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Canonical subgroups and p-adic vanishing cycles on abelian varieties

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This is a report on a joint work with A. Mokrane [1]. Our motivation is to develop a theory of Siegel p-adic modular forms (and for other Shimura varieties) on the model of the elliptic theory developed by Dwork [8], Katz [9], Coleman [5, 6], ... The first step, achieved in [1], provides analogues of the compact Atkin operator U.

Let k be an algebraically closed field of characteristic $p>0$, $W=W(k)$ be the ring of Witt vectors with coefficients in k and σ be the Frobenius endomorphism of k or W. Let A be an ordinary abelian variety over k of dimension g and let $\mathfrak{M}$ be the formal moduli space of deformations of A over artinien W-algebras with residue field k. By Serre-Tate theorem, there exists a canonical isomorphism of formal W-schemes

$$\mathfrak{M} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}_p}(T_pA(k)\otimes T_p\hat{A}(k), \hat{\mathbb{G}}_m),$$

where $\hat{A}$ is the dual abelian variety of A and T_p is the Tate module. Dwork developed another approach to this structure theorem. He proved that a toric formal Lie group structure on $\mathfrak{M}$ is imposed by a W-morphism $\Phi:\mathfrak{M}\to\mathfrak{M}^{(\sigma)}$ lifting the Frobenius. In particular, the group structure of Serre-Tate is completely determined by the canonical lifting of the Frobenius $\Phi_{\mathrm{can}}:\mathfrak{M}\to\mathfrak{M}^{(\sigma)}$ defined as follows. Let $\mathbf{A}/\mathfrak{M}$ be the universal formal abelian scheme, ${}_p\mathbf{A}$ be the kernel of multiplication by p and ${}_p\mathbf{A}^\circ\subset{}_p\mathbf{A}$ be the neutral connected component. Notice that ${}_p\mathbf{A}^\circ$ is the unique closed subgroup scheme of ${}_p\mathbf{A}$, finite and flat over $\mathfrak{M}$ of rank p^g, that lifts the kernel of the isogeny of Frobenius $A\to A^{(\sigma)}$. Then the morphism Φ_{can} is defined by the isomorphism of formal abelian schemes $\Phi^*_{\mathrm{can}}(\mathbf{A}^{(\sigma)})\simeq\mathbf{A}/{}_p\mathbf{A}^\circ$.

In a global situation, Dwork conjectured that the canonical lifting of the Frobenius is *overconvergent*. This problem is known as the excellent lifting problem. Deligne, Dwork [7] and Lubin-Tate [9] proved this conjecture for families of elliptic curves. Then Dwork [8] used it to prove that the unit L function of the Legendre family of ordinary elliptic curves has a meromorphic continuation to $\mathbb{C}_p$. In [1], we prove the overconvergence for higher dimensions under the assumption $p \geq 3$ and we deduce an application to the study of unit L functions attached to Siegel modular varieties.

In this report, we will review only the overconvergence result. We start by reformulating the problem in modular terms. Let K be a complete discrete valuation field of characteristic 0, with perfect residue field k of characteristic $p > 0$, $\mathcal{O}_K$ be its ring of integers and v_p be its valuation normalized by $v_p(p) = 1$. We put $S = \mathrm{Spec}(\mathcal{O}_K)$ and $S_1 = \mathrm{Spec}(\mathcal{O}_K/p\mathcal{O}_K)$. Let M be a φ-$\mathcal{O}_{S_1}$-module, i.e. a free $\mathcal{O}_{S_1}$-module of finite type equiped with a semi-linear endomorphism $\varphi : M \to M$. We define the Hodge height of M as the (truncated) p-adic valuation of the determinant of a matrix of φ. It is a well defined rational number between 0 and 1. Let A be an S-abelian scheme of relative dimension g, $A_1 = A \times_S S_1$ and ${}_pA$ be the kernel of multiplication by p. The Frobenius of A_1 makes $H^1(A_1, \mathcal{O}_{A_1})$ as a φ-$\mathcal{O}_{S_1}$-module. The problem is to construct, under the assumption that the Hodge height of $H^1(A_1, \mathcal{O}_{A_1})$ is strictly less than a rational number $b(g) > 0$, a *canonical* closed subgroup scheme $H_{\mathrm{can}} \subset {}_pA$, finite and flat over S of rank p^g. If A_k is ordinary, we require that H_{can} is the neutral connected component of ${}_pA$. We solve this problem by studying the ramification of finite flat group schemes over S using the ramification theory of Abbes-Saito [2, 3]. Let G be a finite flat S-group scheme. We define on G a canonical exhaustive decreasing filtration $(G^a, a \in \mathbb{Q}_{\geq 0})$ by closed subgroup schemes, finite and flat over S. For a real number $a \geq 0$, we put $G^{a+} = \cup_{b>a} G^b$ (where $b \in \mathbb{Q}$).

Theorem 1 *Assume that $p \geq 3$ and let e be the absolute ramification index of K and $j = e/(p-1)$. Let A be an S-abelian scheme of relative dimension g such that the Hodge height of $H^1(A_1, \mathcal{O}_{A_1})$ is strictly less than*

$$\inf\left(\frac{1}{p(p-1)}, \frac{p-2}{(p-1)(2g(p-1)-p)}\right).$$

Then the level ${}_pA^{j+}$ of the canonical filtration of ${}_pA$ is finite and flat over S of rank p^g. Moreover, if A_k is ordinary, then ${}_pA^{j+}$ is the neutral connected component of ${}_pA$.

Let $\overline{K}$ be an algebraic closure of K, $\mathcal{O}_{\overline{K}}$ be the integral closure of $\mathcal{O}_K$ in $\overline{K}$, $\overline{S} = \mathrm{Spec}(\mathcal{O}_{\overline{K}})$ and $\overline{s}$ and $\overline{\eta}$ be its closed and generic points. In order to prove Theorem 1, we give a description of the canonical filtration of ${}_pA$ using differential forms. We proceed in two steps. First, we describe the dual filtration on $H^1(A_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$ via the spectral sequence of p-adic vanishing cycles, in terms of filtration by symbols ([4] Section I). Then by a syntomic calculus, we deduce a description of the level ${}_pA^{j+}(\overline{K})^{\perp}$. In particular, we prove that ${}_pA^{j+}(\overline{K})^{\perp} = \ker(\theta(-1))$, where

$$\theta : H^1(A_{\overline{K}}, \mathbb{Z}/p\mathbb{Z}(1)) \longrightarrow H^0(A, \Omega^1_{A/S}) \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}/p\mathcal{O}_{\overline{K}}$$

is a classical homomorphism in Kummer theory. Notice that this simple description is not enough to comput the rank of ${}_pA^{j+}$.

Finally we review the result on p-adic vanishing cycles. Let $\overline{A} = A \times_S \overline{S}$. Consider the cartesian diagram

$$\begin{array}{ccccc} A_{\overline{s}} & \xrightarrow{\overline{i}} & \overline{A} & \xleftarrow{\overline{j}} & A_{\overline{\eta}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{s} & \longrightarrow & \overline{S} & \longleftarrow & \overline{\eta} \end{array}$$

and the étale sheaves on $A_{\overline{s}}$

$$\Psi^q = \overline{i}^* R^q \overline{j}_*(\mathbb{Z}/p\mathbb{Z}(q)).$$

The Kummer exact sequence $0 \to \mu_p \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ on $A_{\overline{\eta}}$ induce a symbol map

$$h_{\overline{A}} : \overline{i}^* \overline{j}_* \mathcal{O}^\times_{A_{\overline{\eta}}} \longrightarrow \Psi^1.$$

We put $U^0\Psi^1 = \Psi^1$ and $U^a\Psi^1 = h_{\overline{A}}(1 + \mathfrak{m}_a \overline{i}^*(\mathcal{O}_{\overline{A}}))$ for a rational number $a > 0$, where $\mathfrak{m}_a = \{x \in \mathcal{O}_{\overline{K}}; v(x) \geq a\}$ and the valuation v is normalized by $v(K) = \mathbb{Z}$.

There is a spectral sequence

$$E_2^{\ell,t} = H^\ell(A_{\overline{s}}, \Psi^t)(-t) \Rightarrow H^{\ell+t}(A_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$$

that induces the exact sequence

$$0 \longrightarrow H^1(A_{\overline{s}}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(A_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{u} H^0(A_{\overline{s}}, \Psi^1)(-1)$$

Theorem 2 *Let $e' = ep/(p-1)$. Under the canonical pairing*

$${}_pA(\overline{K}) \times H^1(A_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathbb{Z}/p\mathbb{Z},$$

we have, for any rational number $a > 0$,

$${}_pA^{a+}(\overline{K})^\perp = \begin{cases} u^{-1}(H^0(A_{\overline{s}}, U^{e'-a}\Psi^1)(-1)) & \text{si } 0 \leq a < e', \\ H^1(A_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z}) & \text{si } a \geq e'. \end{cases}$$

References

[1] A. Abbes, A. Mokrane, *Sous-groupes canoniques et cycles évanescents p-adiques pour les variétés abéliennes*, preprint 2003.

[2] A. Abbes, T. Saito, *Ramification of local fields with imperfect residue fields*, American Journal of Math **124** (2002), 879-920.

[3] A. Abbes, T. Saito, *Ramification of local fields with imperfect residue fields II*, preprint (2002).

[4] S. Bloch, K. Kato, *p-adic étale cohomology*, IHES **63** (1986), 107-152.

[5] R. Coleman, *Classical and overconvergent modular forms*, Invent. Math. **124** (1996), 215-241.

[6] R. Coleman, *p-adic Banach spaces and families of modular forms*, Invent. Math. **127** (1997), 417-479.

[7] B. Dwork, *p-adic cycles*, IHES **37** (1969), 27–115.

[8] B. Dwork, *On Hecke polynomials*, Invent. Math. **12** (1971), 249–256.

[9] N. Katz, *p-adic properties of modular schemes and modular forms*, dans *Modular functions of one variable, III* (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), LNM **350**, p. 69–190.

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