$$

where A a given set (controls), $(a_{ij}^\alpha(x)) \in \mathbf{S}^N$ $(\alpha \in A)$ non-negative matrices such that there exist $N \times k$ matrices Σ^α $(\alpha \in A)$

$$(a_{ij}^\alpha(x)) = \Sigma^\alpha(x)^T \Sigma^\alpha(x) \qquad \forall x \in \Omega,$$

$b^\alpha(x) = (b_i^\alpha(x)) \in \mathbf{R}^N$, $c^\alpha(x) \in \mathbf{R}$, such that (a_{ij}^α), b^α, c^α, $f^\alpha \in W^{2,\infty}(\mathbf{R}^N)$ for $\forall \alpha \in A$, and there exists a constant $C_2 > 0$ such that

$$\sup_{\alpha \in A} |a_{ij}^\alpha|_{W^{2,\infty}(\mathbf{R}^N)}, |b_i^\alpha|_{W^{2,\infty}(\mathbf{R}^N)}, |c^\alpha|_{W^{2,\infty}(\mathbf{R}^N)}, |f^\alpha|_{W^{2,\infty}(\mathbf{R}^N)} < C_2 \qquad \forall \alpha \in A. \tag{7}$$

(A4) F is directionally coercive in the following sense, i.e.

$$\lim_{|p''|\to\infty} F(x,u,p,A) = \infty \qquad \text{for} \quad p = (p',p''), \quad p' \in \mathbf{R}^m, \quad p'' \in \mathbf{R}^n$$
$$\text{uniformly in} \quad (x,u,p',A) \in \Omega \times \mathbf{R} \times \mathbf{R}^m \times \mathbf{S}^N. \qquad (8)$$

In some case, we assume the following stronger condition than (A1).

(A1)' There exists a constant $\nu' > 0$, $0 < \rho < 2$ and C_0 such that

$$F(x,u,p,A) \le F(x,u,p,B) - \nu Tr(A' - B') + C_0(|p|^\rho + 1) \qquad (9)$$
$$\forall x \in \overline{\Omega}, \qquad \forall u \in \mathbf{R}, \quad \forall p \in \mathbf{R}^N, \quad \forall A, B \in \mathbf{S}^N,$$
such that $\operatorname{Tr}(A' - B') \ge 0 (A', B' \in \mathbf{S}^m)$,
$$A = \begin{pmatrix} A' & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B' & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Under the above assumtions, we consider

$$F(x,u,\nabla u,\nabla^2 u) = 0 \qquad \text{in} \quad \Omega, \qquad (10)$$

and assume the boundedness of u, i.e. there exists a constant $M > 0$ such that

$$\sup_{x\in\Omega} |u| \le M. \qquad (11)$$

In [5], some sufficient confitions for (11) is given. Now, we state the main results of this paper. Remark that the operator in Example 1.1 satisfies (A1)-(A4).

Proposition 1.1 *Let F satisfy (A1) and (A2), u be a solution of (10) and assume that (11) holds. Then, u satisfies the followings.*
(i) Let $\Omega = \mathbf{R}^N$. For any $\theta \in (0,1]$, there exisis a constant $C > 0$ which depends on θ, F and M, such that

$$|u(x',x'') - u(y',x'')| \le C|x' - y'|^\theta \qquad \forall x', y' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n. \qquad (12)$$

(ii) Let Ω be a bounded open domain in $\mathbf{R}^N$, and assume that (A1)' holds in place of (A1). For any $\theta \in (0,1]$, for any $\delta > 0$, and for $\Omega_\delta = \{x \in$

$\Omega | dist(x, \partial\Omega) \geq \delta\}$, there exists a constant $C_\delta > 0$ which depends on θ, δ, F and $M > 0$,

$$|u(x', x'') - u(y', x'')| \leq C_\delta |x' - y'|^\theta \qquad \forall x', y' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n,$$
$$\text{such that} \quad (x', x''), \quad (y', x'') \in \quad \Omega_\delta. \qquad (13)$$

(iii) Let Ω be an open domain in $\mathbf{R}^N$, and assume that there exists $\theta \in (0, 1]$ and a constant $C_\theta > 0$ such that

$$|u(x', x'') - u(y', x'')| \leq C_\theta |x' - y'|^\theta \qquad \forall x', y' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n,$$
$$\text{such that} \quad (x', x''), \quad (y', x'') \in \quad \partial\Omega. \qquad (14)$$

Then, there exists a constant $C > 0$ which depends on F, $M > 0$ and C_θ, such that,

$$|u(x', x'') - u(y', x'')| \leq C |x' - y'|^\theta \qquad \forall x', y' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n,$$
$$\text{such that} \quad (x', x''), \quad (y', x'') \in \quad \overline{\Omega}. \qquad (15)$$

Theorem 1.2 *Let $\Omega = \mathbf{R}^N$, F satisfy (A1), (A2), (A3). Let $u = u(x', x'')$ be a solution of (10), continuous in $x'' \in \mathbf{R}^n$, and assume that (11) holds. Assume also that there exists a large enough number $\mu \geq 0$ such that*

$$c^\alpha(x) \geq \mu \qquad \forall x \in \mathbf{R}^N, \quad \alpha \in A. \qquad (16)$$

Then, u satisfies the following.

(i) There exists a constant $C > 0$ which depends on F, M and μ, such that

$$|u(x' + h', x'') + u(x' - h', x'') - 2u(x', x'')| \leq C |h'|^2$$
$$\forall x', h' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n. \qquad (17)$$

(ii) Let F satisfy (A4). For any $\theta \in (0, 1]$, there exists a constant $C > 0$ which depends on θ, F, M and μ, such that

$$|u(x) - u(y)| \leq C |x - y|^\theta \qquad \forall x, y \in \mathbf{R}^N. \qquad (18)$$

Remark 1.2. The number $\mu \geq 0$ in Theorem 1.2 depends on M in (4) and C_2 in (7). In some special cases, Theorem 1.2 holds with $\mu = 0$.

Next, as for the uniform interior gradient estimate for solutions of general class of systems of second-order partial differential inequalities, we state the following results.

Theorem 1.3 *Let Ω be a domain in $\mathbf{R}^N$, let $A = (A_{ij})_{1 \leq i,j \leq N}$, where $A_{ij}(x) \in L^\infty(\Omega)$ $(1 \leq i, j \leq N)$ real valued functions defined in $x \in \Omega$ which satisfy the following conditions.*

$$\sup_{x \in \Omega} |A_{ij}(x)| \leq C_1 \qquad 1 \leq i, j \leq N, \tag{19}$$

$$|detA|^{-1} \leq C_2, \tag{20}$$

where C_1, $C_2 > 0$ are constants. Suppose that a real valued function $u(x) \in C^2(\Omega)$ such that $suppu \subset\subset \Omega$ satisfies the following inequalities

$$-\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{N} A_{ij} \frac{\partial u}{\partial x_j}\right)(x) \leq C_3 \quad in \quad x \in \Omega, \quad for \quad 1 \leq i \leq N, \tag{21}$$

where $C_3 > 0$ is a constant. Then, there exists a constant $\overline{C} > 0$ depending on the matrix (A_{ij}) and the constant $C_3 > 0$ such that

$$\sup_{x \in \Omega} |\nabla u(x)| \leq \overline{C}. \tag{22}$$

Theorem 1.4 *Let Ω be a N dimensional torus $T^N = \mathbf{R}^N/\mathbf{Z}^\mathbf{N} = [0,1]^N$, let $A = (A_{ij})_{1 \leq i,j \leq N}$, where $A_{ij} = A_{ij}(x) \in L^\infty(\Omega)$ $(1 \leq i, j \leq N)$ real valued periodic functions defined in $x \in \Omega$ which satisfy the following conditions.*

$$\sup_{x \in \Omega} |A_{ij}(x)| \leq C_1 \qquad 1 \leq i, j \leq N, \tag{23}$$

$$|detA|^{-1} \leq C_2, \tag{24}$$

where C_1, $C_2 > 0$ *are constants. Suppose that a real valued function* $u(x) \in C^2(\Omega)$ *is periodic and satisfies the following inequalities*

$$-\frac{\partial}{\partial x_i}(\sum_{j=1}^{N} A_{ij}\frac{\partial u}{\partial x_j})(x) \leq C_3 \quad in \quad x \in \Omega, \quad for \quad 1 \leq i \leq N, \tag{25}$$

where $C_3 > 0$ *is a constant. Then, there exists a constant* $\overline{C} > 0$ *depending on the matrix* (A_{ij}) *and the constant* $C_3 > 0$ *such that*

$$\sup_{x\in\Omega} |\nabla u(x)| \leq \overline{C}. \tag{26}$$

We remark that Example 1.2 is a special case of Theorem 1.3.
Some regularity results for degenerate elliptic second-order P.D.E.s are known in works of N.V. Krylov [11], P.-L. Lions [12]. See M. Arisawa [1], [2], [4], I. Capuzzo-Dolcetta and A. Curti [8], too. Different from the uniformly elliptic second-order P.D.E.s, as in D. Gilbarg and N.S. Trudinger [9] and X. Cabre and L. Caffarelli [7], there seems to be no general theory to treat regularities of degenerate elliptic cases. In below, we give the proof of Proposition 1.1 for the class of operators satisfying (A1)-(A4) (occassionally (A1)'). For the proofs of Thorems 1.2, 1.3 and 1.4, and the other detailed results related to this paper, we refer the readers to M. Arisawa and I. Capuzzo-Dolcetta [3], and M. Arisawa [1]. We use the comparison argument of viscosity solutions introduced by H.Ishii and P.-L. Lions in [10], to study the regularities in Proposition 1.1 and Theorem 1.2. (See also [6].)

2 Directional Holder (Lipschitz) continuities.

The proof of Proposition 1.1. is given in this section.

Proof of Proposition 1.1.
(i) We prove the directional Lipschitz continuities of u (i.e.$\theta = 1$ in (12)). In (A2), we may assume that $\mu(r) \geq r$ for any $r \geq 0$. Put $l(r) = \int_0^r ds \int_0^s \frac{\mu(\sigma)}{\sigma} d\sigma$ for $r \geq 0$. Since $l'(r) = \int_0^r \frac{\mu(\sigma)}{\sigma} d\sigma$ is monotone increasing in $r \geq 0$, there exists $r_0 > 0$ such that $l'(r_0) = \frac{1}{2}$. Let $K > 0$ be an arbitrarily fixed number

to be determined later. Remark that for $r_0 > 0$, since $rl'(r) \geq l(r)$,

$$\frac{K|x'|}{2} \geq l(K|x'|) \qquad \forall x' \in \mathbf{R}^m \quad \text{such that} \quad K|x'| \leq r_0.$$

Now, let $r = r_0/K$ and choose $C' > 0$ so that $2M \leq C' r_0/2$. It is clear that

$$u(x', x'') - u(y', x'') \leq \frac{C'}{2} K|x' - y'| \tag{27}$$

$$\text{for} \quad \forall x', y' \in \mathbf{R}^m, x'' \in \mathbf{R}^n, \quad \text{such that} \quad |x' - y'| \geq r.$$

Therefore, if there exists $K > 0$ such that

$$u(x', x'') - u(y', x'') \leq C' K|x' - y'| - C' l(K|x' - y'|) \tag{28}$$

$$\forall x', y' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n, \quad \text{such that} \quad |x' - y'| \leq r,$$

the proof ends, since we may take $C = C'K$ in (12). We prove (28) by a contradiction argument, and thus assume that for any $K > 0$,

$$\sup_{x', y' \in \mathbf{R}^m, x'' \in \mathbf{R}^n, |x' - y'| \leq r} \{u(x', x'') - u(y', x'') - C' K|x' - y'| \tag{29}$$

$$- C' l(K|x' - y'|)\} > 0,$$

and shall look for a contradiction. For $\alpha > 0$, $\beta > 0$, put

$$\begin{aligned} \Phi_{\alpha\beta}(x', x'', y', y'') = {} & u(x', x'') - u(y', y'') - C' K|x' - y'| + C' l(K|x' - y'|) \\ & - \alpha|x'' - y''|^2 - \beta(|x|^2 + |y|^2) \end{aligned} \tag{30}$$

$$\text{in} \quad \forall x = (x', x''), y = (y', y'') \in \mathbf{R}^N \quad \text{such that} \quad |x' - y'| < r.$$

Let $(x'_{\alpha\beta}, x''_{\alpha\beta})$, $(y'_{\alpha\beta}, y''_{\alpha\beta})$ be its maximum point. From (27), remark that $|x'_{\alpha\beta} - y'_{\alpha\beta}| < r$, $x'_{\alpha\beta} \neq y'_{\alpha\beta}$, and that for $\beta > 0$ fixed small enough, from (29)

$$\alpha|x''_{\alpha\beta} - y''_{\alpha\beta}|^2 = 0 \qquad \text{as} \quad \alpha \to \infty,$$

$$u(x'_{\alpha\beta}, x''_{\alpha\beta}) - u(y'_{\alpha\beta}, y''_{\alpha\beta}) - C' K|x'_{\alpha\beta} - y'_{\alpha\beta}| + C' l(K|x'_{\alpha\beta} - y'_{\alpha\beta}|) > 0.$$

Put

$$\varphi(x', x'', y', y'') = C' K|x' - y'| - C' l(K|x' - y'|) + \alpha|x'' - y''|^2 + \beta(|x|^2 + |y|^2),$$

and calculate at $x_{\alpha\beta} = (x'_{\alpha\beta}, x''_{\alpha\beta})$, $y_{\alpha\beta} = (y'_{\alpha\beta}, y''_{\alpha\beta})$ ($x'_{\alpha\beta} \neq y'_{\alpha\beta}$),

$$p' = \nabla_{x'}\varphi(x'_{\alpha\beta}, x''_{\alpha\beta}, y'_{\alpha\beta}, y''_{\alpha\beta})$$
$$= C'K\frac{x'_{\alpha\beta} - y'_{\alpha\beta}}{|x'_{\alpha\beta} - y'_{\alpha\beta}|} - C'Kl'(K|x'_{\alpha\beta} - y'_{\alpha\beta}|)\frac{x'_{\alpha\beta} - y'_{\alpha\beta}}{|x'_{\alpha\beta} - y'_{\alpha\beta}|} + 2\beta x'_{\alpha\beta},$$
$$p'' = \nabla_{x''}\varphi(x'_{\alpha\beta}, x''_{\alpha\beta}, y'_{\alpha\beta}, y''_{\alpha\beta}) = 2\alpha(x''_{\alpha\beta} - y''_{\alpha\beta}) + 2\beta x''_{\alpha\beta},$$
$$q' = \nabla_{y'}\varphi(x'_{\alpha\beta}, x''_{\alpha\beta}, y'_{\alpha\beta}, y''_{\alpha\beta})$$
$$= C'K\frac{x'_{\alpha\beta} - y'_{\alpha\beta}}{|x'_{\alpha\beta} - y'_{\alpha\beta}|} - C'Kl'(K|x'_{\alpha\beta} - y'_{\alpha\beta}|)\frac{x'_{\alpha\beta} - y'_{\alpha\beta}}{|x'_{\alpha\beta} - y'_{\alpha\beta}|} - 2\beta y'_{\alpha\beta},$$
$$q'' = \nabla_{y''}\varphi(x'_{\alpha\beta}, x''_{\alpha\beta}, y'_{\alpha\beta}, y''_{\alpha\beta}) = 2\alpha(x''_{\alpha\beta} - y''_{\alpha\beta}) - 2\beta y''_{\alpha\beta},$$

$p = (p', p'')$, $q = (q', q'') \in \mathbf{R}^N$,

$$B' = \nabla^2_{x'x'}\varphi = \frac{C'K}{|x'_{\alpha\beta} - y'_{\alpha\beta}|}I - \frac{C'K}{|x'_{\alpha\beta} - y'_{\alpha\beta}|}\frac{(x'_{\alpha\beta} - y'_{\alpha\beta}) \otimes (x'_{\alpha\beta} - y'_{\alpha\beta})}{|x'_{\alpha\beta} - y'_{\alpha\beta}|^2}$$
$$-\frac{C'Kl'(K|x'_{\alpha\beta} - y'_{\alpha\beta}|)}{|x'_{\alpha\beta} - y'_{\alpha\beta}|}\{I - \frac{(x'_{\alpha\beta} - y'_{\alpha\beta}) \otimes (x'_{\alpha\beta} - y'_{\alpha\beta})}{|x'_{\alpha\beta} - y'_{\alpha\beta}|^{-2}}\}$$
$$-C'K^2l''(K|x'_{\alpha\beta} - y'_{\alpha\beta}|)\frac{(x'_{\alpha\beta} - y'_{\alpha\beta}) \otimes (x'_{\alpha\beta} - y'_{\alpha\beta})}{|x'_{\alpha\beta} - y'_{\alpha\beta}|^{-2}} + 2\beta I \in \mathbf{S}^m. \tag{31}$$

Set

$$B = \begin{pmatrix} B' & O \\ O & 2\alpha I + 2\beta I \end{pmatrix} \in \mathbf{S}^N.$$

From the theory of viscosity solutions, there exist X, $Y \in \mathbf{S}^N$ such that

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \begin{pmatrix} B & -B \\ -B & B \end{pmatrix}, \tag{32}$$

and that

$$F(x_{\alpha\beta}, p, X) \leq 0, \quad F(y_{\alpha\beta}, q, -Y) \geq 0. \tag{33}$$

By writing X, Y as follows

$$X = \begin{pmatrix} X' & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} Y' & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}, \quad X', Y' \in \mathbf{S}^m,$$

we get from (32),

$$\begin{pmatrix} X' & O \\ O & Y' \end{pmatrix} \leq \begin{pmatrix} B' & -B' \\ -B' & B' \end{pmatrix}, \tag{34}$$

$$X' + Y' \leq O, \quad X' + Y' \leq 2B' + 4\beta I.$$

Thus, from (A1),

$$F(x_{\alpha\beta}, q, -Y) - F(y_{\alpha\beta}, p, X) \leq \nu \mathrm{Tr}(X' + Y') + C_0 |p - q|^\rho, \tag{35}$$

and combining this with (33), we have

$$\begin{aligned} 0 &\geq F(x_{\alpha\beta}, p, X) - F(x_{\alpha\beta}, q, -Y) + F(x_{\alpha\beta}, q, -Y) - F(y_{\alpha\beta}, q, -Y) \\ &\geq -\nu \mathrm{Tr}(X' + Y') - C_0 |p - q|^\rho - w(|x_{\alpha\beta} - y_{\alpha\beta}|)|x_{\alpha\beta} - y_{\alpha\beta}|^\tau |p'|^{2+\tau} \\ &\quad -C_1 - \mu(|x_{\alpha\beta} - y_{\alpha\beta}|)||Y'||. \end{aligned} \tag{36}$$

Since there exists a constant $L > 0$ depending only on m such that

$$\begin{aligned} ||Y'|| &\leq L\{|\mathrm{Tr}(X' + Y')| + ||B' + \beta I||^{\frac{1}{2}} |\mathrm{Tr}(X' + Y')|^{\frac{1}{2}}\} \\ &\leq L\{|\mathrm{Tr}(X' + Y')| + ||B' + \beta I||\}. \end{aligned}$$

Thus, from (36),

$$\begin{aligned} 0 \geq &\quad -\nu \mathrm{Tr}(X' + Y') - C_0 |p - q|^\rho - w(|x_{\alpha\beta} - y_{\alpha\beta}|)|x_{\alpha\beta} - y_{\alpha\beta}|^\tau |p'|^{2+\tau} \\ & -C_1 - \mu(|x'_{\alpha\beta} - y'_{\alpha\beta}|)(||B' + \beta I|| + |\mathrm{Tr}(X' + Y')|). \end{aligned} \tag{37}$$

Remark that

$$|p'| \leq C'K(1 - l'(K|x'_{\alpha\beta} - y'_{\alpha\beta}|)) + 2\beta |x'_{\alpha\beta}|,$$

and that there exists a constant $\overline{C} > 0$ such that

$$|B'| \leq \overline{C} \frac{C'K}{|x'_{\alpha\beta} - y'_{\alpha\beta}|}(1 - l'(K|x'_{\alpha\beta} - y'_{\alpha\beta}|) + \mu(K|x'_{\alpha\beta} - y'_{\alpha\beta}|))$$

$$\leq \overline{C} \frac{C'K}{|x'_{\alpha\beta} - y'_{\alpha\beta}|}(1 + \mu(K|x'_{\alpha\beta} - y'_{\alpha\beta}|)).$$

By putting the above into (37), and by letting $\beta \to 0$, we have

$$\frac{\nu}{2}|\mathrm{Tr}(X' + Y')| + \frac{\nu}{2} \frac{C'K^2 \mu(K|x'_{\alpha\beta} - y'_{\alpha\beta}|)}{K|x'_{\alpha\beta} - y'_{\alpha\beta}|} \leq \mu(|x'_{\alpha\beta} - y'_{\alpha\beta}|)|\mathrm{Tr}(X' + Y')|$$

$$+C_1 + \frac{\overline{C}C'K^2\mu(|x'_{\alpha\beta} - y'_{\alpha\beta}|)}{K|x'_{\alpha\beta} - y'_{\alpha\beta}|}(1 + \mu(K|x'_{\alpha\beta} - y'_{\alpha\beta}|))$$
$$+(C'K)^{2+\tau}w(|x'_{\alpha\beta} - y'_{\alpha\beta}|)|x'_{\alpha\beta} - y'_{\alpha\beta}|^{\tau}. \qquad (38)$$

For $P = \frac{(x'_{\alpha\beta}-y'_{\alpha\beta})\otimes(|x'_{\alpha\beta}-y'_{\alpha\beta}|)}{|x'_{\alpha\beta}-y'_{\alpha\beta}|^2}$,

$$\begin{aligned}\mathrm{Tr}(X'+Y') \leq \mathrm{Tr}P(X'+Y') &\leq -C'K^2l''(K|x'_{\alpha\beta} - y'_{\alpha\beta}|)\\ &= -\frac{C'K\mu(K|x'_{\alpha\beta} - y'_{\alpha\beta}|)}{|x'_{\alpha\beta} - y'_{\alpha\beta}|^2},\end{aligned}$$

and taking account this in (38), we get a contradiction for the choice of large enough $K > 0$. Therefore, we proved the directional Lipschitz continuities of u.

(ii) Here, we prove the claim for the case of directional Holder continuities (i.e. $\theta \in (0,1)$ in (13)). The case of directional Lipschitz continuities can be treated similarly as in (i). Let $\delta > 0$ be fixed. Take a point $z \in \partial\Omega_{2\delta}$. We shall prove the existenxe of $C' > 0$ and $L > 0$ such that

$$\begin{aligned}&u(x', x'') - u(y', x'') \leq C'|x' - y'|^{\theta} + L|x - z|^2\\ &\quad \text{for } \forall x', y' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n, \quad |x' - y'| \leq \delta, \quad |x - z| \leq \delta,\end{aligned} \qquad (39)$$

for by putting $z = x$, (39) leads

$$\begin{aligned}&u(x', x'') - u(y', x'') \leq C|x' - y'|^{\theta}\\ &\qquad \text{for } \forall x = (x', x''), \quad y = (y', x'') \in U_\delta(z).\end{aligned} \qquad (40)$$

Since we can take a finite number of points $z_i \in \partial\Omega_{2\delta}$ $(1 \leq i \leq k)$, such that $\cap_{i=1}^k U_\delta(z_i) \supset \partial\Omega_{2\delta}$, (40) for each z_i $(1 \leq i \leq k)$ leads (13) in Ω_δ. Take $C' > 0$ and $L > 0$ so that

$$C'\delta^{\theta} \geq 2M, \quad L\delta^2 \geq 2M, \quad C'\theta(1-\theta) > L. \qquad (41)$$

We use the argument by contradiction, and thus assume the existence of two points $x = (x', x'')$, $y = (y', x'')$ such that $|x' - y'| \leq \delta$, $|x - z| \leq \delta$,

$$u(x', x'') - u(y', x'') > C'|x' - y'|^{\theta} + L|x - z|^2. \qquad (42)$$

From the choices of $C' > 0$ and $L > 0$, clearly

$$|x' - y'| < \delta, \quad |x - z| < \delta.$$

Put

$$\Delta_\delta = \{(x, y) \in \mathbf{R}^N \times \mathbf{R}^N \,||x' - y'| \le \delta, \quad |x - z| \le \delta\},$$

and let $(\overline{x}', \overline{x}'')$, $(\overline{y}', \overline{x}'')$ be a maximum point of

$$u(x', x'') - u(y', x'') - C'|x' - y'|^\theta - L|x - z|^2$$
$$\text{in} \quad \forall x', y' \in \mathbf{R}^m, \quad x'' \in \mathbf{R}^n, \quad \text{such that} \quad |x' - y'|, \quad |x - z| \le \delta, \tag{43}$$

where the maximum value is positive. For $\alpha > 0$, put

$$\Phi(x', x'', y', y'') = u(x', x'') - u(y', y'') - C'|x' - y'|^\theta - L|x - z|^2$$
$$-\alpha|x'' - y''|^2 \quad \text{in} \quad \overline{\Delta}_\delta, \tag{44}$$

and let (x'_α, x''_α), (y'_α, y''_α) be its maximum point in $\overline{\Delta}_\delta$. The usual argument leads: there exist $\overline{x}'$, $\overline{y}' \in \mathbf{R}^m$, $\overline{y}'' \in \mathbf{R}^n$ such that

$$x'_\alpha \to \overline{x}', \quad y'_\alpha \to \overline{x}', \quad \alpha|x''_\alpha - y''_\alpha|^2 \to 0, \quad x''_\alpha, y''_\alpha \to \overline{x}'', \qquad \text{as} \quad \alpha \to \infty.$$

Put

$$\varphi(x', x'', y', y'') = C'|x' - y'|^\theta + L|x - z|^2 + \alpha|x'' - y''|^2.$$

Calculate at $x_\alpha = (x'_\alpha, x''_\alpha)$, $y_\alpha = (y'_\alpha, y''_\alpha)$, the following

$$p = \nabla_x\varphi(x'_\alpha, x''_\alpha), \quad q = \nabla_y\varphi(y'_\alpha, y''_\alpha),$$

$$\begin{array}{rcl}
\nabla^2_{x'x'}\varphi &=& C'\theta(\theta-2)\frac{(x'_\alpha - y'_\alpha)\otimes(x'_\alpha - y'_\alpha)}{|x'_\alpha - y'_\alpha|^{4-\theta}} + C'\theta|x'_\alpha - y'_\alpha|^{\theta-2}I_m + 2LI_m, \\
\nabla^2_{x'_\alpha x''_\alpha}\varphi &=& O, \\
\nabla^2_{x''_\alpha x''_\alpha}\varphi &=& 2LI_n + 2\alpha I_n,
\end{array}$$

and set

$$B' = C'\theta(\theta - 2)\frac{(x'_\alpha - y'_\alpha)\otimes(x'_\alpha - y'_\alpha)}{|x'_\alpha - y'_\alpha|^{4-\theta}} + C'\theta|x'_\alpha - y'_\alpha|^{\theta-2}I_m.$$

From the theory of viscosity solutions, there exist $X, Y \in \mathbf{S}^N$ such that

$$\begin{pmatrix} X & O \\ O & Y \end{pmatrix} \le \begin{pmatrix} B' + 2LI_m & O & -B' & O \\ O & 2LI_n + 2\alpha I_n & O & -2\alpha I_n \\ -B' & O & B' & O \\ O & -2\alpha I_n & O & 2\alpha I_n \end{pmatrix}, \tag{45}$$

and that

$$F(x_\alpha, p, X) \leq 0, \quad F(y_\alpha, q, -Y) \geq 0. \tag{46}$$

Writing

$$X = \begin{pmatrix} X' & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad Y = \begin{pmatrix} Y' & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}, \qquad X', Y' \in \mathbf{S}^m,$$

from (45),

$$\begin{pmatrix} X' & O \\ O & Y' \end{pmatrix} \leq \begin{pmatrix} B' + 2LI_m & -B' \\ -B' & B' \end{pmatrix}, \tag{47}$$

and,

$$X' + Y' - 2LI_m \leq O, \quad X' + Y' - 2LI_m \leq 2B'.$$

Hence,

$$\mathrm{Tr}(X' + Y' - 2LI_m) \leq C'\theta(\theta - 1)|x'_\alpha - y'_\alpha|^{\theta-2},$$

and from (41),

$$\mathrm{Tr}(X' + Y') \leq 2Lm + C'\theta(\theta-1)|x'_\alpha - y'_\alpha|^{\theta-2} \leq r^{-2}(2Lr^2m + C'\theta(\theta-1)r^\theta). \tag{48}$$

From (46), (A1), (A1)', and (A2),

$$\begin{aligned} 0 &\geq F(x_\alpha, p, X) - F(y_\alpha, q, -Y) \\ &= F(x_\alpha, p, X) - F(x_\alpha, p, -Y) + F(x_\alpha, p, -Y) - F(y_\alpha, q, -Y) \end{aligned}$$

$$\geq -\nu'\mathrm{Tr}(X' + Y') - C_1 - w(|x_\alpha - y_\alpha|)|x_\alpha - y_\alpha|^\tau |p'|^{2+\tau} - \mu(|x_\alpha - y_\alpha|)|Y'|,$$

and we obtain

$$\begin{aligned} \nu'|\mathrm{Tr}(X' + Y')| \leq C''\{1 + w(|x_\alpha - y_\alpha|)|x_\alpha - y_\alpha|^\tau |p'|^{2+\tau} + \mu(|x_\alpha - y_\alpha|)(\|B'\|^{\frac{1}{2}} \\ +|\mathrm{Tr}(X + Y - 2LI)|^{\frac{1}{2}})|\mathrm{Tr}(X + Y - 2LI)|^{\frac{1}{2}}\}. \end{aligned}$$

As in (i), by comparing the order in r of the both hand sides of the inequality, we get a contradiction. Thus, we proved (40).

(iii) The proof can be obtained by repeating a similar argument as in (ii), by taking z in (39) on $\partial\Omega$, and we do not write it here.

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