Some Special Cases of Non-Amorphous Association Schemes with A.V.Ivanov's Condition

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In this paper, we study one of the generalizations of Edwin van Dam's result. In his paper, he considered strongly-regular decompositions of the complete graph. He found the matrix form of the 1st eigenmatrix $P$ of non-amorphous association schemes of class $d = 4$. We generalize his results and study a combinatorial structure of the 1st eigenmatrix $P$ of non-amorphous association schemes with some condition.

In general, the special cases of the 1st eigenmatrix $P$ of non-amorphous association schemes with some condition have relation to the incidence matrix of a Balanced Incomplete Block Design.

The second is a joint work with Akihiro Munemasa (Kyushu University). We present a new example of non-amorphous association schemes over the finite field. And, we mention possible new infinite series containing our new example different from van Dam's infinite series.

§1. Preliminaries

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a symmetric association scheme of class $d$ over a finite set $X$ of cardinality $n$. We refer [2] for notations and general theory of association schemes.

Let $P = (p_{ij})_{0 \leq i \leq d}$ and $Q = (q_{ij})_{0 \leq i \leq d}$ be the 1st and 2nd eigenmatrices of $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ respectively.

Let $\{\Lambda_j\}_{0 \leq j \leq d'}$ be a partition of $\{0, 1, \ldots, d\}$ with $\Lambda_0 = \{0\}$. We define $R_{\Lambda_j} = \bigcup_{\ell \in \Lambda_j} R_{\ell}$. If $\tilde{\mathcal{X}} = (X, \{R_{\Lambda_j}\}_{0 \leq j \leq d'})$ satisfies the conditions of association schemes, then we call $\tilde{\mathcal{X}} = (X, \{R_{\Lambda_j}\}_{0 \leq j \leq d'})$ a fusion scheme of $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$. $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is called amorphous if $\tilde{\mathcal{X}} = (X, \{R_{\Lambda_j}\}_{0 \leq j \leq d'})$ is a fusion scheme of $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ for any partition $\{\Lambda_j\}_{0 \leq j \leq d'}$ with $\Lambda_0 = \{0\}$.

There is a simple criterion in terms of $P$ for a given partition $\{\Lambda_j\}_{0 \leq j \leq d'}$ to give rise to a fusion scheme (due to E.Bannai [1], M.Muzychuk [9]): There exists a partition $\{\Delta_j\}_{0 \leq j \leq d'}$ of $\{0, 1, \ldots, d\}$ with $\Delta_0 = \{0\}$ such that each $(\Delta_i, \Lambda_j)$-block of $P$ has a constant row sum. The constant row sum turns out to be the $(i, j)$ entry of the 1st eigenmatrix of the fusion scheme.

In terms of the 2nd eigenmatrix $Q$, the above criterion becomes the following:

There exists a partition $\{\Delta_j\}_{0 \leq j \leq d'}$ of $\{0, 1, \ldots, d\}$ with $\Delta_0 = \{0\}$ such that each $(\Lambda_i, \Delta_j)$-block of $Q$ has a constant row sum.

In [5] Edwin R. van Dam considered strongly-regular demposition of the complete graph. The following result is due to him.
Theorem 1 (Edwin R. van Dam) Let \((\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)\) be a commutative strongly-regular decomposition of the complete graph on \(X\). Let \((X, \Gamma_i)\) have valency \(k_i\) and restricted eigenvalues \(r_i\) and \(s_i\) (where we do not assume that \(r_i > s_i\)), Then \((\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)\) is (i) an amorphous association scheme; or (ii) an association scheme in which three of the graphs, say \(\Gamma_2, \Gamma_3, \Gamma_4\), have the same parameters and which has eigenmatrix given by

\[
P = \begin{pmatrix}
1 & k_1 & k_2 & k_3 & k_4 \\
1 & s_1 & r_2 & r_3 & r_4 \\
1 & r_1 & s_2 & s_3 & r_4 \\
1 & r_1 & s_2 & r_3 & s_4 \\
1 & r_1 & r_2 & s_3 & s_4
\end{pmatrix},
\]

or (iii) it is not an association scheme, in which case the eigenmatrix is given by

\[
P = \begin{pmatrix}
1 & k_1 & k_2 & k_3 & k_4 \\
1 & s_1 & s_2 & r_3 & r_4 \\
1 & s_1 & r_2 & s_3 & r_4 \\
1 & s_1 & r_2 & r_3 & s_4 \\
1 & r_1 & s_2 & s_3 & s_4 \\
1 & r_1 & r_2 & s_3 & s_4
\end{pmatrix},
\]

where possibly one row is removed.

When we restrict his results to the framework of association schemes, only the cases (i) and (ii) appear.

For the case (ii), van Dam presents the following two examples:

For the case (ii), van Dam presents the following two examples:

\[
P = \begin{pmatrix}
1 & v - 4 & 1 & 1 & 1 \\
v - 4 & 1 & 1 & 1 \\
1 & 0 & -1 & -1 & 1 \\
1 & 0 & -1 & 1 & -1 \\
1 & 0 & 1 & -1 & -1
\end{pmatrix}, \quad (1)
\]

\[
P = \begin{pmatrix}
1 & 3276 & 273 & 273 & 273 \\
1 & -52 & 17 & 17 & 17 \\
1 & 12 & 17 & -15 & -15 \\
1 & 12 & -15 & 17 & -15 \\
1 & 12 & -15 & -15 & 17
\end{pmatrix}, \quad (2)
\]

(1) is the wreath product of the complete graph and \(L_{1,1,1}(2)\). (2) is constructed as a fusion scheme of the cyclotomic scheme of class \(d = 45\) over \(GF(2^{12})\). (1) is an imprimitive association scheme, and (2) is a primitive one.

§2. Motivation

The International Conference on Algebraic Combinatorics was held at Vladimir, near Moscow, in 1991. A.V. Ivanov presented the following conjecture:
A.V. Ivanov's Conjecture: Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a symmetric association scheme. We assume that for any $i(\neq 0)$, $\Gamma_i = (X, R_i)$ is a strongly-regular graph. Then $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is an amorphous association scheme.

Since the examples (1) and (2) exist, the result of Edwin R. van Dam is a counterexample of this conjecture ($d = 4$). We want to investigate A.V. Ivanov's conjecture and want to generalize van Dam's results to larger class $d$.

We call the following condition A.V. Ivanov's condition:

A.V. Ivanov's condition: Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a symmetric association scheme. For any $i(\neq 0)$, $\Gamma_i = (X, R_i)$ is a strongly-regular graph.

The next question arises naturally:

Question: Classify symmetric association schemes with A.V. Ivanov's condition.

However, in this stage, it is difficult to solve this problem. Later, we supplement one condition about the matrix form $P$ to get one of the generalizations of van Dam's result. In particular, we are interested in the shape of the $1$st eigenmatrix $P$ of an association scheme with A.V. Ivanov's condition.

§3. One of the generalizations

From now on, we assume a symmetric association scheme $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ satisfies A.V. Ivanov's condition.

Let $P_0$ be the right-lower $d \times d$ submatrix of $P$ and call it the principal part of $P$:

$$P = \begin{pmatrix} 1 & k_1 & \cdots & k_d \\ 1 & & \ddots & \\ \vdots & & & P_0 \\ 1 & & & \end{pmatrix}.$$  

Note that the rows of $P$ are indexed by the primitive idempotents of the adjacency algebra of the association scheme, and the columns of $P$ are indexed by the set of relations $\{R_i\}_{0 \leq i \leq d}$. Hence, $P_0$ is determined up to permutations of rows and columns.

From A.V. Ivanov's condition, for any $i$ ($i \neq 0$) we can make a fusion scheme $\tilde{\mathcal{X}} = (X, \{R_0, R_i, (X \times X) - R_0 - R_i\})$. This fusion scheme must be a strongly regular graph. From this fact, in each column of $P_0$ there are exactly 2 distinct values $a_i$ and $b_i$.

Using these facts and permuting the rows and columns suitably, or exchanging $a_i$ and $b_i$, we may start from the following general form without loss of generality:

$$P = \begin{pmatrix} 1 & k_1 & k_2 & \cdots & k_d \\ 1 & a_1 & a_2 & \cdots & a_d \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a_1 & & \cdots & \end{pmatrix}.$$
In this paper, we only treat the following matrix form $P$:

$$P = \begin{pmatrix}
1 & k_1 & k_2 & \cdots & k_d \\
1 & a_1 & a_2 & \cdots & a_d \\
1 & b_1 & a_2 & \cdots & b_d \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & b_1 & b_2 & \cdots & \tilde{T} \\
1 & b_1 & b_2 & \cdots & 1
\end{pmatrix},$$

(3)

where $\tilde{T}_{ij} \in \{a_j, b_j\}$.

This is an important general form. For this we have two reasons: One is that we investigated symmetric association schemes with A.V. Ivanov's condition with class $d$, concretely, where $d \leq 6$. Then the matrix form (3) is only left alive as a feasible case. Another is that (3) is one of the generalizations of the matrix form given by the paper of van Dam. If we restrict this matrix to the case $d = 4$, we can get van Dam's matrix form (Theorem 1 (ii)). Therefore, we treat this general form.

For the submatrix $\tilde{T}$ in (3), we define the $\{0, 1\}$-matrix $T = (T_{ij})_{3 \times 3}$ as:

$$T_{ij} = \begin{cases}
1, & \text{if } \tilde{T}_{ij} = a_j, \\
0, & \text{otherwise}.
\end{cases}$$

In the next theorem, we mention the combinatorial structure for the incidence matrix $T$.

**Theorem 2** Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a symmetric association scheme with the 1st eigenmatrix (3). Then, we get the followings:

(i) $k_3 = \ldots = k_d$,

(ii) $m_3 = \ldots = m_d$,

(iii) $a_3 = \ldots = a_d = k_3a_2k_2^{-1}$,

(iv) $b_3 = \ldots = b_d = k_3(m_1a_1k_1^{-1} + m_2b_1k_1^{-1} - m_1a_2k_2^{-1})k_2^{-1}$,

(v) $T$ is the incidence matrix of a symmetric balanced incomplete block design.

From Theorem 2 (v), since the number of 1 appearing in each row and column in $T$ is independent of the choice of the rows and columns of $T$, we denote this number by $\ell$:

For $3 \leq i \leq d$,

$$\ell = \# \{j \mid T_{ij} = 1\}.$$

**Proposition 3** Under the same hypothesis as in Theorem 2, the entries of $P$ and $Q$ are expressed as follows:

$$m_2 = \frac{(d - \ell - 2)((d - 3)\ell k_1 + (d - 2)^2 k_2)}{(d - 2)^2(d - 3)\ell},$$
\[ m_3 = \frac{(d-3)\ell k_1 + (d-2)^2k_2}{(d-2)^2\ell}, \]
\[ k_3 = \frac{(d-\ell)(d-2)k_2}{(d-3)\ell}, \]
\[ b_1 = \frac{(d-3)\ell k_1 ((d-2)^2 + \ell a_1)}{(\ell - (d-2)^2)((d-3)\ell k_1 + (d-2)^2 k_2)}, \]
\[ a_2 = \frac{(d-3)\ell(a_1 + 1)}{\ell - (d-2)^2}, \]
\[ b_2 = \frac{-\ell((d-3)\ell k_1 + (d-\ell)(d-2)(d-2)k_2)\ell a_1 + (d-3)((d-3)\ell k_1 - (d-2)^2 k_2)}{(\ell - (d-2)^2)((d-3)\ell k_1 + (d-2)^2 k_2)}, \]
\[ b_3 = \frac{(d-3)((d-2)k_2 + \ell k_1)\ell a_1 + (d-3)\ell^2 k_1 - (d-2)^2(d-\ell)(d-2)k_2}{(\ell - (d-2)^2)((d-3)\ell k_1 + (d-2)^2 k_2)}. \]

Moreover, \( a_1 \) must satisfy the following equation:
\[ s_2 a_1^2 + s_1 a_1 + s_0 = 0, \]
where,
\[ s_2 = -(d-3)^2((d-3)\ell k_1 + \ell k_2 - (d-2)^2 k_2), \]
\[ s_1 = 2\ell \ell^2(d-3)^2 k_1, \]
\[ s_0 = -(d-2)^2(\ell - (d-2)^2)k_2^2 \]
\[ -\ell(d-3)(\ell - (d-2)^2)((d-2)^2(k_1 + 1) + \ell k_1)k_2 + \ell^3(d-3)^2 k_1. \]

Next, we consider the trivial case for \( \tilde{T} \), that is,
\[ \tilde{T} = a_3I_{d-2} + b_3(J_{d-2} - I_{d-2}). \]
Then, we get \( \ell = 1 \). From Theorem 2 and Proposition 3 we get the following:

**Proposition 4** In addition to the assumption of Theorem 2, we assume that in (3)
\[ \tilde{T} = a_3I_{d-2} + b_3(J_{d-2} - I_{d-2}). \]
Then, \( k_2 = k_3, a_2 = a_3 \) and \( b_2 = b_3 \). In other words, we have
\[
P = \begin{pmatrix}
1 & k_1 & k_2 & \cdots & k_2 \\
1 & a_1 & a_2 & \cdots & a_2 \\
1 & b_1 & a_2 & \cdots & b_2 \\
1 & b_1 & b_2 & a_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \\
1 & b_1 & b_2 & \cdots & a_2
\end{pmatrix}.
\]
Moreover, \(n, a_1, b_1, a_2, b_2, k_1, k_2, m_1, m_2\) are expressed as follows:

\[
\begin{align*}
n &= \frac{((d - 2)^2 m_2 + d - a_2 - 3)^2}{a_2^2 - 2(d - 3)a_2 + (d - 3)((d - 2)^2 m_2 + d - 3)}, \\
k_1 &= (d - 2)^2 m_1, \\
k_2 &= \frac{(d - 2)^2 m_2 a_2}{a_2^2 - 2(d - 3)a_2 + (d - 3)((d - 2)^2 m_2 + d - 3)}, \\
m_1 &= m_2 \cdot \frac{-(d - 1)a_2^2 - 2a_2 + (d - 2)^2 m_2 + d - 3}{a_2^2 - 2(d - 3)a_2 + (d - 3)((d - 2)^2 m_2 + d - 3)}, \\
a_1 &= -(d - 1)a_2 - 1, \\
b_1 &= \frac{(d - 3 - a_2)m_2((d - 1)a_2^2 + 2a_2 - (d - 2)^2 m_2 - d + 3)}{a_2^2 - 2(d - 3)a_2 + (d - 3)((d - 2)^2 m_2 + d - 3)}, \\
b_2 &= \frac{a_2^2 - (d - 4)a_2 - (d - 2)^2 m_2 - d + 3}{a_2^2 - 2(d - 3)a_2 + (d - 3)((d - 2)^2 m_2 + d - 3)},
\end{align*}
\]

where,

\[
m_2 > \frac{(d - 1)a_2^2 + 2a_2 - (d - 3)}{(d - 2)^2}.
\]

If we assume \(a_2 = d - 3\) and \(m_2 = (d - 3)s\ (s \in \mathbb{Z}_{>0})\) in the previous proposition, then (4) reduces to

\[
P = \begin{pmatrix}
1 & (d - 2)^2(s - 1) & d - 3 & d - 3 & \cdots & d - 3 \\
1 & -(d - 2)^2 & d - 3 & d - 3 & \cdots & d - 3 \\
1 & 0 & d - 3 & -1 & \cdots & -1 \\
1 & 0 & -1 & d - 3 & \cdots & -1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & -1 & -1 & \cdots & d - 3
\end{pmatrix}.
\] (5)

In (5), \((X, R_1)\) is a complete multipartite graph, and \((X, R_i)\ (2 \leq i \leq d)\) is of Latin square type \(L_1(d - 2)\). Therefore, \(\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})\) with the 1st eigenmatrix (5) is the wreath product of the complete graph and \(\underbrace{L_1, \ldots, L_1}_{d-1}(d - 2)\). This is a generalization of...
§4. A New Example and Possible Infinite Series

This is a joint work with Akihiro Munemasa (Kyushu University). The aim of this section is to present a new example of a non-amorphous association scheme and possible infinite series.

Edwin R. van Dam gives the following infinite series with respect to Theorem 1 (ii):

\[
P = \begin{pmatrix}
1 & 2^{3t} - 4 - 3(2^{2t}) - 3(2^t) & 2^{2t} + 2^t + 1 & 2^{2t} + 2^t + 1 & 2^{2t} + 2^t + 1 \\
1 & -4 - 3(2^t) & 1 + 2^t & 1 + 2^t & 1 + 2^t \\
1 & -4 + 2^t & 1 - 2^t & 1 - 2^t & 1 - 2^t \\
1 & -4 + 2^t & 1 - 2^t & 1 + 2^t & 1 - 2^t \\
1 & -4 + 2^t & 1 - 2^t & 1 - 2^t & 1 + 2^t
\end{pmatrix}.
\]

In the case $t = 4$, he gives one example of non-amorphous association schemes as a fusion scheme of the cyclotomic scheme of class $d = 45$ over finite field $GF(2^{12})$:

\[
P = \begin{pmatrix}
1 & 3276 & 273 & 273 & 273 \\
1 & -52 & 17 & 17 & 17 \\
1 & 12 & 17 & -15 & -15 \\
1 & 12 & -15 & 17 & -15 \\
1 & 12 & -15 & -15 & 17
\end{pmatrix}.
\]

However, there are no comments for the other values $t$.

We tried to construct another example in van Dam's infinite series. However, we could not find one!

The next table presents the feasible parameters $n, k_1, k_2, m_1, m_2, a_1, b_1, a_2, b_2$ calculated in Proposition 4 ($d = 4$). (We use the automatic E-mail interface to Professor A.E. Brouwer’s database of distance-regular graphs.)

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We introduce the following theorem. This theorem is due to A.E. Brouwer, R.M. Wilson, and Q. Xiang ([4]).

**Theorem 5** (A.E. Brouwer, R.M. Wilson, and Q. Xiang)

Let $p$ be a prime and $q = p^e$. Let $e| (q-1)$. We assume that there exists $h > 0$ such that

$$h := \min \{ h > 0; p^f \equiv -1 \mod e \}.$$

We put $t_1 = 2e\kappa^{-1}$. Then, for any $u_1 \ (1 \leq u_1 \leq e - 1)$, strongly-regular graphs with the following eigenvalues exist:

$$k = \frac{q-1}{e}u_1 \text{ with multiplicity 1},$$
\[ \theta_1 = \frac{u_1}{e}(-1 + (-1)^{t_1}\sqrt{q}) \text{ with multiplicity } q - 1 - k, \]
\[ \theta_2 = \frac{u_1}{e}(-1 + (-1)^{t_1}\sqrt{q}) + (-1)^{t_1+1}\sqrt{q} \text{ with multiplicity } k. \]

If we assume, in Proposition 4, that the strongly-regular graph \((X, R_1)\) is equal to A.E. Brouwer, R.M. Wilson, and Q. Xiang's strongly-regular graphs, then the parameters \(e\) and \(t_1\) in Theorem 5 satisfy the following:

Lemma 1
\[ e = \frac{d^2 - 4d + 5}{(d-2)^2}u_1, \quad t_1 = 1. \]

 Lemma 2 If \(d = 4\), the size \(n\) is equal to a power of 16, i.e., \(n = 16^{2h+1}(= 2^{4(2h+1)})\).

Let \(n = q^{2h+1}\) and \(q = 16\). Under the assumption of Lemmas 1 and 2, we guess the existence of the following possible infinite series:

\[ P = \begin{pmatrix} 1 & 12 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 1 \end{pmatrix}. \tag{6} \]

We give the reason that (6) is natural infinite series.
If \(n = 16^1 (h = 0)\) in (6), then we get the following:
\[ P = \begin{pmatrix} 1 & 12 & 1 & 1 & 1 \\ 1 & -4 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & -1 & 1 \end{pmatrix}. \]

This matrix belongs to the matrix form (1).
If \(n = 16^3 (h = 1)\) in (6), then we get the following:

This matrix is an example given by the paper of van Dam ([5]).
If \(n = 16^5 (h = 2)\) in (6), then we can construct the following new example as a fusion scheme of the cyclotomic scheme of class \(d = 75\) over \(GF(2^{20})\) using the computer:
\[ P = \begin{pmatrix} 1 & 838860 & 69905 & 69905 & 69905 \\ 1 & -820 & 273 & 273 & 273 \\ 1 & 12 & 273 & -239 & -239 \\ 1 & 204 & 273 & -239 & -239 \\ 1 & 204 & -239 & 273 & -239 \end{pmatrix}. \]
This is a new example of non-amorphous association schemes of class $d = 4$.

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References


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