POLYLOGARITHMS AND MIXED MOTIVES

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§1. The category of mixed motives $D_{\text{finite}}(k)$.

Let $k$ be a field, $\square^1 = \mathbb{P}^1_k - \{1\}$ and $\square^n = (\square^1)^n$ with coordinates $(x_1, \ldots, x_n)$. Faces of $\square^n$ are intersections of codimension one faces, and the latter are divisors of the form $\square_{i, a}^{n-1} = \{x_i = a\}$ where $a = 0$ or $\infty$. A face of dimension $m$ is canonically isomorphic to $\square^m$.

Let $\Sigma_n$ be the permutation group of the set $\{1, \cdots, n\}$; it acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permutations. Let $G_n$ be the semi-direct product of $(\mathbb{Z}/2\mathbb{Z})^n$ and $\Sigma_n$; an element is of the form $(i_1, \cdots, i_n; \tau)$ with $i_1, \cdots, i_n \in \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ and $\tau \in \Sigma_n$. $\Sigma_n$ acts on $\square^n$ by permutations of coordinates, and $(\mathbb{Z}/2\mathbb{Z})^n$ acts by: $(i_1, \cdots, i_n) \cdot (x_1, \cdots, x_n) = (x_{i_1}, \cdots, x_{i_n})$. So $G_n$ acts of $\square^n$. There is a homomorphism $sgn : G_n \to \{\pm 1\}$, $(i_1, \cdots, i_n; \tau) \mapsto i_1 \cdots i_n \cdot sgn \tau$.

Let $X$ be an equi-dimensional variety (or a scheme) over a field $k$. Let $Z^r(X, n)$ be the $\mathbb{Q}$-vector space of $\mathbb{Q}$-cycles $z$ of codimension $r$ on $X \times \square^n$ such that

(i) each irreducible component of $z$ meets each face $X \times \square^m$ properly, and
(ii) $z$ is alternating with respect to $G_n$, namely, for any $\sigma \in G_n$, $\sigma(z) = sgn(\sigma)z$.

The inclusions of codimension one faces $\delta_{i,a} : \square_{i,a}^{n-1} \hookrightarrow \square^n$ induce the map

$$\partial = \sum (-1)^{i + a} \delta_{i,a}^* : Z^r(X, n) \to Z^r(X, n - 1)$$

(namely, $\partial$ sends alternating cycles to alternating ones) and $(Z^r(X, \cdot), \partial)$ is a homology complex. We call this the cycle complex (of codimension $r$) of $X$. By definition the (rational) higher Chow groups are the homology groups of this complex:

$$CH^r(X, n) = H_n Z^r(X, \cdot).$$

Note $CH^r(X, 0) = CH^r(X)$, the rational Chow group of $X$.

The following notion will be of frequent use. A $C$-complex (of abelian groups) consists of

(i) Complexes $A^m = (A^{m, \cdot}, d_{A^m})$ (each of which is not necessarily bounded) for $m \in \mathbb{Z}$, such that for all but finitely many $m$'s, $A^m = 0$; and
(ii) For $m < n$, maps of graded groups

$$F^{m, n} : A^{m, \cdot} \to A^{n, \cdot - (n - m - 1)}$$

subject to the condition

$$F^{m, n} \circ (-1)^m d_{A^m} + (-1)^n d_{A^{n, \cdot}} F^{m, n} + \sum_{m < \ell < n} F^{\ell, n} \circ F^{m, \ell} = 0$$

as a map $A^{m, \cdot} \to A^{n, \cdot - n + m + 2}$.
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One can associate with it a complex, the *total complex* \( \text{Tot}(A) = (\text{Tot}(A)^\bullet, \mathbf{d}) \), defined as:

\[
\text{Tot}(A)^p = \bigoplus_{p \in \mathbb{Z}} A^{m,p-m}
\]

(which is a finite sum), and the differential \( \mathbf{d} \) is the direct sum, for \( m \), of \((-1)^m d_{A^m} + F^{m,n} : A^{m,\bullet} \to \oplus_{n \geq m} A^{n,*(n-m-1)} \). The condition in (ii) is equivalent to \( \mathbf{d} \) being a differential.

Let \( X \) and \( Y \) be smooth projective. For an element \( f \in Z^s(X \times Y, \ell) \), one has the partially defined map (of graded vector spaces)

\[
f_* : Z^r(X, n) \to Z^r+s(Y, n + \ell), \quad f_*(z) = p_{Y*}[(f \times \square^n) \circ (z \times Y \times \square^\ell)].
\]

Here \((f \times \square^n) \circ (z \times Y \times \square^\ell)\) denotes the alternation (with respect to the action of \( G_{n+\ell} \)) of the cycle theoretic intersection \((f \times \square^n) \cup (z \times Y \times \square^\ell)\), and \(p_Y : X \times Y \times \square^{n+\ell} \to Y \times \square^{n+\ell}\) is the projection.

Similarly there is the partially defined map

\[
Z^r(X \times Y, \cdot) \otimes Z^s(Y \times Z, \cdot) \to Z^{r+s-\dim Y}(X \times Z, \cdot)
\]

which sends \( u \otimes v \in Z^r(X \times Y, n) \otimes Z^s(Y \times Z, m) \) to

\[
v \circ u := \text{the alternation of } \ p_{XZ*}[(X \times u \times \square^n) \cup (u \times Z \times \square^m)].
\]

For each smooth projective variety \( X \) there is a collection of *distinguished subcomplexes* of \( Z^r(X, \cdot) \) satisfying:

(i) For a distinguished subcomplex \( Z^r(X, \cdot)' \), the inclusion into \( Z^r(X, \cdot) \) is a quasi-isomorphism;

(ii) For any cycle \( f \in Z^s(X \times Y, \ell) \) and a distinguished subcomplex \( Z^{r+s-\dim Y}(Y, \cdot)' \) there is a distinguished subcomplex \( Z^{r,\cdot}'(X, \cdot) \) on which \( f_* \) is defined and induces a map \( f_* : Z^r(X, \cdot)' \to Z^{r+s-\dim X}(Y, \cdot)' \);

(iii) The intersection of a finite collection of distinguished subcomplexes is again distinguished.

For details on the category \( D(k) \) of mixed motives, we refer the reader to [Ha, II]. We will only need the subcategory \( D_{\text{finite}}(k) \) of mixed motives of finite type; the definitions are briefly recalled.

A *finite symbol* is a formal sum

\[
\bigoplus_{\alpha \in I} (X_\alpha, r_\alpha)
\]

where \( X_\alpha \) is a smooth projective variety, \( I \) a finite index set and \( r_\alpha \in \mathbb{Z} \). We write 0 for the corresponding symbol when \( I \) is an empty set.

Define dual, tensor product, and inner Hom of (a) finite symbol(s) as:

\[
(\bigoplus (X_\alpha, r_\alpha))^\vee = \bigoplus (X_\alpha, \dim X_\alpha - r_\alpha).
\]

\[
(\bigoplus (X_\alpha, r_\alpha)) \otimes (\bigoplus (X'_\alpha, r'_\alpha)) = \bigoplus (X_\alpha \times X'_\alpha, r_\alpha + r'_\alpha);
\]

and

\[
\text{Hom}(\bigoplus (X_\alpha, r_\alpha), \bigoplus (X'_\alpha, r'_\alpha)) = (\bigoplus (X_\alpha, r_\alpha))^\vee \otimes (\bigoplus (X'_\alpha, r'_\alpha)).
\]
Define the cycle complex of a finite formal symbol by

$$Z^0(\oplus(X_\alpha, r_\alpha), \cdot) = \bigoplus Z^{r_\alpha}(X_\alpha, \cdot).$$

One also uses cohomological notation $Z^0(\oplus(X_\alpha, r_\alpha))^{-n} = Z^0(\oplus(X_\alpha, r_\alpha), n)$.

Note there is a partially defined map

$$Z^0(\text{Hom}((X_1, r_1), (X_2, r_2)), \cdot) \otimes Z^0(\text{Hom}((X_2, r_2)(X_3, r_3)), \cdot)$$

given by the composition of correspondences

$$u \otimes v \mapsto v \circ u = p_{13*}[\mu X_3 \cdot (X_1 \times v)].$$

By definition, an object of $D_{\text{finite}}(k)$ is a set of data $K = (K^m) = (K^m, f^{m,n})$ where

(i) For each integer $m$, $K^m = \oplus_{\alpha \in I(m)}(X_\alpha, r_\alpha)$, a finite symbol.
(ii) For $(m, n)$ with $m < n$, given $f^{m,n} = (f^{m,n}_{\alpha\beta}) \in Z^0(\text{Hom}(K^m, K^n))^{-n+m+1}$, which are subject to the conditions:

For $f^{m_k,m_{k+1}} \in Z^0(\text{Hom}(K^{m_k}, K^{m_{k+1}}))^{-m_{k+1}+m_k+1}$ ($k = 1, 2, \ldots, r$) one has

$$f^{m_r,m_{r+1}} \circ f^{m_r,m_{r-1}} \circ \cdots \circ f^{m_1,m_2}$$

is defined and $\in Z^0(\text{Hom}(K^{m_1}, K^{m_{r+1}}))^{-m_{r+1}+m_1+r}$.

For $m < n$, one has

$$(-1)^n \partial f^{m,n} + \sum_{m < \ell < n} f^{\ell,n} \circ f^{m,\ell} = 0,$$

On the left side the compositions of the correspondences are required to be defined.

There is the functor of cycle complexes $Z^0$ from $D_{\text{finite}}(k)$ to the derived category of $\mathbb{Q}$-vector spaces. To define $Z^0(K, \cdot)$, for each $m$ and $\alpha \in I(m)$, take a distinguished subcomplex

$$Z^0((X_\alpha, r_\alpha), \cdot)'$$

so that each $f^{m,n}_{\alpha\beta}$ induces the map $f^{m,n}_{\alpha\beta*} : Z^0((X_\alpha, r_\alpha), \cdot)' \rightarrow Z^0((X_\beta, r_\beta), \cdot)'$.

We then let

$$Z^0(K^m, \cdot)' := \oplus_{\alpha} Z^0((X_\alpha, r_\alpha), \cdot)'$$

and have $f^{m,n}_{\alpha\beta*} : Z^0(K^m, \cdot)' \rightarrow Z^0(K^n, \cdot' + (n - m - 1))'$ is defined. We define $Z^0(K, \cdot)$ to be the total complex $\text{Tot}(Z^0(K^m, \cdot)'(f^{m,n}_{\alpha\beta*}), d')$ namely the complex $(\mathbb{K}, d)$ with

$$\mathbb{K}^i = \bigoplus_{j \geq i} Z^0(K^j, j - i)' ,$$

and

$$d^i = \sum_j ((-1)^j \partial_j + \sum_{j < \ell} f^{j,\ell}_{\alpha\beta*}).$$

Let $(K, f)$ and $(L, g)$ be objects in $D_{\text{finite}}(k)$. The function cycle complex $\text{Hom}(K, L)^*$ is defined as follows. Let $Z^0(\text{Hom}(K^m, L^{m'}), \cdot)'$ be distinguished subcomplexes such that

For $u \in Z^0(\text{Hom}(K^m, L^{m'}), \cdot)$, both $u \circ f^{m,n}$ and $g^{m',n'} \circ u$ are defined.
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(This is possible since there are only finitely many non-zero $f^{n,m}$’s and $g^{m',n'}$’s.) The cohomological complex to be defined has the group of $N$-cochains

$$\text{Hom}(K, L)^N = \bigoplus_{-m + m' - p = N} \mathcal{Z}^0(\text{Hom}(K^m, L^{m'}), p)'$$

The differential of this complex, which we denote by $D$, is the sum of the three kinds of maps:

$$(-1)^{p + m' + n + 1} (\circ f^{n,m}) : \mathcal{Z}^0(\text{Hom}(K^m, L^{m'}), p)' \to \mathcal{Z}^0(\text{Hom}(K^n, L^{m'}), p + n' - m' - 1)'$$,

$$(-1)^{m' + n'} (g^{m',n'} \circ) : \mathcal{Z}^0(\text{Hom}(K^m, L^{m'}), p)' \to \mathcal{Z}^0(\text{Hom}(K^m, L^{n'}), p + n' - m' - 1)'$$,

and

$$(-1)^{m'} \partial : \mathcal{Z}^0(\text{Hom}(K^m, L^{m'}), \cdot)' \to \mathcal{Z}^0(\text{Hom}(K^m, L^{m'}), \cdot - 1)'$$.

Given three objects $K, L$ and $M$, the partially defined composition map

$$\text{Hom}(K, L) \otimes \text{Hom}(L, M) \rightarrow \text{Hom}(K, M)$$

$u \otimes v \mapsto v \circ u$; \hspace{1cm} \((v \circ u)^{m,n} = \sum_{\ell \in \mathbb{Z}} u^\ell,n \circ f^{m,\ell} + \sum_{\ell \in \mathbb{Z}} g^{\ell,n} \circ u^{m,\ell}\)

satisfies the Leibniz formula

$$D(v \circ u) = Du \circ v + (-1)^{\deg v} v \circ Du$$,

where $\deg v$ is the total degree of $v$ in the cohomological complex. There is a quasi-isomorphic subcomplex of $\text{Hom}(K, L) \otimes \text{Hom}(L, M)$ on which the composition is defined. See [Ha, II, §1].

By definition

$$\text{Hom}_{D_{\text{finite}}(k)}(K, L) = H^0 \mathcal{Z}^0(\text{Hom}(K, L))'$$.

The composition of morphisms is induced from the composition of the function complexes. A morphism $u : K \rightarrow L$ is represented by $u^{m,n} \in \text{Hom}(K^m, L^n)^{-n+m}$ (non-zero only for $m \leq n$) subject to the condition

$$(-1)^{n} \partial u^{m,n} - \sum (-1)^{m+\ell} u^{\ell,n} \circ f^{m,\ell} + \sum (-1)^{\ell+n} g^{\ell,n} \circ u^{m,\ell} = 0$$.

It defines the zero morphism if there exist $U^{m,n} \in \text{Hom}(K^m, L^n)^{-n+m-1}$ (non-zero only for $m \leq n - 1$) such that

$$u^{m,n} = (-1)^{n} \partial U^{m,n} + \sum (-1)^{m+\ell} U^{\ell,n} \circ f^{m,\ell} + \sum (-1)^{\ell+n} g^{\ell,n} \circ U^{m,\ell}$$.

We have the following. Let $\mathbb{Q}(r) = (pt, r)[2r]$, the Tate objects.
(1.1) **Theorem.** The category $D_{\text{finite}}(k)$ has a structure of triangulated category. Moreover
(1) $D_{\text{finite}}(k)$ has dual, tensor product, inner Hom, the unit object $\mathbb{Q}$, and the Tate objects $\mathbb{Q}(r)$.
(2) There is a contravariant functor $h: (\text{Smooth Proj.}/k) \to D_{\text{finite}}(k)$.
(3) If $X$ is smooth and projective, one has
$$\text{Hom}_{D_{\text{finite}}(k)}(\mathbb{Q}, h(X)(r)[2r - ml]) = K_m(X)^{(r)}_q.$$ Here the right hand side is an Adams-graded piece of the $K$-group of $X$.
(4) There is the cycle complex functor $\mathbb{Z}^0: D_{\text{finite}}(k) \to D(\mathbb{Q})$.

For the rest we will simply write $D(k)$ for $D_{\text{finite}}(k)$.

§2. **Etale realization.**

For a smooth projective variety $X$ over a field $k$ and $\ell \neq \text{ch} k$, we have the $\ell$-adic cohomology $H^*(X, \mathbb{Q}_\ell(r))$. We use complexes calculating the etale cohomology, which behave well with respect to composition of correspondences. Let $X, Y$ be smooth projective, and $D$ any variety. One can define a complex of $\mathbb{Q}_\ell$-vector spaces $\text{Hom}(X, Y)_D(r)$ satisfying the following properties (cf. [Ha, II, §5] for details in case of Betti cohomology).

1. $H^i \text{Hom}(X, Y)_D(r) = H^{i+2r}(X \times Y \times D, \mathbb{Q}_\ell(r))$. If the first variety of the pair is $pt = \text{Spec} \ k$, $\text{Hom}(pt, X)_{pt}(r) = \Gamma(X, C^*(\mathbb{Q}_\ell(r)))[2r] := \lim \Gamma(X, C^*(\mathbb{Z}/\ell^r(r))) \otimes \mathbb{Q}_\ell[2r]$. Here $C^*$ denotes the Godement resolution.
2. There is a map of complexes
$$\text{Hom}(X, Y)_D(r) \otimes \text{Hom}(Y, Z)_D(s) \to \text{Hom}(X, Z)_D(r + s - \dim Y)$$
which gives rise to the composition of correspondences
$$H^*(X \times Y \times D, \mathbb{Q}_\ell(r)) \otimes H^*(Y \times Z \times D, \mathbb{Q}_\ell(r)) \to H^*(X \times Z, \mathbb{Q}_\ell(r + s - \dim Y)).$$

The map is associative.

3. To a map $\alpha: D' \to D$, there corresponds to a map of complexes $\alpha^*: \text{Hom}(X, Y)_D(r) \to \text{Hom}(X, Y)_{D'}(r)$.

There is also the supported theory. Given a closed set $V \subset X \times Y \times D$, there is a subcomplex $\text{Hom}(X, Y)_V, D(r) \subset \text{Hom}(X, Y)_D(r)$ satisfying $H^i \text{Hom}(X, Y)_V, D(r) = H^{i+2r}_V(X \times Y \times D, Q_\ell(r))$, and properties analogous to (2) and (3).

Now we take as $D$ the cubical scheme $\square^*$. For $r, s \in \mathbb{Z}$, we define a double complex $C((X, r), (Y, s))^{a, b}$ as follows.
$$C((X, r), (Y, s))^{a, b} = \text{Hom}(X, Y)_{\square^*}(s - r + \dim X)^{\text{Alt}}$$
where $\text{Alt}$ denotes the alternating part with respect to the action of $G_{-b}$ on $\square^{\ast - b}$. Define
$$\partial: C^*(X, r), (Y, s) \to C^{*+1}(X, r), (Y, s)$$
to be the alternating sum of the maps induced by the face maps $\square^{\ast - b-1} \to \square^{\ast - b}$. Denote the associated simple complex by $C((X, r), (Y, s))$ with differential $D = d + (-1)^b \partial$.

The above definitions can be extended to finite symbols. For $K$ and $L$ finite symbols we have the complex $C(K, L)$. Set $C((X, r)) = C((pt, 0), (X, r)) = \text{Hom}(pt, X)(r)$, and $C(K) = C((pt, 0), K)$. There is composition map
$$C(K, L) \otimes C(L, M) \to C(K, M), \quad f \otimes g \mapsto g \circ f,$$
satisfying associativity. An element $F \in C(K, L)^n$ induces a map $F_* : C(K) \to C(L)[n]$.

Given $Z \in Z^r(X, n)$ define a subcomplex

$$C_{|Z|}(X, r)^{a,b} \subset C(X, r)^{a,b}$$

where $C_{|Z|}(X, r)^{a,b}$ is defined by the support condition with respect to $\cup_{\delta} |\delta^* Z|$, $\delta$ varying over the face maps $\delta : \square^{-b} \to \square^n$. Similarly given $f \in \text{Hom}(K, L)^{-n}$ there is a subcomplex $C_{|f|}(K, L)$. Here $|f|$ denotes the support of $f$ (we sometimes write just $f$). An element $f \in \text{Hom}(K, L)^{-n}$ has cycle class $cl(f) \in H^0 C_{-n}(K, L)$.

(2.1) Proposition. Given an object $(K^m, f^{m,n})$ of $D(k)$, there exist, for $m < n$, elements

$$F^{m,n} \in \bigoplus_{a+b=-(n-m-1), a \leq 0} C_{|f|}^{a,b}(K^m, K^n)$$

such that its $(0, -n + m + 1)$-component $0F^{m,n} \in C_f^{-n-m-1}(K^m, K^n)$ satisfies

$$[0F^{m,n}]_d = cl(f^{m,n}) \in H^0 C_f^{-n-m-1}(K^m, K^n)$$

and one has the relation

$$(*) \quad (-1)^n D(F^{m,n}) + \sum_{m < \ell < n} F^{\ell,n}_{\circ} F^{m,\ell} = 0$$

in

$$\bigoplus_{a+b=-(n-m-2), a \leq 0} C_f^{a,b}(K^m, K^n).$$

We call such $(F^{m,n})$ a representative of $(cl(f^{m,n}))$.

Choose $(F^{m,n})$; then the maps

$$F^{m,n}_{\bullet} : C(K^m) \to C(K^n)[-(n-m-1)]$$

satisfy

$$(-1)^n D_{\circ} F^{m,n}_{\bullet} + F^{m,n}_{\circ} (-1)^m D + \sum F^{\ell,n}_{\circ} F^{m,\ell}_{\bullet} = 0$$

where $D$ is the differential of $C(K^m)$ or $C(K^n)$, namely $(C(K^m), F^{m,n}_{\circ})$ is a $C$-complex. So we have the total complex,

$$C(K) := \text{Tot} (\oplus C(K^m), D + \sum F^{m,n}_{\bullet}).$$

It can be shown that $C(K)$ is well-defined independent of the choice of representatives $F^{m,n}$. By definition $H^*(K) = H^*(K, \mathbb{Q}_\ell) := H^*C(K)$.

(2.2) Theorem. We have the functor of $\ell$-adic etale cohomology

$$H^*(-, \mathbb{Q}_\ell) : D(k) \to \text{Vec}_{\mathbb{Q}_\ell}.$$
For $K$ in $D(k)$, we have $K \otimes_k \bar{k}$ in $D(\bar{k})$; the cohomology

$$H^*(K \otimes_k \bar{k}; \mathbb{Q}_\ell)$$

is a $G(\bar{k}/k)$-module. We have the functor to the category of Galois modules

$$D(k) \rightarrow (G(\bar{k}/k) - \text{Vec}_{\mathbb{Q}_\ell})$$

§3. Polylogarithmic objects.

We define the category of mixed Tate motives to be the triangulated subcategory generated by $\mathbb{Q}(r)$, $r \in \mathbb{Z}$.

We give two types of algebraic cycles, each parametrized by $a \in k^*$. For $a \in k^*$, $f_a := \text{Alt}\{t = a\} \in Z^1(pt, 1)$, the alternation of the cycle $\{t = a\}$. For $a \in k^* - \{1\}$ and $r \geq 2$, using cubical coordinates and parameters $t_1, \cdots, t_{r-2}$,

$$V_a^r := [t_1, t_2, \cdots, t_{r-1}, 1 - t_1, 1 - \frac{t_2}{t_1}, \cdots, 1 - \frac{t_{r-1}}{t_{r-2}}]$$

and $C_a^r := \text{Alt}V_a^r \in Z^r(pt, 2r - 1)$. This was considered by Totaro ($r = 2$) and Bloch ($r \geq 3$). Note

$$\partial C_a^r = \begin{cases} f_{1-a} \circ f_a & \text{if } r \geq 3 \\ f_a \circ f_{1-a} & \text{if } r = 2 \end{cases}$$

Define the object

$$L(a) := [(pt, 0) \rightarrow (pt, 1)]$$

where $(pt, 0)$, $(pt, 1)$ are placed in degrees 0 and 2, respectively. More precisely, $L(a) = (L^m, f^{m,n})$ consists of

$$L^{2m} = (pt, m) \text{ for } m = 0, 1, \text{ and } L^n = 0 \text{ otherwise;}$$

$$f^{0,2} = f_a, \ f^{m,n} = 0 \text{ otherwise.}$$

For $p \geq 1$ and $a \in k^* - \{1\}$, define the object $K_p(a)$ (called the polylogarithmic object of weight $p$) by

$$K^{2m} = (pt, m) \text{ for } m = 0, 1, \cdots, p, \ K^n = 0 \text{ otherwise;}$$

$$f^{0,2} = f_{1-a}, \ f^{2m,2m+2} = f_a \text{ for } m = 1, \cdots, p - 1,$$

$$f^{0,2m} = C_a^m \text{ for } m = 2, \cdots, p \text{ and } f^{m,n} = 0 \text{ otherwise.}$$

(3.1) Proposition. (1) $H^\nu(L(a) \otimes \bar{k}, \mathbb{Q}_\ell) = 0$ for $\nu \neq 0$. There is an exact sequence

$$0 \rightarrow \mathbb{Q}_\ell(1) \rightarrow H^0(L(a) \otimes \bar{k}, \mathbb{Q}_\ell) \rightarrow \mathbb{Q}_\ell(0) \rightarrow 0$$

whose extension class is $[a] \in H^1(G(\bar{k}/k), \mathbb{Q}_\ell(1))$.

(2) $H^\nu(K_p(a) \otimes_k \bar{k}, \mathbb{Q}_\ell) = 0$ for $\nu \neq 0$; The cohomology $H^0(K_p(a) \otimes_k \bar{k}, \mathbb{Q}_\ell)$ has a filtration $W_*(\text{the weight filtration}), \ H^0 = W_0 \supset W_{-2} \supset \cdots \supset W_{-2p} \supset W_{-2p-2} = 0$ such that $Gr_{W_q}^W = \mathbb{Q}_\ell(q)$ for $q = 0, \cdots, p$ and the extension class of the exact sequence

$$0 \rightarrow Gr_{W_{-2q-2}}^W = \mathbb{Q}_\ell(q + 1) \rightarrow W_{-2q}/W_{-2q-4} \rightarrow Gr_{W_{-2q}}^W = \mathbb{Q}_\ell(q) \rightarrow 0$$

is $[1 - a]$ for $q = 0$ and $[a]$ for $q = 1, \cdots, p - 1$. 
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REFERENCES


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[Ha, II] is available from www.math.tohoku.ac.jp/hanamura

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