GREEN CURRENTS FOR MODULAR CYCLES IN ARITHMETIC QUOTIENTS OF COMPLEX HYPERBALLS

MASAO TSUZUKI

0. INTRODUCTION AND BASIC NOTATIONS

0.1. Introduction. Let $X$ be a complex manifold and $Y$ its analytic subvariety of codimension $r$. The Green current for $Y$ is defined to be a current $G$ of $(r-1,r-1)$-type on $X$ such that $dd^c G + \delta_Y$ is represented by a $C^\infty$-form of $(r,r)$-type on $X$. In the arithmetic intersection theory developed by Gillet and Soulé, the role played by the algebraic cycles in the conventional intersection theory is replaced with the arithmetic cycles. In a heuristic sense, the Green currents is regarded as the 'archimedean' ingredient of such arithmetic cycles ([2]).

Let us consider the case when $X$ is the quotient of a Hermitian symmetric domain $G/K$ by an arithmetic lattice $\Gamma$ in the semisimple Lie group $G$, and $Y$ is a modular cycle stemming from a modular imbedding $H/H \cap K \hookrightarrow G/K$, where $H$ is a reductive subgroup of $G$ such that $H \cap K$ is maximally compact in $H$. Then inspired by the classical works on the resolvent kernel functions of the Laplacian on Riemannian surfaces and also by a series of works of Miatello and Wallach ([5], [6]), T. Oda posed a plan to construct a Green current for $Y$ making use of a 'secondary spherical function' on $H \backslash G$, giving an evidence for divisorial case with some conjectures. Among many possible choices of the Green currents for a modular cycle $Y$, this construction may provide a way to fix a natural one. If $r = 1$, namely $Y$ is a modular divisor, we already obtained a satisfactory result by introducing the secondary spherical functions properly ([7]). Here we focus on the case when $G/K$ is an $n$-dimensional complex hyperball and $H/H \cap K$ is a complex sub-hyperball of codimension $r > 1$, and show that the same method also works well.

Thanks are due to Professor Takeshi_KO for his interest in this work, a constant encouragement and fruitful discussions which always stimulate the author.

0.2. Notations. The Lie algebra of a Lie group $G$ is denoted by $\text{Lie}(G)$. For a complex matrix $X = (x_{ij})_{ij}$, put $X^* := (\bar{x}_{ji})_{ij}$.

1. INVARIANT TENSORS

Let $n$ and $r$ be integers such that $2 \leq r < n/2$.

Let us consider the two involutions $\sigma$ and $\theta$ in the Lie group $G = U(n,1) := \{g \in GL_{n+1}(\mathbb{C})|g^* I_{n,1} g = I_{n,1}\}$ defined by $\theta(g) = I_{n,1} g I_{n,1}$ and $\sigma(g) = S g S$ respectively. Here $I_{n,1} := \text{diag}(I_n, -1)$ and $S = \text{diag}(I_{n-r}, -I_r, 1)$. Then $K := \{g \in G|\theta(g) = g\} \cong U(n) \times U(1)$ is a maximal compact subgroup in $G$ and $H := \{g \in G|\sigma(g) = g\} \cong U(n-r,1) \times U(r)$ is a symmetric subgroup of $G$ such that $K_H := H \cap K \cong U(n-r) \times U(r) \times U(1)$ is maximally compact in $H$. 
The Lie group $G$ acts transitively on the complex hyperball

$$\mathcal{D} = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n | \sum_{i=1}^{n} |z_i|^2 < 1 \}$$

by the fractional linear transformation $g \cdot z = g_{11} z + z_{12}, g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \in G, \quad z \in \mathbb{C}^n$. (Here the matrix $g \in \text{GL}_{n+1}(\mathbb{C})$ is partitioned into blocks so that $g_{11}$ is an $n \times n$-matrix and $g_{22}$ is a scalar.) Since $K$ is the stabilizer of the origin $0 \in \mathcal{D}$, we have the identification $G/K \cong \mathcal{D}$ of $G$-manifolds assigning the point $z = g \cdot 0$ to $g \in G$. Then $H/K_H \subset G/K$ corresponds to the $H$-orbit of $0$ in $\mathcal{D}$, that is $\mathcal{D}^H := \{ z \in \mathcal{D} | z_{n-r+1} = \cdots = z_n = 0 \}$. In particular the real codimension of $H/K_H$ in $G/K$ is $2r$.

The Lie algebra $\mathfrak{g} := \text{Lie}(G)$ is realized in its complexification $\mathfrak{g}_c = \mathfrak{g}_{n+1}(\mathbb{C})$ as an $\mathbb{R}$-subalgebra of all $X \in \mathfrak{g}_{n+1}(\mathbb{C})$ such that $X^* 1_{n+1} + 1_{n+1} X = 0_{n+1}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{t} := \text{Lie}(K)$ in $\mathfrak{g}$ with respect to the $G$-invariant, non-degenerate bi-linear form $\langle X, Y \rangle = 2^{-1} \text{tr}(XY)$ on $\mathfrak{g}$. For $1 \leq i, j \leq n + 1$, let $E_{i,j} := (\delta_{ui}\delta_{uj})_{uv} \in \mathfrak{g}_{n+1}(\mathbb{C})$ be the matrix unit. The operator $J := \text{ad}(\tilde{Z}_0) \mid \mathfrak{p}$ with $\tilde{Z}_0 := \frac{\sqrt{-1}}{n+1} (\sum_{i=1}^{n} E_{n+1,i} - nE_{n+1,n+1})$ gives a $K$-invariant complex structure of $\mathfrak{p}$, which induces the $K$-invariant decomposition $\mathfrak{p}_c = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with $\mathfrak{p}_\pm$ the $(\pm \sqrt{-1})$-eigenspace of $J$ in $\mathfrak{p}_c$. Since $\mathfrak{p}$ is identified with the tangent space of $G/K$ at $K$, we can extend $J$ to the $G$-invariant complex structure of $G/K$ making the identification $G/K \cong \mathcal{D}$ bi-holomorphic. Put $X_i := E_{i,n+1} (1 \leq i \leq n)$, $X_0 := E_{n,n+1}$. Then $\mathfrak{p}_+ = \sum_{i=0}^{n} \mathbb{C}X_i$, $\mathfrak{p}_- = \sum_{i=0}^{n} \mathbb{C}X_i$ with $X_i = E_{n+i,n+1}$, $X_0 = E_{n+1,n+1}$. Let $\{ \omega_i \}$ and $\{ \overline{\omega}_i \}$ be the basis of $\mathfrak{p}_+^*$ and $\mathfrak{p}_-^*$ dual to $\{ X_i \}$ and $\{ X_i \}$ respectively.

The exterior algebra $\Lambda \mathfrak{p}_c^*$ is decomposed to the direct sum of subspaces $\Lambda^{\rho,q} \mathfrak{p}_c^* := (\Lambda^\rho \mathfrak{p}_+^*) \cap (\Lambda^q \mathfrak{p}_-^*)$ ($\rho, q \in \mathbb{N}$). Put

$$\omega := \sqrt{-1} \frac{1}{2} \sum_{i=0}^{n-1} \omega_i \wedge \overline{\omega}_i \quad (\in \Lambda_{1,1} \mathfrak{p}_c^* \cap \wedge \mathfrak{p}_c^*), \quad \text{vol} := \frac{1}{n!} \omega^n \quad (\in \Lambda_{n,n} \mathfrak{p}_c^* \cap \wedge \mathfrak{p}_c^*).$$

The inner product $\langle X, Y \rangle$ on $\mathfrak{p}$ yields the Hermitian inner product $\langle \cdot, \cdot \rangle$ of $\Lambda \mathfrak{p}_c^*$ in the standard way. Then the Hodge star operator $\ast$ is defined to be the $\mathbb{C}$-linear automorphism of $\Lambda \mathfrak{p}_c^*$ such that $\ast \alpha = \overline{\alpha}$ and such that $\langle \alpha, \beta \rangle \text{vol} = \alpha \wedge \ast \beta$, $(\forall \alpha, \beta \in \Lambda \mathfrak{p}_c^*)$. For $\alpha \in \Lambda \mathfrak{p}_c^*$, let us define the endomorphism $e(\alpha) : \Lambda \mathfrak{p}_c^* \to \Lambda \mathfrak{p}_c^*$ by $e(\alpha) \beta = \alpha \wedge \beta$. As usual, we have the Lefschetz operator $L := e(\omega)$ and its adjoint operator $\Lambda$ acting on the finite dimensional Hilbert space $\Lambda \mathfrak{p}_c^*$ ([8, Chap. V]).

Put $\mathfrak{h} = \text{Lie}(H)$. Then $g$ restricts to a Cartan involution of $\mathfrak{h}$ giving the decomposition $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{t}) \oplus (\mathfrak{h} \cap \mathfrak{p})$. The complex structure $J$ of $\mathfrak{p}$ induces that of $\mathfrak{h} \cap \mathfrak{p}$ by restriction giving the decomposition $(\mathfrak{h} \cap \mathfrak{p})_\pm = \mathfrak{h}_c \cap \mathfrak{p}_\pm = \sum_{i=1}^{n+r} \mathbb{C}X_i$ and $(\mathfrak{h} \cap \mathfrak{p})_- = \mathfrak{h}_c \cap \mathfrak{p}_- = \sum_{i=1}^{n-r} \mathbb{C}X_i$. We introduce two tensors $\omega_H$ and $\eta$ as

$$\omega_H := \frac{\sqrt{-1}}{2} \sum_{i=1}^{n-r} \omega_i \wedge \overline{\omega}_i, \quad \eta := \frac{\sqrt{-1}}{2} \sum_{j=n-r+1}^{n-1} \omega_i \wedge \overline{\omega}_i = \omega - \omega_H - \frac{\sqrt{-1}}{2} \omega_0 \wedge \overline{\omega}_0.$$

The coadjoint representation of $K$ on $\mathfrak{p}_c^*$ is extended to the unitary representation $\tau : K \to \text{GL}(\Lambda \mathfrak{p}_c^*)$ in such a way that $\tau(k)(\alpha \wedge \beta) = \tau(k)\alpha \wedge \tau(k)\beta$ holds for all $\alpha, \beta \in \Lambda \mathfrak{p}_c^*$ and $k \in K$. The differential of $\tau$ is also denoted by $\tau$. 

The irreducible decomposition of the $K$-invariant subspaces $\bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^{*}$ is well-known.

**Lemma 1.** Let $p, q$ be non-negative integers such that $p + q \leq n$. Put

$$F_{p,q} := \{ \alpha \in \bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^{*} | \Lambda(\alpha) = 0 \}.$$  

Then $F_{p,q}$ is an irreducible $K$-invariant subspace of $\bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^{*}$. The $K$-homomorphism $L$ induces a linear injection $\bigwedge^{p-1,q-1}\mathfrak{p}_{\mathbb{C}}^{*} \rightarrow \bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^{*}$ whose image is the orthogonal complement of $F_{p,q}$ in $\bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^{*}$, i.e.,

$$\bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^{*} = F_{p,q} \bigoplus L(\bigwedge^{p-1,q-1}\mathfrak{p}_{\mathbb{C}}^{*}).$$

The $\mathbb{R}$-subspace $a$ of $\mathfrak{g}$ generated by the element $Y_0 := X_0 + \overline{X}_0 \in \mathfrak{p}$ is a maximal abelian subalgebra in $q \cap \mathfrak{p}$ with $q$ the $(-1)$-eigenspace of $d\sigma$, the differential of $\sigma$. Since $(G, H)$ is a symmetric pair, by the general theory, the group $G$ is a union of double cosets $H a_t K$ ($t \geq 0$) with

$$a_t := \exp(t Y_0) = \text{diag} \left( I_{n-1}, \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \right), \quad t \in \mathbb{R}.$$  

Put $A = \{ a_t | t \in \mathbb{R} \}$. Let $M$ be the group of all the elements $k \in H \cap K$ such that

$$\text{Ad}(k)Y_0 = Y_0$$

and put $M = M_0 \cap H$. Then

$$M = \{ \text{diag}(u_1, u_2, u_0, u_0) | u_1 \in U(n-r), u_2 \in U(r-1), u_0 \in U(1) \}.$$  

**Proposition 1.** Let $p$ be an integer such that $0 < p < r$. Put

$$v_0^{(p)} = \frac{1}{n-p+1} \sum_{j=0}^{p} c_{p-j}^{(p)} L^{p-j} \left( (n-p-j+1)\eta^j + \frac{\sqrt{-1}}{2} j(r-j) \omega_0 \Lambda \overline{\omega}_0 \Lambda \eta^{j-1} \right),$$

$$v_1^{(p)} = \frac{-1}{p(n-2p+1)} \sum_{j=0}^{p} c_{p-j}^{(p)} L^{p-j} \left( (p-j)\eta^j + \frac{\sqrt{-1}}{2} j(r-j) \omega_0 \Lambda \overline{\omega}_0 \Lambda \eta^{j-1} \right)$$

with

$$c_{p-j}^{(p)} = (-1)^j \binom{p}{j} \binom{n-p+1}{j} \binom{r-1}{j}, \quad 0 \leq j \leq p.$$  

Then $F_{p,p}^M$ is a two-dimensional space generated by $v_0^{(p)}$ and $v_1^{(p)}$.

For convenience, we put $v_0^{(0)} = 1, v_1^{(0)} = 0$; these are elements of $F_{0,0} = \mathbb{C}$.

2. Secondary spherical functions

Before we state the main theorem of this section, we put a lemma which is important not only here but in the 'global theory' to be developed in §4.

**Lemma 2.** For each integer $p$ with $1 \leq p \leq r$, there exists a unique holomorphic function $s \mapsto \nu_s^{(p)}$ on the domain $\mathbb{C} - L_p$ with

$$L_p = \{ s \in \sqrt{-1} \mathbb{R} | |\text{Im}(s)| \leq 2 \sqrt{(r-p)(n-p-r+2)} \}$$
which takes a positive real value for $s > 0$ and such that

$$\{\nu_s^{(p)}\}^2 = s^2 + 4(r - p)(n - p - r + 2).$$

We have the functional equation $\nu_{-s}^{(p)} = -\nu_s^{(p)}$, $(s \in \mathbb{C} - L_p)$. If $\text{Re}(s) > 0$, then we have $\text{Re}(\nu_s^{(p)}) > \text{Re}(\nu_s^{(p+1)}) > |\text{Re}(s)|$.

For convenience, we put

$$\mu = r - 1, \quad \lambda = n - 2r + 2.$$

Consider the holomorphic function

$$d(s) := \prod_{p=1}^{r} \Gamma(\nu_s^{(p)})^{-1} \Gamma(2^{-1}(\nu_s^{(p)} - \lambda) + 1)^{-1}, \quad s \in \mathbb{C} - L_1$$

and put

$$D = \{s \in \mathbb{C} - L_1 \mid d(s) \neq 0\}, \quad \tilde{D} = \bigcap_{p=1}^{\mu} \{s \in D \mid \text{Re}(\nu_s^{(p)}) + \text{Re}(\nu_s^{(p+1)}) > 4\}.$$

**Theorem 1.** There exists a unique family of $C^\infty$-functions $\phi_s : G - HK \to \wedge^{\mu,\mu} \mathfrak{p}_\mathbb{C}^*$ $(s \in \tilde{D})$ with the following conditions.

(i) For each $g \in G - HK$, the function $s \mapsto \phi_s(g)$ is holomorphic.

(ii) $\phi_s$ has the $(H, K)$-equivariance

$$\phi_s(hgk) = \tau(k)^{-1} \phi_s(g), \quad h \in H, k \in K, g \in G - HK.$$

(iii) $\phi_s$ satisfies the differential equation

$$\Omega \phi_s(g) = (s^2 - \lambda^2) \phi_s(g), \quad g \in G - HK$$

(iv) We have

$$\lim_{t \to +0} t^{2\mu} \phi_s(a_t) = (\omega - \omega_H)^\mu.$$

(v) If $\text{Re}(s) > n$, then $\phi_s(a_t)$ decays exponentially as $t \to +\infty$.

We call the function $\phi_s$ the secondary spherical function.

2.1. Construction of $\phi_s$. We set

$$c(s) := \frac{\Gamma(s + 1) \Gamma(\mu + 2)}{\Gamma((s + n)/2 + 1) \Gamma((s - \lambda)/2 + 1)},$$

and

$$h_s(z) := _2F_1\left(-\frac{s - n}{2} + 1, -\frac{s + \lambda}{2} + 1; \mu + 2; z\right),$$

$$H_s(z) := _2F_1\left(\frac{s - n}{2}, \frac{s + \lambda}{2}; s + 1; 1 - z\right).$$
Proposition 2. Let \( \{\gamma_p\}_{p=0}^{\mu} \) be the sequence of real numbers defined by the recurrence relation:
\[
\gamma_{\mu} = \frac{1}{c_{0}^{(\mu)}}, \quad \gamma_{j} c_{0}^{(j)} = - \sum_{p=j+1}^{\mu} \gamma_{p} c_{p-j}^{(p)}, \quad (0 \leq j < \mu).
\]

Then we have
\[
\phi_{s}(ha_{t}k) = \mu \tau \left\{ \sum_{p=1}^{\mu} \frac{\gamma_{p} (n-p-r+1)p}{c(n_{s}^{(p+1)}) c(n_{s}^{(p)})} \tau(k)^{-1} (f_{01}^{(p)}(s; \tanh^2 t) v_{0}^{(p)} + f_{11}^{(p)}(s; \tanh^2 t) v_{1}^{(p)}) 
+ \frac{\gamma_{0}}{c(n_{s}^{(1)})} f_{01}^{(0)}(s; \tanh^2 t) v_{0}^{(0)} \right\} \quad \forall (h, t, k) \in H \times (0, \infty) \times K.
\]

Here the functions \( f_{ij}^{(p)} \) are given as follows.
- For \( p > 0 \),
  \[
  f_{01}^{(p)}(s; z) = f_{00}^{(p)}(s; z) a_{01}^{(p)}(s; z) + f_{01}^{(p)}(s; z) a_{11}^{(p)}(s; z),
  \]
  \[
  f_{11}^{(p)}(s; z) = f_{10}^{(p)}(s; z) a_{01}^{(p)}(s; z) + f_{11}^{(p)}(s; z) a_{11}^{(p)}(s; z)
  \]

with
\[
 a_{01}^{(p)}(s; z) = \frac{\nu_{s}^{(p+1)} + \nu_{s}^{(p)}}{(n-p-r+1)p} \times \left( z(1-z) \frac{d}{dz} + \frac{r-p}{n-2p+1} (1-z) \right) H_{\nu_{s}^{(p+1)}}(z),
\]
\[
 a_{11}^{(p)}(s; z) = \frac{\nu_{s}^{(p+1)} + \nu_{s}^{(p)}}{1-z} \times \left( (1-z) \frac{d}{dz} + \frac{r-p}{n-2p+1} (1-z) \right) H_{\nu_{s}^{(p+1)}}(z)
\]
and
\[
 f_{10}^{(p)}(s; z) = (1-z)^{(-\nu_{s}^{(p+1)}+n)/2+1} H_{\nu_{s}^{(p+1)}}(z),
\]
\[
 f_{11}^{(p)}(s; z) = z^{(-\nu_{s}^{(p+1)}+n)/2+1} H_{\nu_{s}^{(p+1)}}(z),
\]
\[
 f_{00}^{(p)}(s; z) = \frac{(1-z)^{(-\nu_{s}^{(p+1)}+n)/2}}{(n-p-r+1)p} \times \left( z(1-z) \frac{d}{dz} + \frac{r-p}{n-2p+1} (1-z) \right) H_{\nu_{s}^{(p+1)}}(z),
\]
\[
 f_{01}^{(p)}(s; z) = - \frac{z^{(-\nu_{s}^{(p+1)}+n)/2}}{(n-p-r+1)p} \times \left( z(1-z) \frac{d}{dz} + \frac{r-p}{n-2p+1} (1-z) \right) H_{\nu_{s}^{(p+1)}}(z).
\]

- For \( p = 0 \),
  \[
  f_{01}^{(0)}(s; z) = \frac{2 z^{(-\nu_{s}^{(1)}+n)/2}}{\nu_{s}^{(1)}+n} 2F1 \left( \frac{\nu_{s}^{(1)} - n}{2} + 1, \frac{\nu_{s}^{(1)} + \lambda}{2}; \nu_{s}^{(1)} + 1; 1-z \right).
  \]
2.2. Some properties of the secondary spherical function.

**Theorem 2.** Let \( \phi_s(s \in \tilde{D}) \) be the secondary spherical function constructed in Theorem 1.

- There exist \( \mu \) polynomial functions \( a_\alpha(s) \) with values in \( (\wedge^{\mu,\mu} \mathfrak{p}_{\mathbb{C}}^{*})^{M} \), positive number \( \epsilon \) and \( (\wedge^{\mu,\mu} \mathfrak{p}_{\mathbb{C}}^{*})^{l \vee I} \)-valued holomorphic functions \( b_i(s, z) (i = 0, 1, 2) \) on \( \{(s, z)|s \in \tilde{D}, |z| < \epsilon\} \) such that
  \[
  a_0(s) = (\omega - \omega_H)^\mu, \\
  a_\alpha(-s) = a_\alpha(s), \quad \deg(a_\alpha(s)) \leq 2\alpha
  \]
  and such that
  \[
  \phi_s(a_i) = \sum_{\alpha=0}^{\mu-1} \frac{a_\alpha(s)}{z^{\mu-\alpha}} + b_0(s; z) + b_1(s; z) \log z + b_2(s; z) z^{\mu+2}(\log z)^2,
  \]
  \( s \in \tilde{D}, \ z = \tanh^2 t \in (0, \epsilon) \).

- There exists a positive number \( \epsilon' \), \( (\wedge^{\mu,\mu} \mathfrak{p}_{\mathbb{C}}^{*})^{M} \)-valued holomorphic functions \( f^{(p)}(s; y) (0 \leq p \leq \mu) \) on \( \{(s, y)||y| < \epsilon', \Re(s) > n\} \) such that
  \[
  \phi_s(a_i) = \sum_{p=0}^{\mu} y^{(\nu_s^{(p)}+n)/2} f^{(p)}(s; y), \quad \Re(s) > n, \ y = \frac{1}{\cosh^2 t} \in (0, \epsilon')
  \]

2.3. The function \( \psi_s \). For each \( s \in \tilde{D} \), let us define the function \( \psi_s : G - HK \to \wedge^{r,r} \mathfrak{p}_{\mathbb{C}}^{*} \) by

(2)
\[
\psi_s(g) = \sum_{i,j=0}^{n-1} R_{X_i \overline{X}_j} \phi_s(g) \Lambda \omega_i \Lambda \overline{\omega}_j, \quad g \in G - HK.
\]

**Theorem 3.**

- The function \( \psi_s \) is \( C^\infty \) on \( G - HK \) and satisfies
  \[
  \psi_s(hgk) = \tau(k)^{-1} \psi_s(g), \quad \forall h \in H, \forall g \in G - HK, \forall k \in K.
  \]

- There exist \( \mu \) \( (\wedge^{r,r} \mathfrak{p}_{\mathbb{C}}^{*})^{M} \)-valued polynomial functions \( \tilde{c}_\alpha(s) \), positive number \( \epsilon \) and \( (\wedge^{r,r} \mathfrak{p}_{\mathbb{C}}^{*})^{M} \)-valued holomorphic functions \( d_i(s, z) (i = 0, 1, 2) \) on \( \{(s, z)|s \in \tilde{D}, |z| < \epsilon\} \) such that
  \[
  \tilde{c}_0(s) = -\frac{\sqrt{-1}}{2} \frac{(r-1)!}{(n-r)!} (\omega - \omega_H)^r, \\
  \tilde{c}_\alpha(-s) = \tilde{c}_\alpha(s), \quad \deg(\tilde{c}_\alpha(s)) \leq 2\alpha
  \]
  and
  \[
  \psi_s(a_t) = (s^2 - \lambda^2) \sum_{\alpha=0}^{\mu-1} \frac{\tilde{c}_\alpha(s)}{z^{\mu-\alpha}} + d_0(s; z) + d_1(s; z) \log z + d_2(s; z) z^\mu(\log z)^2,
  \]
  \( s \in \tilde{D}, \ z = \tanh^2 t \in (0, \epsilon) \).
• There exists a positive number $\epsilon'$, $(\wedge^r \mathfrak{p}_C)\odot M$-valued holomorphic functions $g^{(p)}(s;y)(0 \leq p \leq r)$ on $\{(s,y)|\text{Re}(s) > n, |y| < \epsilon'\}$ such that

$$\psi_s(a_t) = \sum_{p=0}^{r} y^{(\nu s^{(p)}+n)/2} g^{(p)}(s;y), \text{ Re}(s) > n, y = \frac{1}{\cosh^2 t} \in (0, \epsilon')$$

### 3. Poincaré series

Let $\Gamma$ be a discrete subgroup of $G$. We assume that $(G, H, \Gamma)$ is arranged as follows. There exists a connected reductive $\mathbb{Q}$-group $G$, a $\mathbb{Q}$-subgroup $H$ of $G$ and an arithmetic subgroup $\Delta$ of $G(\mathbb{Q})$ such that there exists a morphism of Lie groups from $G(\mathbb{R})$ onto $G$ with compact kernel which maps $H(\mathbb{R})$ onto $H$ and $\Delta$ onto $\Gamma$.

#### 3.1. Invariant measures. Let $dk$ and $dk_0$ be the Haar measures of compact groups $K$ and $K_H$ with total volume 1. Then we can take a unique Haar measure $dg$ (resp. $dh$) of $G$ (resp. $H$) such that the quotient measure $\frac{dg}{dk}$ (resp. $\frac{dh}{dk_0}$) corresponds to the measure on the symmetric space $G/K$ (resp. $H/K_H$) determined by the invariant volume form $vol$ (resp. $vol_H$).

**Lemma 3.** For any measurable functions $f$ on $G$ we have

$$\int_G f(g) \, dg = \int_H dh \int_K dk \int_0^\infty f(ha_t k) \varrho(t) \, dt$$

with $dt$ the usual Lebesgue measure on $\mathbb{R}$ and

$$\varrho(t) = 2c_r (\sinh t)^{2r-1} (\cosh t)^{2n-2r+1}, \quad c_r = \frac{\pi^{r}}{\mu!}.$$

#### 3.2. Currents defined by Poincaré series. Let $\mathfrak{F}$ denote the set of the families of functions $\varphi_s(s \in \tilde{D})$ or $\varphi_s = \partial_s \psi_s(s \in \tilde{D})$ with some differential operator $\partial_s$ with holomorphic coefficient on $\tilde{D}$.

For $\varphi_s \in \mathfrak{F}$, let us introduce the Poincaré series

$$(3) \quad \tilde{P}(\varphi_s)(g) = \sum_{\gamma \in \Gamma_H \backslash \Gamma} \varphi_s(\gamma g) \quad g \in G,$$

which is the most basic object in our investigation. First of all, we discuss its convergence in a weak sense. Note that $\varphi_s$ takes its values in the finite dimensional Hilbert space $\wedge \mathfrak{p}_C$ with the norm $||\alpha|| = (\alpha|\alpha)^{1/2}$.

**Theorem 4.** The function in $s$ defined by the integral

$$\tilde{P}(||\varphi_s||)(g) := \int_{\Gamma \backslash G} \left(\sum_{\gamma \in \Gamma_H \backslash \Gamma} ||\varphi_s(\gamma g)||\right) \, dg$$

is locally bounded on $\text{Re}(s) > n$. For each $s$ with $\text{Re}(s) > n$, the series (3) converges absolutely almost everywhere in $g \in G$ to define an $L^1$-function on $\Gamma \backslash G$.

If $\Gamma$ is neat, then the quotient space $\Gamma \backslash G/K$ acquires a structure of complex manifold from the one on $G/K \cong \mathbb{D}$. Let $\pi : G/K \rightarrow \Gamma \backslash G/K$ be the natural projection. Let $A(\Gamma \backslash G/K)$ denote the space of $C^\infty$-differential forms on $\Gamma \backslash G/K$ and $A_c(\Gamma \backslash G/K)$ the
of compactly supported forms. Given $\alpha \in \mathcal{A}(\Gamma \backslash G/K)$, we have a unique $C^\infty$-function $\tilde{\alpha} : G \to \bigwedge \mathfrak{p}_C$ such that $\tilde{\alpha}(\gamma g k) = \tau(k)^{-1} \tilde{\alpha}(g)$, $(\gamma \in \Gamma, k \in K)$ and such that

$$\langle (\pi^* \alpha)(g K), (\wedge dL_g)(\xi_o) \rangle = \langle \tilde{\alpha}(g), \xi_o \rangle, \quad g \in G, \xi_o \in \bigwedge \mathfrak{p} = \bigwedge T_o(G/K)$$

holds. Here $L_g$ denotes the left translation on $G/K$ by the element $g$ and we identify $\mathfrak{p}$ with $T_o(G/K)$, the tangent space of $G/K$ at $o = eK$.

For any left $\Gamma$-invariant continuous function $f$ on $G$, put

$$\mathcal{I}_H(f ; g) = \int_{\Gamma \backslash H} f(hg) dh, \quad g \in G.$$ 

We already discussed the convergence problem of this integral in [7, 3.2]. For convenience we recall the result. If $\Gamma$ is co-compact, we take a compact fundamental domain $\Sigma^\Gamma$ for $\Gamma$ in $G$ and $t_{\Sigma^\Gamma}$ the constant function 1. Hence $G = \Gamma \Sigma^\Gamma$ in this case. If $\Gamma$ is not co-compact, then one can fix a complete set of representatives $P^i (1 \leq i \leq h)$ of $\Delta$-conjugacy classes of $\mathbb{Q}$-parabolic subgroups in $G$ together with $\mathbb{Q}$-split tori $G_m \cong A^i$ in the radical of $P^i$ such that an eigencharacter of $\text{Ad}(t) (t \in G_m)$ in the Lie algebra of $P^i$ is one of $t^j (j = 0, 1, 2)$.

For each $i$, let $N^i$ and $N^i$ be the images in $G$ of $A^i(\mathbb{R})$ and the unipotent radical of $P^i(\mathbb{R})$ respectively. Then we can choose a Siegel domain $\Sigma^i$ in $G$ with respect to the Iwasawa decomposition $G = N^i T^i K$ for each $i$ such that $G$ is a union of $\Gamma \Sigma^i (1 \leq i \leq h)$. Let $t_{\Sigma^i} : \Sigma^i \to (0, \infty)$ be the function $t_{\Sigma^i} (n_i t_i k) = t_i , (n_i t_i k \in \Sigma^i)$. Here $t_i$ denote the image of $t \in G_m(\mathbb{R}) \cong A^i(\mathbb{R})$ in $T^i$.

Given $\delta \in (2rn^{-1}, 1)$, let $\mathcal{M}_\delta$ be the space of all left $\Gamma$-invariant $C^\infty$-functions $f : G \to \bigwedge \mathfrak{p}_C$ with the $K$-equivariance $f(gk) = \tau(k)^{-1} f(g)$ such that for any $\epsilon \in (0, \delta)$ and $D \in U(g_\mathbb{C})$ the estimation

$$||R_D \varphi(g)|| < t_{\Sigma^i}(g)^{(2-\epsilon)n}, \quad \forall g \in \Sigma^i, \forall \epsilon$$

holds.

**Proposition 3.** Let $f \in \mathcal{M}_\delta$ with $\delta \in (2rn^{-1}, 1)$ and $D \in U(g_\mathbb{C})$.

- We have

$$\mathcal{I}_H(||R_D f|| ; a_t) < e^{(2-\epsilon)nt}, \quad t \geq 0$$

for any $\epsilon \in (2rn^{-1}, \delta)$. The function $\mathcal{I}_H(f ; g)$ is of class $C^\infty$, belongs to $C_\Gamma^\infty$ and

$$\mathcal{I}_H(R_D f ; g) = R_D \mathcal{I}_H(f ; g), \quad g \in G.$$

- For any $\{ \varphi_s \} \in \mathcal{F}$, the integral

$$\int_{\Gamma \backslash G} |(\tilde{P}(\varphi_s)(g)| R_D f(g))| dg$$

is finite if $\text{Re}(s) > 3n - 2r$. We have

$$\int_{\Gamma \backslash G} (\tilde{P}(\varphi_s)(g)| R_D f(g)) dg = \int_0^\infty \vartheta(t) (\varphi_s(a_t)| R_D \mathcal{I}_H (f ; a_t)) dt.$$

Let $\vartheta(t)$ be a locally constant function on $\mathbb{R}$ that is supported in $((2rn^{-1} - \delta)\mathbb{R})^+$ and $\vartheta(t) = 1$ for $t \in \mathbb{R}$.
Proposition 4. There exists a unique current \( P(\varphi_s) \) on \( \Gamma \backslash \Gamma \backslash G/K \) such that
\[
\langle P(\varphi_s), \alpha \rangle = \int_{\Gamma \backslash G} (\tilde{P}(\varphi_s)(g)|\tilde{\alpha}(g))d\tilde{g}
= \int_0^\infty \varrho(t) (\varphi_s(a_t)|\mathcal{J}_H(\tilde{\alpha};a_t))\,dt,
\alpha \in A_c(\Gamma \backslash G/K)
\]
Let \( \partial_s \) be a holomorphic differential operator on \( \tilde{D} \). Then for any \( \alpha \in A_c(\Gamma \backslash G/K) \), the function \( s \mapsto \langle P(\varphi_s), \alpha \rangle \) is holomorphic on \( \tilde{D} \) and \( \partial_s \langle P(\varphi_s), \alpha \rangle = \langle P(\partial_s \varphi_s), \alpha \rangle \).

Definition
For \( s \in \mathbb{C} \) with \( \text{Re}(s) > n \), we put
\[
\tilde{G}_s := \tilde{P}(\phi_s), \quad \tilde{\Psi}_s := \tilde{P}(\psi_s),
G_s := P(\psi_s), \quad \Psi_s := P(\psi_s).
\]
The current \( G_s \) and \( \Psi_s \) on \( \Gamma \backslash G/K \) are of type \((r-1, r-1)\) and of type \((r, r)\) respectively.

4. Spectral expansion

In this section we investigate the spectral expansion of the functions \( \delta_{j,s} \tilde{G}_s \) with
\[
\delta_{j,s} := \frac{1}{j!} \left( -\frac{1}{2s} \frac{d}{ds} \right)^j, \quad j \in \mathbb{N}
\]
to obtain a meromorphic continuation of the current-valued function \( s \mapsto G_s \), which is already holomorphic on the half plane \( \text{Re}(s) > n \).

4.1. Spectral expansion. In order to describe the spectral decomposition of the function \( \delta_{\mu,s} \tilde{G}_s \) we need some preparations.

For \( q \geq 0 \), let \( L^q_{\mathbb{C}}(\tau) \) denote the Banach space of all measurable functions \( f : G \to \mathbb{C} \) such that \( f(\gamma g k) = \tau(k)^{-1}f(g) \) (\( \forall \gamma \in \Gamma, \forall k \in K \)) and \( \int_{\Gamma \backslash G} \|f(g)\|^{q}d\tilde{g} < \infty \).

For \( 0 \leq d \leq n \), let \( L^q_{\mathbb{C}}(\tau)^{(d)} \) denote the subspace of those functions \( f \in L^q_{\mathbb{C}}(\tau) \) with values in \( \bigwedge^{d,d} \mathfrak{p}_{\mathbb{C}}^* \). The inner product of two functions \( f_1 \) and \( f_2 \) in \( L^q_{\mathbb{C}}(\tau)^{(d)} \) is given as
\[
\langle f_1|f_2 \rangle = \int_{\Gamma \backslash G} (f_1(g)|f_2(g))d\tilde{g}.
\]

Let \( \tilde{\Delta} \) be the operator on \( L^q_{\mathbb{C}}(\tau)^{(d)} \) whose action on the smooth functions in \( L^q_{\mathbb{C}}(\tau)^{(d)} \) is induced by \(-R_\Omega\). For each \( 0 \leq d \leq n \), let \( \{\lambda_n^{(d)}\}_{n \in \mathbb{N}} \) be the increasing sequence of the eigenvalues of the bidegree \((d,d)\)-part of \( \tilde{\Delta} \) such that each eigenvalue occurs with its multiplicity. Choose an orthonormal system \( \{\alpha_n^{(d)}\}_{n \in \mathbb{N}} \) in \( L^2_{\mathbb{C}}(\tau)^{(d)} \) consisting of automorphic forms such that \( \tilde{\Delta} \alpha_n^{(d)} = \lambda_n^{(d)} \alpha_n^{(d)} \) for each \( n \) and put \( L^2_{\mathbb{C}}(\tau)^{(d)} \) to be the closed span of the functions \( \alpha_n^{(d)} \) in \( L^2_{\mathbb{C}}(\tau)^{(d)} \). When \( \Gamma \) is co-compact we have \( L^2_{\mathbb{C}}(\tau)^{(d)} = L^2_{\Gamma}(\tau)^{(d)} \). Otherwise we need the Eisenstein series to describe the orthogonal complement of \( L^2_{\mathbb{C}}(\tau)^{(d)} \).

Recall the parabolic subgroups \( P^i \) used to construct the Siegel domains \( \mathfrak{G}^i \) (see 3.2). Let \( P^i = M_0^i T^i N^i \) be its Langlands decomposition with \( M_0^i := Z_K(T^i) \). For each \( i \) let \( \Gamma_{P^i} = \Gamma \cap P^i \) and \( \Gamma M_0^i = M_0^i \cap (\Gamma_{P^i} N^i) \). Then \( \Gamma M_0^i \) is just a finite subgroup of the compact group \( M_0^i \).
For a vector $u \in V^{(d)} := (\bigwedge^{d,d} \mathfrak{p}_{\mathbb{C}})^{\Gamma} \otimes \overline{\mathfrak{g}}_{\mathbb{C}}$ and a complex number $s$, let us define the function $\varphi_{s}^{i}(u; g)$ on $G$ using the Iwasawa decomposition $G = N^{i}T^{i}K$ by

$$\varphi_{s}^{i}(u; n_{i} t_{k} k) = t^{s+n_{i}} \tau(k)^{-1} u, \quad n_{i} \in N^{i}, t > 0, k \in K.$$  

Then the Eisenstein series associated with $u$ is defined by the infinite series

$$E^{i}(s; u) = \sum_{\gamma \in \Gamma \backslash G} \varphi_{s}^{i}(u; \gamma g), \quad g \in G$$

By the general theory, the series is convergent in $\Re(s) > n$ normally and the function $g \mapsto E^{i}(s; u; g)$ is an automorphic form on $\Gamma \backslash G$. Moreover there exists a family of linear maps $E^{i}(s)$ from $V^{(d)}$ to the space of automorphic forms on $\Gamma \backslash G$, which depends meromorphically on $s \in \mathbb{C}$ and is holomorphic on the imaginary axis, such that $(E^{i}(s)(u))(g) = E^{i}(s; u; g)$ coincides with (4) when $\Re(s) > n$. For each $1 \leq i \leq h$, let $\Omega_{M_{0}^{i}}$ be the Casimir element of $M_{0}^{i}$ corresponding to the invariant form $\langle X, Y \rangle$ on its Lie algebra. Then if $u \in V^{(d)}$ is an eigenvector of $\tau(\Omega_{M_{0}^{i}})$ with eigenvalue $c \in \mathbb{C}$, then $R_{\Omega}E^{i}(s; u) = (s^{2} - n^{2} + c) E^{i}(s, u)$ for any $s \in \mathbb{C}$ where $E^{i}(s)$ is regular.

**Lemma 4.** For $0 \leq p \leq d$ and $\epsilon \in \{0, 1\}$, let $W^{(d)}(p; \epsilon)$ be the eigenspace of $\tau(\Omega_{M_{0}^{i}})$ on $V^{(d)}$ corresponding to the eigenvalue $(2p - \epsilon)(2n - 2p + \epsilon)$. Then we have the orthogonal decomposition

$$V^{(d)} = \bigoplus_{p=0}^{\mu} \bigoplus_{\epsilon \in \{0, 1\}} W^{(d)}(p; \epsilon).$$

For each index $(d, i, p, \epsilon)$, fix an orthonormal basis $B^{(d)}(p; \epsilon)$ of the space $W^{(d)}(p; \epsilon)$.

**4.2. Some properties of Eisenstein period.**

**Propostion 5.** • For $1 \leq i \leq h$ and $u \in V^{(d)}$, there exists a unique $\bigwedge^{d,d} \mathfrak{p}_{\mathbb{C}}$-valued meromorphic function $\mathcal{P}^{i}_{H}(s; u)$ on $\mathbb{C}$ which is regular and has the value given by the absolutely convergent integral $\int_{H} E^{i}(s; u; \epsilon)$ at any regular point $s \in \mathbb{C}$ of $E^{i}(s; u)$ in $|\Re(s)| < 1 - 2rn^{-1}$.

• Let $1 \leq i \leq h$ and $1 \leq p \leq d$. Then for any $u \in W^{(d)}(p; 1)$, we have $\mathcal{P}^{i}_{H}(s; u) = 0$ identically.

**4.3. Meromorphic continuation and functional equations.** Put $w := (\omega - \omega_{H})^{\mu}$.

**Theorem 5.** Let $\Re(s) > 3n - 2r$. Then there exists $\epsilon > 0$ such that the function $\delta_{\mu, s} G_{s}(g)$ belongs to the space $L^{2+\epsilon}(\tau)^{(\mu)}$. The spectral expansion of $\delta_{\mu, s} G_{s}$ is given as

$$\delta_{\mu, s} G_{s} = \sum_{m=0}^{\infty} \frac{4(|J_{H}(\hat{\Delta}_{m}^{(\mu)}; \epsilon)|}{\mu! (\lambda^{2} - \lambda_{m}^{(\mu)} - s^{2})^{r}} \hat{\Delta}_{m}^{(\mu)}$$

$$+ \sum_{p=0}^{\mu} \frac{1}{4n \sqrt{-1}} \int_{\sqrt{-1} \mathbb{R}} \sum_{i=1}^{h} \sum_{u \in B^{(d)}(p; 0)} \frac{4(w|J_{H}(E^{i}(\zeta; u); \epsilon))}{\mu! (\zeta^{2} - (\nu_{p+1}^{(\epsilon)})^{2})^{r}} E^{i}(\zeta; u) d\zeta,$$

where the summations in the right-hand side of this formula are convergent in $L^{2}(\tau)^{(\mu)}$. 

Let $\mathcal{X}_\Gamma(\tau)$ be the space of $C^\infty$-functions $\tilde{\beta} : G \to \bigwedge \mathfrak{p}_G^*$ with compact support modulo $\Gamma$ such that $\tilde{\beta}(\gamma g k) = \tau(k)^{-1}\tilde{\beta}(g) \ (\forall \gamma \in \Gamma, \forall k \in K)$.

**Theorem 6.** Let $L_1$ be the interval on the imaginary axis defined by (1). Let $0 \leq j \leq \mu$. Then for each $\tilde{\beta} \in \mathcal{X}_\Gamma(\tau)$ the holomorphic function $\tilde{\beta} \mapsto \mathcal{S}_j(s, \tilde{\beta}) := \langle \delta_{j,s} \tilde{G}_s | \tilde{\beta} \rangle$ on $\text{Re}(s) > n$ has a meromorphic continuation to the domain $\mathbb{C} - L_1$. A point $s_0 \in \mathbb{C} - L_1$ with $\text{Re}(s_0) \geq 0$ is a pole of the meromorphic function $\mathcal{S}_j(s, \beta)$ if and only if there exists an $m \in \mathbb{N}$ such that $\langle w| J_H(\tilde{\alpha}_m^{(\mu)} ; e) \rangle \neq 0$, $\langle \tilde{\alpha}_m^{(\mu)} | \tilde{\beta} \rangle \neq 0$ and $s_0^2 - \lambda^2 = -\lambda_m^{(\mu)}$. In this case, the function

$$\mathcal{S}_j(s, \beta) - \sum_{m \in \mathbb{N}; \lambda_m^{(\mu)} = s_0^2 - \lambda^2} \frac{4 \langle w| J_H(\tilde{\alpha}_m^{(\mu)} ; e) \rangle \langle \tilde{\alpha}_m^{(\mu)} | \tilde{\beta} \rangle}{\mu! (s_0^2 - s^2)^{j+1}}$$

is holomorphic at $s = s_0$. We have the functional equation

$$\mathcal{S}_j(-s, \tilde{\beta}) - \mathcal{S}_j(s, \tilde{\beta}) = (-1)^\mu \delta_j, s \left( \sum_{p=0}^{\mu} \frac{\langle \tilde{\mathcal{E}}_p^{(\mu)}(\nu^{(p+1)}); \tilde{\beta} \rangle}{2 \nu^{(p+1)}} \right).$$

**5. GREEN CURRENTS**

We put the Kähler form $\omega$ on $\Gamma \backslash G/K$ such that $\tilde{\omega}(g) = \omega (\forall g \in G)$. The metric on $\Gamma \backslash G/K$ corresponding to $\omega$ defines the Laplacian $\triangle$, the Lefschetz operator and its adjoint $\Lambda$ acting on the space of forms and currents on $\Gamma \backslash G/K$.

**5.1. Currents defined by modular cycles.** Let $D$ be the image of the map $\Gamma_H \backslash H/K_H - \Gamma \backslash G/K$ induced by the natural holomorphic inclusion $H/K_H \hookrightarrow G/K$. Then $D$, a closed complex analytic subset of $\Gamma \backslash G/K$, defines an $(r, r)$-current $\delta_D$ on $\Gamma \backslash G/K$ by the integration

$$\langle \delta_D, \alpha \rangle = \int_{D_{ns}} j^* \alpha, \quad \alpha \in A_c(\Gamma \backslash G/K).$$

Here $j : D \hookrightarrow \Gamma \backslash G/K$ is the natural inclusion and $D_{ns}$ is the smooth locus of $D$. Since $\delta_D$ is real and closed, it defines a cycle on $\Gamma \backslash G/K$ of real codimension $2r$ ([4, p.32–33]).

**5.2. Differential equations.**

**Theorem 7.** Let $\text{Re}(s) > n$. Then we have

$$(\triangle + s^2 - \lambda^2) G_s = -4\Lambda \delta_D,$$

$$\triangle \Psi_s = (\lambda^2 - s^2)(\Psi_s - 2\sqrt{-1} \delta_D),$$

$$\partial\bar{\partial} G_s = \Psi_s - 2\sqrt{-1} \delta_D.$$
5.3. Main theorem. Let $A^{p,q}_{(2)}(\Gamma \backslash G/K)$ be the Hilbert space of the measurable $(p, q)$-forms on $\Gamma \backslash G/K$ with the finite $L^2$-norm $\|\alpha\| := \int_{\Gamma \backslash G/K} \alpha \wedge *\overline{\alpha}$. For each $c \in \mathbb{C}$, let $A^{p,q}_{(2)}(\Gamma \backslash G/K ; c)$ be the $c$-eigenspace of the Laplacian $\Delta$ acting on $A^{p,q}_{(2)}(\Gamma \backslash G/K)$. In particular, $\mathcal{H}^{p,q}_{(2)}(\Gamma \backslash G/K) := A^{p,q}_{(2)}(\Gamma \backslash G/K ; 0)$ is the space of the harmonic $L^2$-forms of $(p, q)$-type. For each $p$, let $\mathcal{E}^{(\mu)}_{p}(\nu)$ be the $C^\infty$-form of $(\mu, \mu)$-type on $\Gamma \backslash G/K$ corresponding to the function $\tilde{\mathcal{E}}^{(\mu)}_{p}(\nu)$ on $G$ defined by (5). Then Theorem 6 immediately gives us the following theorem.

**Theorem 8.** There exists a meromorphic family of $(\mu, \mu)$-currents $G_s (s \in \mathbb{C} - L_1)$ on $\Gamma \backslash G/K$ with the following properties.

- For $s \in \mathbb{C}$ with $\text{Re}(s) > n$, it is given by
  $$\langle G_s, *\overline{\alpha} \rangle = \frac{1}{(r-1)\pi^r} \int_0^\infty \varrho(t) \langle \phi_s(a_t) | J_H(\tilde{\alpha}; a_t) \rangle \, dt, \quad \alpha \in A_c(\Gamma \backslash G/K).$$

- A point $s_0 \in \mathbb{C} - L_1$ with $\text{Re}(s) \geq 0$ is a pole of $G_s$ if and only if there exists an $L^2$-form $\alpha \in A^{r-1,r-1}_{(2)}(\Gamma \backslash G/K ; (n-2r+2)^2 - s_0^2)$ such that
  $$\int_D j^*(\omega \wedge \overline{\alpha}) \neq 0.$$

In this case $s_0$ is a simple pole with the residue
  $$\text{Res}_{s=s_0} G_s = \frac{2}{s_0} \sum m \left( \int_D j^*(\omega \wedge \overline{\alpha}_m) \right) \cdot \alpha_m.$$

Here $\{\alpha_m\}$ is an arbitrary orthonormal basis of $A^{r-1,r-1}_{(2)}(\Gamma \backslash G/K ; (n-2r+2)^2 - s_0^2)$.

- The functional equation
  $$G_{-s} - G_s = (-1)^{r-1} \sum_{p=0}^{r-1} \frac{\mathcal{E}^{(r-1)}_{p}(\nu_{s}^{(p+1)})}{2\nu_{s}^{(p+1)}}, \quad s \in \mathbb{C} - L_1$$
holds.

**Theorem 9.** There exists a meromorphic family of $(r, r)$-currents $\Psi_s (s \in \mathbb{C} - L_1)$ on $\Gamma \backslash G/K$ with the following properties.

- For $s \in \mathbb{C}$ with $\text{Re}(s) > n$, it is given by
  $$\langle \Psi_s, *\overline{\alpha} \rangle = \frac{1}{(r-1)\pi^r} \int_0^\infty \varrho(t) \langle \psi_s(a_t) | J_H(\tilde{\alpha}; a_t) \rangle \, dt, \quad \alpha \in A_c(\Gamma \backslash G/K).$$

- $\Psi_s$ is holomorphic at $s = n - 2r + 2$.

**Definition**

We define the $(r - 1, r - 1)$-current $\mathcal{G}$ on $\Gamma \backslash G/K$ to be the quarter of the constant term of the Laurent expansion of $G_s$ at $s = \lambda$. Namely, if $\{\alpha_m\}$ is any orthonormal basis of $\mathcal{H}^{r-1,r-1}_{(2)}(\Gamma \backslash G/K)$, then we put
  $$\mathcal{G}(x) = \frac{1}{4} \lim_{s \to \lambda} \left( G_s(x) - \frac{2}{n - 2r + 2} \sum_m \int_D j^*(\omega \wedge \overline{\alpha}_m) \frac{\alpha_m(x)}{s - (n - 2r + 2)} \right).$$
Theorem 10. We have the equation

$$dd_{c}G = \frac{\sqrt{-1}}{2}\Psi_{n-2r+2} + \delta_{D}, \quad \Delta \Psi_{n-2r+2} = 0$$

The current $\Psi_{n-2r+2}$ is represented by an element of $A^{r,r}(\Gamma \backslash G/K)$.

6. THE CURRENT $\Psi_{s}$

We remark that $\ast \text{vol}_{H} = \frac{1}{r!} (\omega - \omega_{H})^{r}$ with $\text{vol}_{H} = \frac{1}{(n-r)!} \omega_{H}^{n-r}$ the ‘volume form’ of $H/K_{H}$.

Theorem 11. Let $\text{Re}(s) > 3n - 2r$. Then there exists $\varepsilon > 0$ such that the function $\delta_{\mu,s}((s^{2} - \lambda^{2})^{-1}\tilde{\Psi}_{s})$ belongs to the space $L^{2+\varepsilon}_{\Gamma}(\tau)^{(r)}$. The spectral expansion of $\delta_{\mu,s}((s^{2} - \lambda^{2})^{-1}\tilde{\Psi}_{s})$ is given as

$$\delta_{\mu,s}(\frac{\tilde{\Psi}_{s}}{s^{2} - \lambda^{2}}) = \sum_{m=0}^{\infty} \frac{2\sqrt{-1}(\ast \text{vol}_{H}|\mathcal{J}_{H}(\tilde{\alpha}_{m}^{(r)}; e))}{(\lambda^{2} - \lambda_{m}^{(r)} - s^{2})^{r}} \tilde{\alpha}_{m}^{(r)} + \sum_{p=0}^{r} \frac{2\sqrt{-1}(\ast \text{vol}_{H}|\mathcal{J}_{H}(\mathcal{E}^{i}(\zeta; u); e))}{(\zeta^{2} - \nu_{s}^{(p+1)})^{r}} \mathcal{E}^{i}(\zeta; u) d\zeta,$$

where the summations in the right-hand side of this formula are convergent in $L^{2}_{\Gamma}(\tau)^{(r)}$.

Theorem 12. Let $L_{1}$ be the interval on the imaginary axis defined by (1). Let $0 \leq j \leq \mu$. Then for each $\tilde{\beta} \in \mathcal{K}_{r}(\tau)$ the holomorphic function $s \mapsto \mathcal{F}_{j}(s, \tilde{\beta}) := \langle \delta_{j,s}(s^{2} - \lambda^{2})^{-1}\tilde{\Psi}_{s}|\tilde{\beta}\rangle$ on $\text{Re}(s) > n$ has a meromorphic continuation to the domain $\mathbb{C} - L_{1}$.

A point $s_{0} \in \mathbb{C} - L_{1}$ with $\text{Re}(s_{0}) > 0$ is a pole of the meromorphic function $\mathcal{F}_{j}(s, \tilde{\beta})$ if and only if there exists an $m \in \mathbb{N}$ such that $(\ast \text{vol}_{H}|\mathcal{J}_{H}(\tilde{\alpha}_{m}^{(r)}; e)) \neq 0$, $\langle \tilde{\alpha}_{m}^{(r)}|\tilde{\beta}\rangle \neq 0$ and $s_{0}^{2} - \lambda^{2} = -\lambda_{m}^{(r)}$. In this case, the function

$$\mathcal{F}_{j}(-s, \tilde{\beta}) - \mathcal{F}_{j}(s, \tilde{\beta}) = (-1)^{\mu} \delta_{j,s} \left( \sum_{p=0}^{r} \frac{(\tilde{\mathcal{E}}_{p}^{(r)}(\nu; g))}{2\nu_{s}^{(p+1)}} \right) \mathcal{F}_{j}(s, \tilde{\beta} - \sum_{m \in \mathbb{N}, \lambda_{m}^{(r)} = \lambda^{2} - s_{0}^{2}} 2\sqrt{-1}(\ast \text{vol}_{H}|\mathcal{J}_{H}(\tilde{\alpha}_{m}^{(r)}; e))\langle \tilde{\alpha}_{m}^{(r)}|\tilde{\beta}\rangle (s_{0}^{2} - s^{2})^{j+1}$$

is holomorphic at $s = s_{0}$. We have the functional equation

$$\mathcal{F}_{j}(-s, \tilde{\beta}) - \mathcal{F}_{j}(s, \tilde{\beta}) = (-1)^{\mu} \delta_{j,s} \left( \sum_{p=0}^{r} \frac{(\tilde{\mathcal{E}}_{p}^{(r)}(\nu; g))}{2\nu_{s}^{(p+1)}} \right) \mathcal{F}_{j}(s, \tilde{\beta})$$

with

$$\tilde{\mathcal{E}}_{p}^{(r)}(\nu; g) := -2\sqrt{-1} \sum_{i=1}^{h} \sum_{u \in \mathcal{B}^{(r)}(p; 0)} \ast \text{vol}_{H}|\mathcal{J}_{H}(\mathcal{E}^{i}(\nu; u); e)) \mathcal{E}^{i}(\nu; u; g), \quad g \in G.$$

Theorem 13. A point $s_{0} \in \mathbb{C} - L_{1}$ with $\text{Re}(s_{0}) > 0$, $s_{0} \neq n - 2r + 2$ is a pole of $\Psi_{s}$ if and only if there exists an $L^{2}$-form $\alpha \in A_{(2)}^{r,r}(\Gamma \backslash G/K; (n-2r+2)^{2} - s_{0}^{2})$ such that

$$\int_{D} j^{*}\alpha \neq 0.$$
In this case $s_0$ is a simple pole with the residue
\[ \text{Res}_{s=s_0} \Psi_s = \frac{\sqrt{-1}(s_0^2 - (n - 2r + 2)^2)}{s_0} \sum_m \left( \int_D j^* \alpha_m \right) \cdot \alpha_m. \]

Here $\{\alpha_j\}$ is an arbitrary orthonormal basis of $A_{(2)}^{\ell,\ell}(\Gamma\backslash G/K; (n - 2r + 2)^2 - s_0^2)$.

- We have
  \[ \Psi_{n-2r+2} = 2\sqrt{-1} \sum_m \left( \int_D j^* \beta_m \right) \cdot \beta_m \]
  with $\{\beta_m\}$ an arbitrary orthonormal basis of $H_{(2)}^{\ell,\ell}(\Gamma\backslash G/K)$. In particular $\Psi_{n-2r+2} \in H_{(2)}^{\ell,\ell}(\Gamma\backslash G/K)$.

The equations in Theorem 10 means the fundamental class $[\delta_D] \in H^{r,r}(\Gamma\backslash G/K; \mathbb{C})$ of $D$ has the harmonic $L^2$-representative $\Psi_{n-2r+2}$.

REFERENCES


Masao TSUZUKI
Department of Mathematics
Sophia University, Kioi-cho 7-1 Chiyoda-ku Tokyo, 102-8554, Japan
E-mail: tsuzuki@mm.sophia.ac.jp