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The final works of Rich Pelz, a man of warm-hearted intelligence

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Abstract

Only four months after face-to-face acquaintance, I lost a dear friend and my professional partner who shared three months in Kyoto with myself. The scientific community lost an irreplaceable spirit. Rich had not only a heaven-sent creativity, but a personality that warmed the lives of everyone who had met and known him. While I will never get over losing him at such an early age, we can take comfort in the marvelous papers he left us all. Their striking feature is an ideal combination of analytical and computational skills. Because of this his works have led and will continue to guide the scientific community. That was a privilege he was calmly proud of, as I am proud of having worked with him. He is truly missed.

I. INTRODUCTION

I was at a loss when I logged in the computer system of our Institute to see that a job was properly running for a joint work with Rich. What I found was an email message which carried his sudden death.

I have been steady with him for a couple of years online. We exchanged our preprints and opinions, mainly on the problem of blowup of solutions of hydrodynamical equations. At the same time we sought for a possibility of our joint work. I was very happy when he was kind enough to accept our offer of a visiting professorship at our Institute. He stayed here during May 19 - August 19, 2003.

After discussing a possible theme for our collaboration during his stay in Kyoto, we reached one problem which we called 'the sine flow project'. This is a main subject we worked on in Kyoto. It is about the effects of the pressure (or its Hessian) upon the dedingularization of blow-up observed in a linear strain flow. The latter flow has no boundary conditions and infinite amount of total energy.

Another problem we considered was a singularity formation in MHD equations. This is in a sense more controversial than the corresponding problem for the Euler equations.

I am not in a position to write an obituary about him as he passed away only 4 months after face-to-face acquaintance with myself. I describe below an overview of his scientific avtivity during the stay, deferring detailed account to forthcoming papers. On leaving Kyoto he cleared most of his files from our computer system. Some were left unremoved for a smooth continuation of our joint project. As Appendices I include his writings that he left on the system. These were made available to me thanks to the understanding of system managers.

II. THE 'SINE FLOW' PROJECT

It was him who proposed to compare the behavior of a linear strain flow against that of a flow with periodic boundary conditions. We compared a flow developing from the linear straining flow,

$$\boldsymbol{u} = -(y + z, z + x, x + y),$$

and a flow whose initial condition is locally similar to it, namely,

$$\boldsymbol{u} = -(\sin y + \sin z, \sin z + \sin x, \sin x + \sin y).$$

The linear straining flow is not in $L^2(\mathbb{R}^3)$ (energy is not bounded), it is boundary-free (no conditions are imposed at infinity) and it has non-unique solutions, some of which blow up in finite time, e.g.

$$\boldsymbol{u}(x, y, z, t) = \left(\frac{y+z}{t-1}, \frac{z+x}{t-1}, \frac{x+y}{t-1}\right),$$
$$p(x, y, z, t) = -\frac{x^2+y^2+z^2}{(t-1)^2}.$$

The latter flows match the linear flow to first order at the origin, but are in $L^2(\mathbb{T}^3)$ and have outer boundary conditions. I prepared a pseudo-spectral code for simulations of the latter. Using both pseudospectral and power series in time, we found that the latter flow solutions remain smooth at least for a time well beyond the critical time of the linear flow.

The nonuniqueness in the boundary-free flow is interpreted as an arbitrariness of the homogeneous solution of the pressure Poisson equation. The $(1-t)^{-1}$ blowup follows from the inclusion of the particular solution only. Strong growth in the anisotropic components of the pressure Hessian, which is required for desingularization, is exhibited in the solution of the bounded flows. It is also found from the time series analysis, that the particular solution exhibits secular behavior. A perturbation theory to include the homogeneous solution properly is attempted, but unfortunately has not been completed while he was alive.

He prepared some scripts of Maple to solve the Euler equations as a series solution in time. I also wrote such a script and we checked that our results agree with each other. For him writing Maple scripts of this sort is trivial, for me it was nontrivial. For him writing C codes with arbitrarily high precision (as used in [1]) was 'something' , while for me it is beyond my capability. He was quite tolerant of my slow progress in our joint project!

An account of this work will be included in Kokyuroku. A longer article is in preparation.

III. HIGH-SYMMETRIC MHD FLOW

The equations of magnetohydrodynamics (MHD) for incompressible, inviscid and perfectly conducting fluid read as follows:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}.$$

Here $\boldsymbol{u}$,$\boldsymbol{B}$ and p denote the velocity, magnetic and pressure fields, respectively. They conserve total energy

$$\int (|\boldsymbol{u}|^2 + |\boldsymbol{B}|^2) d\boldsymbol{x}$$

and

$$\int \boldsymbol{u} \cdot \boldsymbol{B} d\boldsymbol{x}, \int \boldsymbol{A} \cdot \boldsymbol{B} d\boldsymbol{x}$$

which are called cross and magnetic helicities ($\boldsymbol{B} = \nabla \times \boldsymbol{A}$).

The blowup of the Euler equations ($\boldsymbol{B} \equiv 0$) is an unsolved problem. It has been controversial for a long time.

The blowup problem of MHD equations is also controversial, e.g. [2, 3]. The well known Beale-Kato-Majda constraint for a possible blowup of the Euler equations has been extended to

$$\int_0^t (|\boldsymbol{\omega}|_\infty(t') + |\boldsymbol{J}|_\infty(t'))\, dt' \to \infty, \text{ as } t \to T$$

in [4] ($\boldsymbol{J} = \nabla \times \boldsymbol{B}$), for a possible breakdown time T. This has been obtained by using the Elsässer variables.

In [3], formation of current-sheets, i.e. region with high $|\boldsymbol{J}|$ was reported. But after their formation, growth in current became exponential in time, which suggests no blowup. Current sheets are not flat but have the form of potato chips, hence this name. This experiment has raised a following problem; Is it possible for two potato-chips (rounded material surfaces) to approach with each other and collide over a region of finite (no matter small) area in finite time ? In response to this query, under the assumption of controlled velocity, that is,

$$\int_0^T |\boldsymbol{u}|_\infty dt' < \infty$$

it was proved in [5] that no blow can occur by the scenario of potato-chip singularity.

He published papers on the blowup of the Euler equations for a special class of symmetrized initial conditions [6, 7]. This research was originally motivated by Kida's high symmetric flow. A natural question we had is; What happens if we add magnetic fields to the high-symmetric flow? Does the Eulerian singularity persist or not?

He was very ambitious to start yet another project during his stay. He began modifying his pseudo-spectral code for the MHD case. I was preparing Maple scripts to evaluate

Taylor series in time. He did some computations at low resolutions. In both approaches the computational burden is much heavier than that of the Euler equation.

In practice, after transforming to Elsässer variables $\boldsymbol{u}^{\pm} = \boldsymbol{u} \pm B$ we tried

$$u_1^+ = (1+\alpha)(\cos(3y)\cos(3z) - \cos(y)\cos(3z)),$$

$$u_1^- = (1-\alpha)(\cos(3y)\cos(3z) - \cos(y)\cos(3z))$$

with $\alpha = 1/2$. In the Eulerian case ($\boldsymbol{B} = 0$), 12 vortices (6 vortex pairs) are observed to collapse at the origin. But in the presence of magnetic field, which has the same order of magnitude of velocity, a drastically different was observed by him. He told me something like 'Vortices were dancing happily..' after working out some visualization on his PC.

Unfortunately I have never seen his results. But the overall impression of his preliminary calculations suggested the singularity formation is suppressed or at least delayed by the presence of magnetic fields. On the other hand, the computer algebra for calculating the time series could not go beyond 4th-order, because the required memory and speed was too big. More careful work would be necessary to settle the problem.

IV. SOME OTHER TOPICS WE DISCUSSED IN KYOTO

I have been interested in a framework of viscous fluid, called Eulerian-Lagrangian formalism [8, 9]. I was working on its application to the problem of vortex reconnection and turbulence [10]. After discussing it with Rich he asked me how it is related with the vortex line representation [11]. I could not answer his inquiry satisfactorily but I believe it is worth pursuing such a connection.

The initial condition used in the sine flow project has an isotropic pressure Hessian at the origin:

$$\frac{\partial^2 p}{\partial x_i \partial x_j} = \frac{\delta_{ij}}{3}\Delta p.$$

The ansatz that the pressure Hessian remains isotropic under Eulerian dynamics is known as Leorat-Vieillefosse assumption [12]. Our numerical results show that this ansatz is definitely wrong, in that rapid growth in the off-diagonal elements is crucial for removing the singularity. It was him who turned my attention to E. Tadmor's recent works on the Leorat-Vieillefosse model [13]. That work is nevertheless interesting because the blowup in the restricted flow (the Leorat-Vieillefosse assumption) is shown by spectral characteristics.

V. VISITS TO OTHER UNIVERSITIES AND SOCIAL ASPECTS

During his stay he traveled to Sapporo c/o Dr. Iima of Hokkaido University and to Fukuoka c/o Prof. Fukumoto of Kyushu University. In both of these places he gave a talk on his blowup computations that was welcome by the audience of physicists and mathematicians, with a number of stimulating comments. He told me that he enjoyed these trips socially and academically. He also gave a successful talk at Engineering Department of Kyoto University.

Some RIMS visitors try learning (or picking up) the Japanese language bravely during their stay. As far as I know he is one of the most serious ones. He memorized 5-10 Chinese characters each day. Finally he arrived at a point to realize the meaning of such names as KI-DA, YAMA-DA, and OH-KI-TANI. To him this seemed to be a pleasant surprise. He even tried to eavesdrop my conversation with secretaries in vain and got frustrated for not understanding it very well! He bought some books on Japanese and brought them home to keep the language alive.

He was also interested in Japanese cookery. He bought a rice steamer and prepared meals for himself whenever time was available. When I invited him to my house my wife showed him how to make Sushi, from scratch. He was very happy to do that.

VI. CONCLUDING REMARKS

After he came back to the States, we continued working together on the sine flow project. The last email message I got from him is dated September 20, and 4 days later he passed away. I feel it is a bit ironic that both of the two themes, the sine flow project and the high-symmetric MHD flow (a paper is being prepared for the former) suggest regularity rather than singularity.

He talked about some of numerical papers reporting blowup and about alleged singularity which were refuted later. As far as I understand his works on the singularity formation of symmetrized vortices still deserve serious attention and are one of the most important result of this area. At the same time he kept telling me that he was ready to be refuted either analytically or numerically. This proves that he knew how difficult it is to pursue the topic with numerical methods.

Some people say with mathematical methods only, the progress is too slow (in comparison with our life time) and we need to have recourse on computational methods. Others say that it is impossible to study blowup problems numerically and do nothing about them. While the ultimate solution must be purely analytical, we may get stimulating information or motivation by carefully devised numerical methods. But this is easier said than done. I can safely mention that Rich has been regarded as a good practitioner of the hybrid approach. I am convinced that his death will certainly delay the resolution of hydrodynamical blowup problem he has been anxiously after.

VII. ARCHIVES

What he left in the hard disk of workstations he used at RIMS.

```
./Sine/sine_flow.tex
./Sine/ramp.tex
./Sine/ramp1.tex
./Sine/sine_flow_extra.tex
./Maple_expansion/writeup.tex
./GMP-Vort1/blowup_scenarios.tex
./GMP-Vort1/Eb.tex
./GMP-Vort1/problem_statement.tex
./GMP-Vort1/writeup_vort.tex
./GMP-Vort1/writeup_vort_sym.tex
./GMP-Vort1/writeup_vort_sym1.tex
./kida/MHD/writeup.tex
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APPENDIX A: SINE_FLOW.TEX

The following is a verbatim record, i.e. an unfinished and unedited draft which I got from him on August 6, 2002. It has a fair number of typo's which I left uncorrected to show 'the making' of our research project as faithfully as possible. The corresponding abstract was submitted for the DPP02 Meeting of APS on July 16, 2002. as a contribution to MINI-CONFERENCE ON SINGULARITIES. The talk was not delivered. (KO)

Linear strain flows with and without boundaries

R.B.Pelz and K.Ohkitani

Abstract

Whether or not the Euler equations for incompressible flow admit solutions with finite-time singularities, it is clear that the nonlocal action of pressure (non-isotropic Hessian terms) plays a critical role. To address this question we contrast the boundary-free, linear strain flow $u = -(y+z, z+x, x+y)$ that has nonunique solutions including some which blowup in finite time, and some bounded flows with similar behavior near the origin, eg, $u = -(\sin y + \sin z, \sin z + \sin x, \sin x + \sin y)$. Using both pseudospectral and power series in time, it is found that there is no evidence for blowup of the bounded flows. The nonuniqueness in the boundary-free flow is interpreted as an arbitrariness of the homogeneous solution of the pressure Poisson equation. The $(1-t)^{-1}$ blowup follows from the inclusion of the particular solution only. In expanding about the origin, it is found that only the first spherical harmonic contributes to the non-isotropic Hessian. Strong growth in this mode, which is required for desingularization, is exhibited in the solution of the bounded flows. It is also found from the time series analysis, that the particular solution exhibits secular behavior. In analogy with perturbation theory, proper inclusion of the homogeneous solution extends the convergence of the series.

1. Introduction

In this work the linear straining flow,

$$(u_0, v_0, w_0) = -(y+z, z+x, x+y),$$

and a number of flows that are locally similar to it, namely,

$$u = -(\sin y + \sin z, \sin z + \sin x, \sin x + \sin y)$$

$$u = -[\sin(y+z), \sin(z+x), \sin(x+y)]$$

and

$$u = -[\sin y \cos 2z + \sin z \cos 2y, \sin z \cos 2x + \sin x \cos 2z, \sin x \cos 2y + \sin y \cos 2x]$$

are examined. The linear straining flow is not in $L^2\mathcal{R}^3$ (energy is not bounded), it is boundary-free (no conditions are imposed at infinity) and it has non-unique solutions, some of which blow up in finite time. The latter flows match the linear flow to first order at the origin, but are in $L^2(T^3)$ and have outer boundary conditions. It is found in this work through simulation of these flows, that the latter flow solutions remain smooth at least for a time well beyond the critical time of the linear flow. The overall goal is to explore whether or not a finite-energy solution to the Euler equations of hydrodynamics can blow up in finite time. The immediate goal is to understand the difference in these two types of flows, how outer boundary conditions effect the solution near the origin, and in particular what the role of pressure is. An answer to the latter question may shed some light on the first.

If the diagonal transformation, defined as $(x,y,z)^T = A(x',y',z')^T$, $(u',v',w')^T = A(u,v,w)^T$ with

$$A = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix},$$

is used, the straining flow is written

$$(u_0', v_0', w_0') = (x', y', -2z')$$

and the strain rate matrix, $D_{i,j} = 1/2(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, is $D_0 = diag(1,1,-2)$.

Many such boundary-free fields have exact solutions to the Euler and Navier-Stokes equations and have been examined by [1, 2, Hopf 52, Majda 86, Ohkitani 90, Bhattacharjee & Wang 92, Okamoto 97]. They are especially interesting because of their blowup behavior.

Since the linear straining flow is not in $L_2(\mathcal{R}^3)$, it should be considered as a flow local to the origin. [2, Bhattacharjee & Wang 92] realizing this, examined a number of flows, and found under certain conditions the local flow solution possessed a finite-time singularity. There was not a way, however, to consistently account for outer boundary conditions. In the present work, the opposite approach is taken; a number of specific flows with local behavior being the linear flow but with bona fide outer conditions are examined.

2. The global solution

The linear straining flow solution can be examined as a special exact solution of the strain-rate equation. Following [Majda 86], taking the symmetric derivative of the Euler equation gives

$$\frac{dD}{dt} + D^2 + \Omega^2 = -P$$

where D is the strain rate or deformation tensor, Ω is the vorticity tensor $\epsilon_{ijk}\omega_k$, P is the pressure Hessian and d/dt is the substantial derivative.

Letting $\Omega = 0$ and assuming the velocity remains linear, there are an infinite family of solutions of the form $D(t) = b(t)D_0$, $b(0) = 1$ provided that the Hessian has the form

$$P = -diag[b^2 + b', b^2 + b', b^2 + b' - 3(b' - b^2)].$$

Let D_0 have no units, and b have units of 1/time. If the following definitions are made

$$d \equiv b^2, \quad e \equiv 2(b' - b^2)$$

so that the Hessian is written

$$P = -diag(2d + e/2, 2d + e/2, 2d - e).$$

The problem can be viewed as for a given e, the solution b can be found by solving the Ricatti equation $b' = b^2 + e/2$.

A solution can be found in which the Hessian is isotropic. Setting $e = 0$, the Ricatti equation $b' = b^2$ has the solution

$$b = \frac{1}{1/b_0 - t}.$$

This solution is the one that is selected as an example of blowup.

[Relate to the Restricted Euler Equations (Cantwell, Viefosse, Tadmor)]

A nonzero e does not necessarily mean the solution does not blow up. If e has the form $-ab^2$. where $0 < a < 1$. The solution is

$$b = \frac{1}{1/b_0 - (1-a)t}$$

which simply delays blowup to a time $1/[b_0(1-a)]$.

In order to desingularize the solution, the growth in e must be stronger than b^2. If e has the form

$$e = 2(b_0 t)^\alpha b^2$$

for $\alpha > 0$, then the smooth solution is

$$b = \frac{1}{1/b_0 - [1-(b_0 t)^\alpha]t}$$

Clearly, the arbitrariness of e is the origin of the non-uniqueness of solutions. In this formulation, however, the physical nature of e is missing.

3. The series solution

Some understanding of the solution with isotropic Hessian is gained by examining the time power series solution.

If a power series in time is assumed

$$u = \sum_{p=0}^{\infty} u_p t^p$$

then each term can be found in the following three steps

$$\tilde{u} = -\sum_{q=0}^{p-1} u_q \cdot \nabla u_{p-1-q}, \quad \Delta P = \nabla \cdot \tilde{u}, \quad u_p = (\tilde{u} - \nabla P)/p.$$

Let u_0 be the linear straining flow. In the first stage of computing u_1, $\tilde{u}$ is $2x + y + z$ (the diagonal frame is assumed). The pressure equation is then $\Delta P = -6$. If the particular solution is taken, $P = -r^2$, the results is

$$u_1 = u_0.$$

In fact, using the same particular solution, each subsequent term is u_0 and so

$$u = u_0(1 + t + t^2 + \ldots) = \frac{u_0}{1-t}.$$

The particular blowup solution with isotropic pressure Hessian given in the previous section, is the same as the series solution where only the particular solution to the pressure Poisson problem is selected.

The homogeneous solution, which allows the general solution to satisfy boundary conditions, cannot be constructed in this boundary-free flow.

In addition, the constant right-hand-side of the pressure Poisson equation is a solution to Laplaces equation. While valid only for solutions in L_2, the Fredholm Alternative for $Lx = f$ states that if the differential operator, L is singular, f must be orthogonal to the space of homogeneous solution in order for L_2 solutions to exist. If the right-hand-side is not orthogonal, secular behavior is expected.

When such resonance behavior develops in regular perturbation problems, the secular term limits the convergence of the perturbation series solution. When proper homogeneous solutions are included and the series is summed, or a method of multiple scales is used, the correct behavior is recovered (Bender & Orszag). Both methods rely, however, on specific boundary conditions begin given.

4. The discrete groups

A way of introducing the idea of locality of this solution and to bring in the effect of outer boundary conditions is to cast the linear straining flow as the first order term in an expansion about the origin. Thus the linear strain flow is one element of a group of flows. It is useful first to examine the symmetries of these flows and classify them as elements of a particular discrete group.

To do this we begin with the more general flow in [2]. They considered a local flow with the following expansion

$$u(x, y, z) = \sum_{l,m,n \geq 0} a_{lmn} x^l y^m z^n$$

where $a_{lmn} = 0$ if $l + m + n$ is even. From incompressibility $a_{100} = 0$, and taking $a_{010} = a_{001} = -1$ we have the linear strain flow (in original coordinates).

This flow is an element of the S_2 group possessing only the identity and inverse symmetries. (The S_n improper reflection operation can be considered as an n-fold rotation about an axis and then a reflection with respect to the plane normal to this axis.)

Next the permutation symmetry of the velocity vector field is added,

$$v(x,y,z)=u(y,z,x),\quad w(x,y,z)=u(z,x,y).$$

Using the diagonal transformation introduced in section 1, allows this symmetry to be classified as a C_3 three-fold symmetry about the z' axis.

A flow with both the inverse and permutation symmetries is then restricted to be an element of the S_6 group; thus [2] considered flows in the S_6 group. Since the Euler equations are invariant to translations, rotations and reflections, if the initial flow is in this group, the subsequent flows in the evolution are in this group also.

When a rotational symmetry about the diagonal $y=z$ is added, for example if u has the initial condition

$$u=y+z,\quad \text{or}\quad u=\sin y+sinz$$

this further restricts the flow to be in the group D_{3d}. Thus the flows considered in this paper are in this group. A molecule that is in D_{3d} group is the staggered ethane molecule, C_2H_6, with CH_3 centered on the positive z' axis and the other methyl group on the negative z' axis but rotated 60 degrees.

By knowing the group, the basis functions are established that are involved in the spatial expansion of analytic functions about the origin. They are called the integral rational basis of invariants, and at second order are z'^2 and $x'^2+y'^2$ for D_{3d}.

The homogeneous solutions to the Poisson equation (solutions to Laplace equation) involve those spherical harmonics that are invariant under D_{3d}. They can be constructed from the basis of invariants. The first few terms are

$$\begin{aligned} p &= a_0+a_{2,0}(z'^2-x'^2/2-y'^2/2)+a_{4,0}[z'^4+(3/8)(x'^2+y'^2)^2-3z'^2(x'^2+y'^2)] \\ &+ a4,3z'[x'^3-3x'(x'^2+y'^2)]+b4,3z'[y'^3-3y'(x'^2+y'^2)]+\dots \end{aligned}$$

The second order term, which we shall define as Y_2 is $z'^2-x'^2/2-y'^2/2$. Y_2 is clearly the most influential homogeneous mode; $2\nabla Y_2=-x',-y',2z'$ which is the linear strain flow, and $\nabla\nabla Y_2=diag(-1/2,-1/2,1)$ which is the matrix that multiplies e in the pressure Hessian.

5. Multiple Scale Analysis

The idea that the linear straining field is valid at early times in an inner region and that far-field boundary conditions can be introduced in an outer region suggests that the method of multiple scales may be useful in understanding whether strong growth in e in the inner region can be explained.

Let the velocity and pressure coefficients of the power series be functions of the fast and slow spatial variables x and tx. (Note tx can be normalized by b_0.) The expansion is then

$$u = u_0(x, tx) + tu_1(x, tx) + t^2u_2(x, tx)\ldots$$

$$P = P_0(x, tx) + Pu_1(x, tx) + P^2u_2(x, tx)\ldots$$

The nonlinear term (nl) and pressure Laplacian then become

$$\begin{aligned} u \cdot \nabla u &= u_0 \cdot \nabla u_0 + (u_0 \cdot \nabla u_1 + u_1 \cdot \nabla u_0 + u_0 \cdot \bar{\nabla} u_0)t \\ &+ (u_0 \cdot \nabla u_2 + u_2 \cdot \nabla u_0 + u_1 \cdot \nabla u_1 + u_0 \cdot \bar{\nabla} u_1 + u_1 \cdot \bar{\nabla} u_0)t^2 + \ldots \end{aligned}$$

$$\Delta P = \Delta P_0 + (\Delta P_1 + 2\bar{\nabla} \cdot \nabla P_0)t + (\Delta P_2 + \bar{\Delta} P_0 + 2\bar{\nabla} \cdot \nabla P_1)t^2 + \ldots$$

where do we go from here?

6. The Sine Flow

A group of incompressible flows that have the D_{3d} symmetry in a unit cell and are embedded in a cubic lattice can be defined. These flows are then in the domain T^3, have periodic boundary conditions and can be represented by Fourier series in space. One such flow is

$$u = -\sin y - \sin z, \; v = -\sin z - \sin x, \; w = -\sin x - \sin y.$$

The flow was investigated by Childress and Spiegel (private comm).

To first order around the origin, the sine flow is equivalent to the linear straining flow. As with all the flows considered, the Riccati equation $b' = b^2 - e$, where b is $\partial u/\partial y$ holds at the origin. In this case, e is generated consistently with the evolution of the complete flow.

The Sine flow, hence, provides an example of a local linear-straining flow with outer boundary conditions.

Expanding, it is easy to show that in addition to $a_{010} = -1$, the terms $a_{0i0} = (-1)^{(i+1)/2}/i!$ for $i = 3, 5, \ldots$ are active.

FIG. 1: Evolution of $b(t)$ and $e(t)$ for $u_0 = -\sin y - \sin z$. The solid line is the pseudo-spectral solution, the short dashed line is the 20-term Taylor series solution, the long dashed line is the [10,10] Padé approximant of the series solution, and the dotted line shows $1/(1-t)$ behavior of the linear straining flow.

The initial value problem – Euler equations for incompressible flow with periodic boundary conditions and Sine initial condition – was solved by two methods: a Fourier pseudo-spectral method and a power series in time.

Koji fill in details.

A power series was assumed for velocity

$$u = \sum_{p=0}^{\infty} u_p t^p$$

and the recursion for the pth term can be written

$$pu_p = -\left(1 + \frac{\partial}{\partial x}\Delta^{-1}\frac{\partial}{\partial x}\right)\sum_{q=0}^{p-1}\left(u(x,y,z)\frac{\partial u}{\partial x} + u(y,z,x)\frac{\partial u}{\partial y} + u(z,x,y)\frac{\partial u}{\partial z}\right).$$

Maple was used to find the first 20 terms of the series with the Fourier coefficients as rational numbers (exact precision).

Figure 1 show the evolution of $b(t)$ and $e(t)$. The pseudo-spectral solution, the solid line, the short dashed line is the 20-term Taylor series solution, the long dashed line is the [10,10] Padé approximant of the series solution, and the dotted line shows $1/(1-t)$ behavior of the linear straining flow.

7. The Second Period Flow

Similar methods of simulation are used starting with the field

$$u = -[\sin(y+z), \sin(z+x), \sin(x+y)].$$

This flow, with cross terms $a_{012} = 1/2$, has a smaller region around the origin that approximates the linear flow. It is expected that the effect of the outer boundary conditions will be cause an earlier divergence of this solution from the linear flow as compared to the

8. The Third Period Flow

The next periodic flow of increasing complexity about the origin is

$$u = -[\sin y \cos 2z + \sin z \cos 2y, \sin z \cos 2x + \sin x \cos 2z, \sin x \cos 2y + \sin y \cos 2x].$$

A similar analysis is done with this flow.

Here, the cross term is $a_{012} = 2$, so it is expected that there will be a larger difference between this solution and the linear flow, compared to the other two flow.

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APPENDIX B: SINE_FLOW_EXTRA.TEX

1. Implicit Outer Boundary Conditions

Problem: secular behavior of particular solution: this mode grows too fast. pert theory suggests that the homogeneous solution quench secular behavior.

But without specific outer boundary conditions, how can the general solution be introduced?

Conjecture: Since any outer boundary condition will not match the growth of the secular term, satisfying the outer boundary condition necessary means the secular growth is matched by a homogeneous term of the same order.

The secular term is $P_p = x'^2 + y'^2 + z'^2$; the only homogeneous mode is $P_h = (2z'^2 - x'^2 - y'^2)$. Terms of order r^2 will not be generated from homogeneous modes of r^4 and higher. The second order mode when written in original coordinates is $xy + yz + yz$ and clearly will be the term setting the off-diagonal elements of the Hessian. This mode, as can be seen from its gradient $-x, -y, 2z$ which is the same as the linear flow, will also act directly on the velocity in constructive or destructive way.

2. Extra Stuff

Central to the blowup question in the linear straining flow is the nonuniqueness of solutions to the pressure problem. In order to obtain the blowup solution above, only the particular solution solution is used. We will show that the time evolution of the pressure is just a solution or a Riccati equation and that homogeneous solutions give rise to forcing functions. If the forcing function is strong enough, however, desingularization occurs.

Setting e restores uniqueness. What determines e? It will be shown that e is the coefficient of a low order spherical harmonic, hence, it is part of the homogeneous solution of the Poisson equation for pressure. It is the only spherical harmonic that is the same order in r as the particular solution, which is a resonance solution. Clearly e is set by the outer boundary conditions in regards to the order matching with the particular solution.

To explore the idea of secular behavior in the Euler equations, the solution is developed as a power series in time. It is conjectured that a similar type of necessary growth of the homogenenous solutions will occur in this situation. Such growth will be monitored in the bounded flow solutions and and related to the forcing function in the Riccati equation.

How the homogeneous solutions are related to e and the resonance, and how this can be related to the bounded flows will be addressed shortly.

The Hessian: In the Cartesian frame, the Hessian has diagonal -d(t) and off-diagonal term -e(t). Note that d and e are positive in time. Transforming into the primed frame, the Hessian is diagonal but not symmetric. It can be decomposed into an isotropic matrix $-dI$ and a traceless diagonal matrix with diagonal $-e/2, -e/2, e$.

For the unbounded flow, $u = -y - z$, the Poisson equation for the pressure has the form

$$\Delta P = -6.$$

A particular solution is

$$P = -(x^2 + y^2 + z^2).$$

The Laplace equation has solutions $r^n Y_n$, that are the homogeneous solutions to the Poisson equation. Furthermore, the right hand side is not orthogonal to Y_0. In the case when the linear operator is singular and the rhs is not orthogonal to the homogeneous solution, neither case of the Fredholm alternative is satisfied and we are not assured of a solution in L_2, but the original flow when take in $\mathcal{R}^3$ is not in L_2.

3. The solution via Spherical harmonics

Since the flow $u = -y - z$ is not in $L_2(\mathcal{R}^3)$, in order for it to be considered as a candidate blowup flow, it must be considered as existing in a finite ball, with a smooth outer flow in $L_2(\mathcal{R}^3)$ connecting to it.

On this sphere, boundary conditions on velocity must be given (or some Schwarz matching can be done there). The influence of velocity boundary conditions will be assumed to be local and play an insignificant role in the inner solution at least for times considered. More important is the boundary condition on pressure, which due to the elliptic nature of incompressible flows, can be felt throughout the domain immediately.

The Poisson equation for the pressure will be considered in this finite domain with some outer boundary conditions, be it Dirichlet, Neumann or mixed. A particular boundary condition is not desired, they will be kept as general as possible by considering the general solution to potential boundary value problem. The general solution of the potential problem in terms of spherical harmonics is

$$P_H = \sum_{n=0}^{\infty} r^n \left[a_{n,0} P_n(\cos\theta) + \sum_{h=1}^{n} (a_{n,h}\cos h\phi + b_{n,h}\sin h\phi) P_n^h(\cos\theta) \right].$$

An important observation of the pressure problem is that the rhs of the Poisson equation is an element of the homogeneous solutions, $\Delta 6 = 0$. When this occurs, the solution is expected to be in resonance with the rhs which manifests itself in the existence of secular behavior in the solution.

The particular solution, $p = x^2 + y^2 + z^2$, which has in the past been considered as the full solution of the pressure problem for this linear flow, is such a secular term. If we consider the solution as a power series in time,

$$P = \sum_{p=0} P_p t^p, \quad u = \sum_{p=0} u_p t^p,$$

and using only this particular solution, then

$$P = (x^2 + y^2 + z^2)(1 + 2t + 3t^t + \ldots) = \frac{x^2 + y^2 + z^2}{(1-t)^2}.$$

A characteristic of secular behavior is that it limits the convergence and long term-behavior of series solutions, which appears to be the case here. In other examples of secular behavior, if a general solution is taken at each level of the series, then the secular behavior is quenched.

4. Homogeneous solution as a Perturbation

One way of introducing the effect of outer boundary conditions into the pressure problem is to simply assume that the homogeneous solutions enter the general solution as a perturbation to the particular solution. For the effect of outer boundary to be constant in time, during the generation of the power series the perturbation is introduced at the first term only, namely P_0.

Case 2: If the solution to the first pressure problem is

$$P_0 = x'^2 + y'^2 + z'^2 + \epsilon(x'^2 + y'^2 - 2z'^2),$$

then subsequent terms will be of the form

$$P_p = f_p(\epsilon)(x'^2 + y'^2 + z'^2) + g_p(\epsilon)(x'^2 + y'^2 - 2z'^2)$$

and

$$u_p = -h_p(\epsilon)(y + z)$$

If for the pressure problem at every level of the expansion, the effect of the boundary is included, for example in the following generator

$$h_p = \frac{1}{p}\sum_{q=0}^{p-1} h_q * h_{p-1-q}(1-\epsilon)$$

, the resulting series is

$$P = \frac{1}{1-(1-\epsilon)t}.$$

which has a singularity time $t_0 = 1/(1-\epsilon)$.

If the homogeneous term is introduced as a constant at each level, the generator is

$$h_p = \frac{1}{p}\left(\sum_{q=0}^{p-1} h_q * h_{p-1-q} - \epsilon\right).$$

This makes the solution remain smooth by moving the singularity off the real time axis and into the complex plane. From Pade analysis, if $\epsilon = 0.3$, the singularity is at $t = 1.24 + .055i$ and if $\epsilon = 0.5$, the singularity is at $t = 1.41 + .316i$.

$\boxed{\text{Series Solution}}$

Initial condition

$$\omega_1 = \cos x_1 \sin 3x_2 \sin x_3 + \cos x_1 \sin x_2 \sin 3x_3 + 2\cos 3x_1 \sin x_2 \sin x_3$$

$$\omega_2(x_1, x_2, x_3) = \omega_1(x_2, x_3, x_1), \quad \omega_3(x_1, x_2, x_3) = \omega_1(x_3, x_1, x_2)$$

Enforce symmetries: O_h at origin, O group at $\pi/2, \pi/2, \pi/2$.

With cyclic symmetry, Euler equations in $\mathcal{R}^3$ reduce to one equation for ω_1.

Call the nonlocal, quadratic operation $\diamond$,

$$\frac{\partial \omega_1}{\partial t} = \omega_1 \diamond \omega_1$$

The vorticity has the following Fourier representation

$$\omega_1 = \sum_{i,j,k \geq j} \tilde{\omega}_1(i,j,k)[\cos ix_1 \sin jx_2 \sin kx_3 - (-1)^i \cos ix_1 \sin kx_2 \sin jx_3]$$

$\{i, j, k\}$ either all even or all odd.

Using these trig identities, the equation of motion is

$$\omega_1 \diamond \omega_1 = \sum_{i,j \geq k} \sum_{l,m \geq n} \tilde{\omega}_1(i,j,k)\tilde{\omega}_1(l,m,m) \sum_{I=1}^{192} case\ I$$

where each case is $f \cos(i_{new}x_1) \sin(j_{new}x_2) \sin(k_{new}x_3)/(i^2 + j^2 + k^2)$ and $f, i_{new}, j_{new}, k_{new}$ are functions of i, j, k, l, m, n.

Formal solution: power series in time,

$$\omega = \sum_{p=0} \omega_p t^p,$$

with ω_0 the initial condition at time $t = 0$.

The pth term of the expansion is

$$p\omega_p = \sum_{q=0}^{p-1} \omega_q \diamond \omega_{p-1-q}.$$

The commutative product, symbol $\circ$, is defined $\omega_q \circ \omega_r \equiv (\omega_q \diamond \omega_r + \omega_r \diamond \omega_q)/2$. The pth field becomes

$$p\omega_p = \sum_{q=0}^{(p-1)/2} (2 - \delta_{q,p-1-q})\, \omega_q \circ \omega_{p-1-q}.$$

$$\dot{\omega} = \omega^2, \;\; \omega(0) = \omega_0$$

Solution:

$$\omega = \frac{\omega_0}{1 - \omega_0 t} = \omega_0 + \omega_0^2 t + \omega_0^3 t^2 + \ldots$$

If series has form $\omega_n = \omega_0^{n+1}$ (power associative algebra) then solution singular.(Walcher)

The first few terms are written

$$\omega_1 = \omega_0 \circ \omega_0 \quad \omega_2 = \omega_0 \circ \omega_1$$

$$\omega_3 = (2\omega_0 \circ \omega_2 + \omega_1 \circ \omega_1)/3$$

ω_0^4 is not defined because $\circ$ not associative

$$(\omega_0 \circ \omega_0) \circ (\omega_0 \circ \omega_0) \neq (\omega_0 \circ (\omega_0 \circ (\omega_0 \circ \omega_0)).$$

Local associativity?

Flows with Discrete Symmetries

Apply Representation Theory of 3-D Groups of Discrete Symmetries

- Use characterization of this well-developed area.
- Connection between flow fields and irreducible representations.
- It may be isolated (point group) or in a lattice (space group).
- 32 point groups, 7 lattices, 230 space groups.

Blow-up occurs about a null point in vorticity.

- Most candidate flows support this.
- Blowup point is part of symmetry element.

Singularity is point-wise in space.

- For NS 1-D Hausdorff measure for space-time singular set is zero (Caffarelli, Kohn & Niremberg, CPAM 35 (1982)).
- Continuum of freqency and wavenumber blowup theorems of Beale-Kato-Majda and Constantin-Fefferman-Majda.
- If on a reflection plane, blowup involves collapse of toroidal vorticity.

Blow-up Point can be positioned at origin

- Singularity is like an atom or molecule.
- Spherical coordinates may isolate r/t normalization.