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POLYLOGARITHMS AND MIXED MOTIVES

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§1. The category of mixed motives $\mathcal{D}_{finite}(k)$.

Let k be a field, $\square^1 = \mathbb{P}^1_k - \{1\}$ and $\square^n = (\square^1)^n$ with coordinates $(x_1, \cdots, x_n)$. Faces of $\square^n$ are intersections of codimension one faces, and the latter are divisors of the form $\square^{n-1}_{i,a} = \{x_i = a\}$ where $a = 0$ or ∞. A face of dimension m is canonically isomorphic to $\square^m$.

Let Σ_n be the permutation group of the set $\{1, \cdots, n\}$; it acts on $(\mathbb{Z}/2\mathbb{Z})^n$ by permutations. Let G_n be the semi-direct product of $(\mathbb{Z}/2\mathbb{Z})^n$ and Σ_n; an element is of the form $(i_1, \cdots, i_n; \tau)$ with $i_1, \cdots, i_n \in \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ and $\tau \in \Sigma_n$. Σ_n acts on $\square^n$ by permutations of coordintates, and $(\mathbb{Z}/2\mathbb{Z})^n$ acts by: $(i_1, \cdots, i_n) \cdot (x_1, \cdots, x_n) = ({x_1}^{i_1}, \cdots, {x_n}^{i_n})$. So G_n acts of $\square^n$. There is a homomorphism $sgn : G_n \to \{\pm 1\}$, $(i_1, \cdots, i_n; \tau) \mapsto i_1 \cdots i_n \cdot sgn\, \tau$.

Let X be an equi-dimensional variety (or a scheme) over a field k. Let $\mathcal{Z}^r(X, n)$ be the $\mathbb{Q}$-vector space of $\mathbb{Q}$-cycles z of codimension r on $X \times \square^n$ such that

(i) each irreducible component of z meets each face $X \times \square^m$ properly, and

(ii) z is alternating with respect to G_n, namely, for any $\sigma \in G_n$, $\sigma(z) = sgn(\sigma) z$.

The inclusions of codimension one faces $\delta_{i,a} : \square^{n-1}_{i,a} \hookrightarrow \square^n$ induce the map

$$\partial = \sum (-1)^{i+a} \delta^*_{i,a} : \mathcal{Z}^r(X, n) \to \mathcal{Z}^r(X, n-1)$$

(namely, ∂ sends alternating cycles to alternating ones) and $(\mathcal{Z}^r(X, \cdot), \partial)$ is a homology complex. We call this the *cycle complex* (of codimension r) of X. By definition the (rational) higher Chow groups are the homology groups of this complex:

$$CH^r(X, n) = H_n \mathcal{Z}^r(X, \cdot)\ .$$

Note $CH^r(X, 0) = CH^r(X)$, the rational Chow group of X.

The following notion will be of frequent use. A *C-complex* (of abelian groups) consists of

(i) Complexes $A^m = (A^{m,\bullet}, d_{A^m})$ (each of which is not necessarily bounded) for $m \in \mathbb{Z}$, such that for all but finitely many m's, $A^m = 0$; and

(ii) For $m < n$, maps of graded groups

$$F^{m,n} : A^{m,\bullet} \to A^{n,\bullet-(n-m-1)}$$

subject to the condition

$$F^{m,n} \circ (-1)^m d_{A^m} + (-1)^n d_{A^n} \circ F^{m,n} + \sum_{m<\ell<n} F^{\ell,n} \circ F^{m,\ell} = 0$$

as a map $A^{m,\bullet} \to A^{n,\bullet-n+m+2}$.

One can associate with it a complex, the *total complex* $Tot\,(A) = (Tot\,(A)^\bullet, \mathbf{d})$, defined as:

$$Tot\,(A)^p = \bigoplus_{p\in\mathbb{Z}} A^{m,p-m}$$

(which is a finite sum), and the differential $\mathbf{d}$ is the direct sum, for m, of $(-1)^m\, d_{A^m} + F^{m,n} : A^{m,\bullet} \to \oplus_{n\geq m} A^{n,\bullet-(n-m-1)}$. The condition in (ii) is equivalent to $\mathbf{d}$ being a differential.

Let X and Y be smooth projective. For an element $f \in \mathcal{Z}^s(X \times Y, \ell)$, one has the partially defined map (of graded vector spaces)

$$f_* : \mathcal{Z}^r(X,n) \dashrightarrow \mathcal{Z}^{r+s}(Y, n+\ell), \qquad f_*(z) = p_{Y*}[(f \times \square^n)\cdot(z \times Y \times \square^\ell)]\ .$$

Here $(f \times \square^n)\cdot(z \times Y \times \square^\ell)$ denotes the alternation (with respect to the action of $G_{n+\ell}$) of the cycle theoretic intersection $(f \times \square^n) \odot (z \times Y \times \square^\ell)$, and $p_Y : X \times Y \times \square^{n+\ell} \to Y \times \square^{n+\ell}$ is the projection.

Similarly there is the partially defined map

$$\mathcal{Z}^r(X \times Y, \cdot) \otimes \mathcal{Z}^s(Y \times Z, \cdot) \dashrightarrow \mathcal{Z}^{r+s-\dim Y}(X \times Z, \cdot)$$

which sends $u \otimes v \in \mathcal{Z}^r(X \times Y, n) \otimes \mathcal{Z}^s(Y \times Z, m)$ to

$$v \circ u := \text{ the alternation of } \quad p_{XZ*}[(X \times v \times \square^n) \odot (u \times Z \times \square^m)]\ .$$

For each smooth projective variety X there is a collection of ***distinguished subcomplexes*** of $\mathcal{Z}^r(X,\cdot)$ satisfying:

(i) For a distinguished subcomplex $\mathcal{Z}^r(X,\cdot)'$, the inclusion into $\mathcal{Z}^r(X,\cdot)$ is a quasi- isomorphism;

(ii) For any cycle $f \in \mathcal{Z}^s(X \times Y, \ell)$ and a distinguished subcomplex $\mathcal{Z}^{r+s-\dim X}(Y,\cdot)'$ there is a distinguished subcomplex $\mathcal{Z}^r(X,\cdot)'$ on which f_* is defined and induces a map $f_* : \mathcal{Z}^r(X,\cdot)' \to \mathcal{Z}^{r+s-\dim X}(Y, \cdot + \ell)'$;

(iii) The intersection of a finite collection of distinguished subcomplexes is again distinguished.

For details on the category $\mathcal{D}(k)$ of mixed motives, we refer the reader to [Ha, II]. We will only need the subcategory $\mathcal{D}_{finite}(k)$ of mixed motives of finite type; the definitions are briefly recalled.

A *finite symbol* is a formal sum

$$\bigoplus_{\alpha\in I}(X_\alpha, r_\alpha)$$

where X_α is a smooth projective variety, I a finite index set and $r_\alpha \in \mathbb{Z}$. We write 0 for the corresponding symbol when I is an empty set.

Define dual, tensor product, and inner Hom of (a) finite symbol(s) as:

$$\left(\oplus(X_\alpha, r_\alpha)\right)^\vee = \oplus(X_\alpha, \dim X_\alpha - r_\alpha)\ .$$

$$(\oplus(X_\alpha, r_\alpha)) \otimes (\oplus(X'_{\alpha'}, r_{\alpha'})) = \oplus(X_\alpha \times X'_{\alpha'}, r_\alpha + r'_{\alpha'})\ ;$$

and

$$\underline{Hom}(\oplus(X_\alpha, r_\alpha), \oplus(X'_{\alpha'}, r_{\alpha'})) = (\oplus(X_\alpha, r_\alpha))^\vee \otimes (\oplus(X'_{\alpha'}, r_{\alpha'}))\ .$$

Define the *cycle complex* of a finite formal symbol by

$$\mathcal{Z}^0(\oplus(X_\alpha, r_\alpha), \cdot) = \bigoplus \mathcal{Z}^{r_\alpha}(X_\alpha, \cdot) .$$

One also uses cohomological notation $\mathcal{Z}^0(\oplus(X_\alpha, r_\alpha))^{-n} = \mathcal{Z}^0(\oplus(X_\alpha, r_\alpha), n)$.

Note there is a partially defined map

$$\mathcal{Z}^0(\underline{Hom}((X_1, r_1), (X_2, r_2)), \cdot) \otimes \mathcal{Z}^0(\underline{Hom}((X_2, r_2)(X_3, r_3)), \cdot)$$

$$-- \to \mathcal{Z}^0(\underline{Hom}((X_1, r_1), (X_3, r_3)), \cdot)$$

given by the composition of correspondences

$$u \otimes v \mapsto v \circ u = p_{13*}[(u \times X_3) \cdot (X_1 \times v)] .$$

By definition, an object of $\mathcal{D}_{finite}(k)$ is a set of data $K = (K^m) = (K^m, f^{m,n})$ where

(i) For each integer m, $K^m = \oplus_{\alpha \in I(m)}(X_\alpha, r_\alpha)$, a finite symbol.

(ii) For (m, n) with $m < n$, given $f^{m,n} = (f^{m,n}_{\alpha\beta}) \in \mathcal{Z}^0(\underline{Hom}(K^m, K^n))^{-n+m+1}$, which are subject to the conditions:

For $f^{m_k, m_{k+1}} \in \mathcal{Z}^0(\underline{Hom}(K^{m_k}, K^{m_{k+1}}))^{-m_{k+1}+m_k+1}$ $(k = 1, 2, \cdots, r)$ one has

$$f^{m_r, m_{r+1}} \circ f^{m_r, m_{r-1}} \circ \cdots \circ f^{m_1, m_2} \text{ is defined and } \in \mathcal{Z}^0(\underline{Hom}(K^{m_1}, K^{m_{r+1}}))^{-m_{r+1}+m_1+r} .$$

For $m < n$, one has

$$(-1)^n \partial f^{m,n} + \sum_{m<\ell<n} f^{\ell,n} \circ f^{m,\ell} = 0 \ , \ .$$

On the left side the compositions of the correspondences are *required* to be defined.

There is the functor of *cycle complexes* $\mathcal{Z}^0$ from $\mathcal{D}_{finite}(k)$ to the derived category of $\mathbb{Q}$-vector spaces. To define $\mathcal{Z}^0(K, \cdot)$, for each m and $\alpha \in I(m)$, take a distinguished subcomplex $\mathcal{Z}^0((X_\alpha, r_\alpha), \cdot)'$ so that each $f^{m,n}_{\alpha,\beta}$ induces the map $f^{m,n}_{\alpha,\beta\,*} : \mathcal{Z}^0((X_\alpha, r_\alpha), \cdot)' \to \mathcal{Z}^0((X_\beta, r_\beta), \cdot)'$. We then let

$$\mathcal{Z}^0(K^m, \cdot)' := \oplus_\alpha \mathcal{Z}^0((X_\alpha, r_\alpha), \cdot)'$$

and have $f^{m,n}{}_* : \mathcal{Z}^0(K^m, \cdot)' \to \mathcal{Z}^0(K^n, \cdot + (n - m - 1))'$ is defined. We define $\mathcal{Z}^0(K, \cdot)$ to be the *total complex* $Tot(\mathcal{Z}^0(K^m, \cdot), f^{m,n}_*)$, namely the complex $(\mathbb{K}, d)$ with

$$\mathbb{K}^i = \bigoplus_{j \geq i} \mathcal{Z}^0(K^j, j - i)' ,$$

and

$$d^i = \sum_j ((-1)^j \partial_j + \sum_{j<\ell} f^{j,\ell}_*) .$$

Let (K, f) and (L, g) be objects in $\mathcal{D}_{finite}(k)$. The *function cycle complex* $\mathrm{Hom}(K, L)^\bullet$ is defined as follows. Let $\mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}), \cdot)'$ be distinguished subcomplexes such that

For $u \in \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}), \cdot)$, both $u \circ f^{n,m}$ and $g^{m',n'} \circ u$ are defined.

(This is possible since there are only finitely many non-zero $f^{n,m}$'s and $g^{m',n'}$'s.) The cohomological complex to be defined has the group of N-cochains

$$\mathrm{Hom}(K,L)^N = \bigoplus_{-m+m'-p=N} \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}), p)'$$

The differential of this complex, which we denote by D, is the sum of the three kinds of maps:

$$(-1)^{p+m'+n+1}(\circ f^{n,m}) : \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}), p)' \to \mathcal{Z}^0(\underline{Hom}(K^n, L^{m'}), p+n'-m'-1)' ,$$

$$(-1)^{m'+n'}(g^{m',n'}\circ) : \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}), p)' \to \mathcal{Z}^0(\underline{Hom}(K^m, L^{n'}), p+n'-m'-1)' ,$$

and

$$(-1)^{m'}\partial : \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}), \cdot)' \to \mathcal{Z}^0(\underline{Hom}(K^m, L^{m'}), \cdot - 1)' .$$

Given three objects K, L and M, the partially defined composition map

$$\mathrm{Hom}(K,L)_\bullet \otimes \mathrm{Hom}(L,M)^\bullet \dashrightarrow \mathrm{Hom}(K,M)^\bullet$$
$$u \otimes v \mapsto v\circ u; \quad (v\circ u)^{m,n} = \sum_{\ell\in\mathbb{Z}} v^{\ell,n}\circ u^{m,\ell}$$

satisfies the Leibniz formula

$$D(v\circ u) = Dv\circ u + (-1)^{\deg v} v\circ Du ,$$

where $\deg v$ is the total degree of v in the cohomological complex. There is a quasi-isomorphic subcomplex of $\mathrm{Hom}(K,L)_\bullet \otimes \mathrm{Hom}(L,M)^\bullet$ on which the composition is defined. See [Ha, II, §1].

By definition

$$\mathrm{Hom}_{\mathcal{D}_{finite}(k)}(K,L) = H^0\mathcal{Z}^0(\underline{Hom}(K,L))^\bullet .$$

The composition of morphisms is induced from the composition of the function complexes. A morphism $u : K \to L$ is represented by $u^{m,n} \in \mathrm{Hom}(K^m, L^n)^{-n+m}$ (non-zero only for $m \le n$) subject to the condition

$$(-1)^n\partial u^{m,n} - \sum(-1)^{m+\ell} u^{\ell,n}\circ f^{m,\ell} + \sum(-1)^{\ell+n} g^{\ell,n}\circ u^{m,\ell} = 0 .$$

It defines the zero morphism if there exist $U^{m,n} \in \mathrm{Hom}(K^m, L^n)^{-n+m-1}$ (non-zero only for $m \le n-1$) such that

$$u^{m,n} = (-1)^n\partial U^{m,n} + \sum(-1)^{m+\ell} U^{\ell,n}\circ f^{m,\ell} + \sum(-1)^{\ell+n} g^{\ell,n}\circ U^{m,\ell} .$$

We have the following. Let $\mathbb{Q}(r) = (pt, r)[2r]$, the Tate objects.

(1.1) Theorem. *The category $\mathcal{D}_{finite}(k)$ has a structure of triangulated category. Moreover*

(1) $\mathcal{D}_{finite}(k)$ has dual, tensor product, inner Hom, the unit object $\mathbb{Q}$, and the Tate objects $\mathbb{Q}(r)$.

(2) There is a contravariant functor $h : (Smooth\,Proj./k) \to \mathcal{D}_{finite}(k)$.

(3) If X is smooth and projective, one has

$$\mathrm{Hom}_{\mathcal{D}_{finite}(k)}(\mathbb{Q}, h(X)(r)[2r-m]) = K_m(X)^{(r)}_{\mathbb{Q}} \ .$$

Here the right hand side is an Adams-graded piece of the K-group of X.

(4) There is the cycle complex functor $\mathcal{Z}^0 : \mathcal{D}_{finite}(k) \to D(\mathbb{Q})$.

For the rest we will simply write $\mathcal{D}(k)$ for $\mathcal{D}_{finite}(k)$.

§2. Etale realization.

For a smooth projective variety X over a field k and $\ell \neq \mathrm{ch}\, k$, we have the ℓ-adic cohomology $H^*(X, \mathbb{Q}_\ell(r))$. We use complexes calculating the etale cohomology, which behave well with respect to composition of correspondences. Let X, Y be smooth projective, and D any variety. One can define a complex of $\mathbb{Q}_\ell$-vector spaces $\mathrm{Hom}(X,Y)_D(r)$ satisfying the following properties (cf. [Ha, II, §5] for details in case of Betti cohomology).

(1) $H^i\,\mathrm{Hom}(X,Y)_D(r) = H^{i+2r}(X \times Y \times D, \mathbb{Q}_\ell(r))$. If the first variety of the pair is $\mathrm{pt} = \mathrm{Spec}\, k$, $\mathrm{Hom}(\mathrm{pt}, X)_{\mathrm{pt}}(r) = \Gamma(X, \mathcal{C}^\bullet(\mathbb{Q}_\ell(r)))[2r] := \varprojlim \Gamma(X, \mathcal{C}^\bullet(\mathbb{Z}/\ell^\nu(r))) \otimes \mathbb{Q}_\ell[2r]$. Here $\mathcal{C}^\bullet$ denotes the Godement resolution.

(2) There is a map of complexes

$$\mathrm{Hom}(X,Y)_D(r) \otimes \mathrm{Hom}(Y,Z)_D(s) \to \mathrm{Hom}(X,Z)_D(r+s-\dim Y)$$

which gives rise to the composition of correspondences

$$H^*(X \times Y \times D, \mathbb{Q}_\ell(r)) \otimes H^*(Y \times Z \times D, \mathbb{Q}_\ell(r)) \to H^*(X \times Z, \mathbb{Q}_\ell(r+s-\dim Y)) \ .$$

The map is associative.

(3) To a map $\alpha : D' \to D$, there corresponds to a map of complexes $\alpha^* : \mathrm{Hom}(X,Y)_D(r) \to \mathrm{Hom}(X,Y)_{D'}(r)$.

There is also the supported theory. Given a closed set $V \subset X \times Y \times D$, there is a subcomplex $\mathrm{Hom}(X,Y)_{V,D}(r) \subset \mathrm{Hom}(X,Y)_D(r)$ satisfying $H^i\,\mathrm{Hom}(X,Y)_{V,D}(r) = H^{i+2r}_V(X \times \times D, \mathbb{Q}_\ell(r))$, and properties analogous to (2) and (3).

Now we take as D the cubical scheme $\square^\bullet$. For $r, s \in \mathbb{Z}$, we define a double complex $C((X,r),(Y,s))^{a,b}$ as follows.

$$C((X,r),(Y,s))^{\bullet,b} = \mathrm{Hom}(X,Y)_{\square^{-b}}(s-r+\dim X)^{Alt}$$

where Alt denotes the alternating part with respect to the action of G_{-b} on $\square^{-b}$. Define

$$\partial : C^{\bullet\, b}((X,r),(Y,s)) \to C^{\bullet\, b+1}((X,r),(Y,s))$$

to be the alternating sum of the maps induced by the face maps $\square^{-b-1} \to \square^{-b}$. Denote the associated simple complex by $C((X,r),(Y,s))$ with differential $D = d + (-1)^b \partial$.

The above definitions can be extended to finite symbols. For K and L finite symbols we have the complex $C(K,L)$. Set $C(X,r) = C((\mathrm{pt},0),(X,r)) = \mathrm{Hom}(\mathrm{pt},X)(r)$, and $C(K) = C((\mathrm{pt},0),K)$. There is composition map

$$C(K,L) \otimes C(L,M) \to C(K,M) \ , \quad f \otimes g \mapsto g \circ f \ ,$$

satisfying associativity. An element $F \in C(K,L)^n$ induces a map $F_* : C(K) \to C(L)[n]$.

Given $Z \in \mathcal{Z}^r(X,n)$ define a subcomplex

$$C_{|Z|}(X,r)^{\bullet\bullet} \subset C(X,r)^{\bullet \geq -n} \subset C(X,r)^{\bullet\bullet}$$

where $C_{|Z|}(X,r)^{a,b}$ is defined by the support condition with respect to $\cup_\delta |\delta^* Z|$, δ varying over the face maps $\delta : \square^{-b} \to \square^n$. Similarly given $f \in \mathrm{Hom}(K,L)^{-n}$ there is a subcomplex $C_{|f|}(K,L)$. Here $|f|$ denotes the support of f (we sometimes write just f). An element $f \in \mathrm{Hom}(K,L)^{-n}$ has cycle class $cl(f) \in H^0 C^{\bullet,b}_{|f|}(K,L)$.

(2.1) Proposition. *Given an object $(K^m, f^{m,n})$ of $\mathcal{D}(k)$, there exist, for $m < n$, elements*

$$F^{m,n} \in \bigoplus_{a+b=-(n-m-1), a\leq 0} C^{a,b}_{|f|}(K^m, K^n)$$

such that its $(0, -n+m+1)$-component ${}^0F^{m,n} \in C_f^{\bullet,-(n-m-1)}(K^m,K^n)$ satisfies

$$[{}^0F^{m,n}]_d = cl(f^{m,n}) \in H^0 C_f^{\bullet,-(n-m-1)}(K^m,K^n)$$

and one has the relation

$$(*) \qquad (-1)^n D(F^{m,n}) + \sum_{m<\ell<n} F^{\ell,n} \circ F^{m,\ell} = 0$$

$$\text{in} \bigoplus_{a+b=-(n-m-2), a\leq 0} C_f^{a,b}(K^m,K^n) \, .$$

We call such $(F^{m,n})$ a representative of $(cl(f^{m,n}))$.

Choose $(F^{m,n})$; then the maps

$$F^{m,n}{}_* : C(K^m) \to C(K^n)[-(n-m-1)]$$

satisfy

$$(-1)^n D \circ F^{m,n}{}_* + F^{m,n}{}_* \circ (-1)^m D + \sum F^{\ell,n}{}_* \circ F^{m,\ell}{}_* = 0$$

where D is the differential of $C(K^m)$ or $C(K^n)$, namely $(C(K^m), F^{m,n}{}_*)$ is a C-complex. So we have the total complex,

$$C(K) := Tot\,(\oplus C(K^m), D + \sum F^{m,n}{}_*) \, .$$

It can be shown that $C(K)$ is well-defined independent of the choice of representatives $F^{m,n}$. By definition $H^*(K) = H^*(K, \mathbb{Q}_\ell) := H^* C(K)$.

(2.2) Theorem. *We have the functor of ℓ-adic etale cohomology*

$$H^*(-, \mathbb{Q}_\ell) : \mathcal{D}(k) \to Vec_{\mathbb{Q}_\ell} \, .$$

For K in $\mathcal{D}(k)$, we have $K \otimes_k \bar{k}$ in $\mathcal{D}(\bar{k})$; the cohomology

$$H^*(K \otimes_k \bar{k}, \mathbb{Q}_\ell)$$

is a $G(\bar{k}/k)$-module. We have the functor to the category of Galois modules

$$\mathcal{D}(k) \to \quad \left(G(\bar{k}/k) - Vec_{\mathbb{Q}_\ell}\right) .$$

§3. Polylogarithmic objects.

We define the category of mixed Tate motives to be the triangulated subcategory generated by $\mathbb{Q}(r)$, $r \in \mathbb{Z}$.

We give two types of algebraic cycles, each parametrized by $a \in k^*$. For $a \in k^*$, $f_a := Alt\{t = a\} \in \mathcal{Z}^1(\mathrm{pt}, 1)$, the alternation of the cycle $\{t = a\}$. For $a \in k^* - \{1\}$ and $r \geq 2$, using cubical coordinates and parameters $t_1, \cdots, t_{r-2}$,

$$V_a^r := [t_1, t_2, \cdots, t_{r-1}, 1 - t_1, 1 - \frac{t_2}{t_1}, \cdots, 1 - \frac{t_{r-1}}{t_{r-2}}]$$

and $C_a^r := AltV_a^r \in \mathcal{Z}^r(\mathrm{pt}, 2r-1)$. This was considered by Totaro ($r = 2$) and Bloch ($r \geq 3$). Note

$$\partial C_a^r = \begin{cases} f_{1-a} \circ f_a & \text{if} \quad r = 2 \\ C_a^{r-1} \circ f_a & \text{if} \quad r \geq 3 \end{cases}$$

Define the object

$$L(a) := [(\mathrm{pt}, 0) \xrightarrow{f_a} (\mathrm{pt}, 1)]$$

where $(\mathrm{pt}, 0)$, $(\mathrm{pt}, 1)$ are placed in degrees 0 and 2, respectively. More precisely, $L(a) = (L^m, f^{m,n})$ consists of

$$L^{2m} = (\mathrm{pt}, m) \quad \text{for} \quad m = 0, 1, \quad \text{and} \quad L^n = 0 \quad \text{otherwise;}$$

$$f^{0,2} = f_a, \quad f^{m,n} = 0 \quad \text{otherwise.}$$

For $p \geq 1$ and $a \in k^* - \{1\}$, define the object $K_p(a)$ (called the polylogarithmic object of weight p) by

$$K^{2m} = (\mathrm{pt}, m) \quad \text{for} \quad m = 0, 1, \cdots, p, \quad K^n = 0 \quad \text{otherwise;}$$

$$f^{0,2} = f_{1-a}, \quad f^{2m,2m+2} = f_a \quad \text{for} \quad m = 1, \cdots, p-1,$$

$$f^{0,2m} = C_a^m \quad \text{for} \quad m = 2, \cdots, p \quad \text{and} \quad f^{m,n} = 0 \quad \text{otherwise.}$$

(3.1) Proposition. *(1) $H^\nu(L(a) \otimes \bar{k}, \mathbb{Q}_\ell) = 0$ for $\nu \neq 0$. There is an exact sequence*

$$0 \to \mathbb{Q}_\ell(1) \to H^0(L(a) \otimes \bar{k}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(0) \to 0$$

whose extension class is $[a] \in H^1(G(\bar{k}/k), \mathbb{Q}_\ell(1))$.

(2) $H^\nu(K_p(a) \otimes_k \bar{k}, \mathbb{Q}_\ell) = 0$ for $\nu \neq 0$; The cohomology $H^0(K_p(a) \otimes_k \bar{k}, \mathbb{Q}_\ell)$ has a filtration $W_\bullet$ (the weight filtration), $H^0 = W_0 \supset W_{-2} \supset \cdots \supset W_{-2p} \supset W_{-2p-2} = 0$ such that $Gr_{-2q}^W = \mathbb{Q}_\ell(q)$ for $q = 0, \cdots, p$ and the extension class of the exact sequence

$$0 \to Gr_{-2q-2}^W = \mathbb{Q}_\ell(q+1) \to W_{-2q}/W_{-2q-4} \to Gr_{-2q}^W = \mathbb{Q}_\ell(q) \to 0$$

is $[1-a]$ for $q = 0$ and $[a]$ for $q = 1, \cdots, p-1$.

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