THE BRIANÇON-SPEDER AND OKA FAMILIES ARE NOT
BILIPSCHITS TRIVIAL

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At the symposium "Several Topics on Singularity Theory," Tadeusz Mostowski gave excellent lectures on Lipschitz equisingularity ([18]). Related to this topic, the author described some facts in the short communications. In this note we explain it in detail.

Let $f$, $g : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be analytic function germs. We say that they are blow-analytically equivalent if there are real modifications $\mu : (M, \mu^{-1}(0)) \rightarrow (\mathbb{R}^n, 0)$, $\mu' : (M', \mu'^{-1}(0)) \rightarrow (\mathbb{R}^n, 0)$ and an analytic isomorphism $\Phi : (M, \mu^{-1}(0)) \rightarrow (M', \mu'^{-1}(0))$ which induces a homeomorphism $\phi : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that $f = g \circ \phi$. A blow-analytic homeomorphism is such a homeomorphism induced by an analytic isomorphism via real modifications. Blow-analytic equivalence is a notion introduced by Tzee-Char Kuo ([14, 15]) as a natural equivalence relation for real analytic function germs. Concerning the blow-analyticity, he posed the following question:

**Question 0.1.** Is a blow-analytic homeomorphism a biLipschitz homeomorphism?

A blow-analytic homeomorphism is an arc-analytic homeomorphism in the sense of K. Kurdyka [16]. Namely, any analytic arc is mapped to an analytic arc by a blow-analytic homeomorphism. The relation between blow-analyticity and arc-analyticity was discussed by E. Bierstone and P. Milman [1]. Originally, Kuo presented the following weak conjecture ([14]):

**Conjecture 0.2.** A blow-analytic homeomorphism preserves the contact order of analytic arcs.

For more properties of blow-analyticity, see [5].

We next consider the Briançon-Speder family ([2]) $f_t : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0)$, $t \in J = (-1 - \epsilon, \infty)$, defined by

$$f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}$$

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where $\epsilon$ is a sufficiently small positive number. Then, $\{f_t\}_{t \in J}$ is a family of functions with isolated singularities. It was shown by T. Fukui [4] that $\{f_t\}_{t \in J}$ admits a blow-analytic trivialisation along $I$ for any closed subinterval $I$ of $J$. At about the same time as Fukui, Koike observed in [10] that $(\mathbb{R}^3, f_0^{-1}(0))$ and $(\mathbb{R}^3, f_{-1}^{-1}(0))$ cannot be equivalent via a homeomorphism which preserves the tangency of analytic arcs contained in the zero-sets. We call such a homeomorphism a strong homeomorphism. Combining the results of Fukui and Koike, we get the fact that

(0.1) a blow-analytic equivalence is not always a “blow-analytic and strong $C^0$” equivalence.

This implies the following:

(0.2) A blow-analytic equivalence is not always a “blow-analytic and biLipschitz” equivalence.

These facts show that there are a negative example to Question 0.1 and a counterexample to Conjecture 0.2. In fact, after Fukui and Koike, L. Paunescu constructed a blow-analytic homeomorphism which is not a biLipschitz one ([24]).

For a family of analytic function germs with isolated singularities, T.C. Kuo ([15]) established a locally finite classification theorem on blow-analytic equivalence, but recently J.-P. Henry and A. Parusiński ([8, 9]) showed the appearance of Lipschitz moduli. Therefore, Question 0.1 is not so interesting any more. On the other hand, a local finiteness theorem was established on biLipschitz equivalence for a family of zero-sets of analytic functions (T. Mostowski [17], A. Parusiński [21, 22, 23]).

**Example 0.3.** Let $f_t : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $t \in I = [-2, 2]$, be a family of weighted homogeneous polynomials defined by

$$f_t(x, y) = x^3 + 3xy^4 + ty^6.$$

Then $\{f_t\}_{t \in I}$ admits a blow-analytic trivialisation along $I$ ([4], T. Fukui and L. Paunescu [6]), and $\{(\mathbb{R}^2, f_t^{-1}(0))\}$ is biLipschitz trivial over $I$. But Lipschitz moduli as function germs appear in the family $\{f_t\}_{t \in I}$ ([8, 9]).

In the next section we introduce the notion of a sea-tangle neighbourhood of a Lipschitz arc. Using some properties of the neighbourhood, we shall show in §2 that the Briançon-Speder family is not biLipschitz trivial as a family of zero-sets, and in §3 that the Oka family [20] is not so, either. But both families of functions are blow-analytically trivial. Therefore (0.2) is improved as follows:

(0.3) Blow-analytic equivalence for functions does not always imply biLipschitz equivalence for the zero-sets.
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1. PRELIMINARIES

By a $C^\omega$ arc at $0 \in \mathbb{R}^n$, we mean the germ of an analytic map $\lambda : [0, \epsilon) \to \mathbb{R}^n$ with $\lambda(0) = 0$, $\lambda(s) \neq 0$, $s > 0$. The set of all such arcs is denoted by $A(\mathbb{R}^n, 0)$.

Let $S^{n-1}$ denote the $(n-1)$-dimensional unit sphere. For $a = (a_1, \cdots, a_n) \in S^{n-1}$, let $L(a) : [0, \delta) \to \mathbb{R}^n$, $\delta > 0$, be a mapping defined by

$$L(a)(t) = (a_1 t, \cdots, a_n t).$$

Then $L(a) \in A(\mathbb{R}^n, 0)$. For any $ \lambda \in A(\mathbb{R}^n, 0)$, there exists unique $a \in S^{n-1}$ such that $\lambda$ is tangent to $L(a)$ at $0 \in \mathbb{R}^n$. Then we write $L(a) = T(\lambda)$.

For any analytic function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, let $C_0(f)$ denote the set of connected components of $f^{-1}(0) - \{0\}$ as germs at $0 \in \mathbb{R}^n$. We put $C_0(f) = \{C_1, \cdots, C_m\}$, $m \in \{0\} \cup \mathbb{N}$.

For each $i$, let $D_i(f)$ be the set of $a \in S^{n-1}$ satisfying the following condition: There exists an arc $\lambda \in A(\mathbb{R}^n, 0)$ such that $\lambda \subset \overline{C_i}$ and $\lambda$ is tangent to $L(a)$ at $0 \in \mathbb{R}^n$.

For $1 \leq i, j \leq m$, $i \neq j$, we define the cardinal number of common directions of $C_i$ and $C_j$ as follows:

$$D_{ij}(f) = \#(D_i(f) \cap D_j(f)).$$

Let $\theta : [0, \epsilon) \to \mathbb{R}^n$ be a Lipschitz map not identically zero such that $\theta(0) = 0$. For $d, C > 0$, define a sea-tangle neighbourhood of $\theta$ of degree $d$ with width $C$ in a small neighbourhood of $0 \in \mathbb{R}^n$:

$$ST_d(\theta; C) = \{x \in \mathbb{R}^n : \text{dist}(x, \theta) \leq C|x|^d\}.$$ 

This notion originated in the horn neighbourhood $H_d(f; C)$ introduced by T.C. Kuo [11, 13].

Let $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a biLipschitz homeomorphism germ, namely, there are positive numbers $K_1$, $K_2 > 0$ with $K_1 \leq K_2$ such that

$$K_1|x_1 - x_2| \leq |\phi(x_1) - \phi(x_2)| \leq K_2|x_1 - x_2|$$

in a small neighbourhood of $0 \in \mathbb{R}^n$. Conversely, we have

$$\frac{1}{K_2}|y_1 - y_2| \leq |\phi^{-1}(y_1) - \phi^{-1}(y_2)| \leq \frac{1}{K_1}|y_1 - y_2|$$

in a small neighbourhood of $0 \in \mathbb{R}^n$. Then we have the following:

**Lemma 1.1.** For $K > 0$, $ST_d(\phi(\theta); \frac{KK_2}{K_1}) \subset \phi(ST_d(\theta; K)) \subset ST_d(\phi(\theta); \frac{KK_2}{K_1})$ in a small neighbourhood of $0 \in \mathbb{R}^n$. 
Proof. We first show the right inclusion. Pick \( x \in ST_d(\theta; K) \). Then there is \( x_0 \in \theta \) such that \( |x - x_0| \leq K|x|^d \). On the other hand, \( K_1|x| \leq |\phi(x)| \leq K_2|x| \). Therefore

\[
\text{dist}(\phi(x), \phi(\theta)) \leq |\phi(x) - \phi(x_0)| \leq K_2|x - x_0| \leq KK_2|x|^d \leq \frac{KK_2}{K_1^d}|\phi(x)|^d.
\]

Thus \( \phi(x) \in ST_d(\phi(\theta); \frac{KK_2}{K_1^d}) \).

We can show the left inclusion similarly. By the above argument, we have

\[
\phi^{-1}(ST_d(\phi(\theta); C)) \subset ST_d(\theta; \frac{C(\frac{1}{K_1})}{(\frac{1}{K_2})^d}).
\]

Set \( \frac{C(\frac{1}{K_1})}{(\frac{1}{K_2})^d} = K \), then \( C = \frac{KK_1}{K_2^d} \).

It is easy to see the following fact.

**Lemma 1.2.** Let \( 0 < C < 1 \) and \( a \in S^{n-1} \). Let \( \theta : [0, \epsilon) \rightarrow \mathbb{R}^n \) be a Lipschitz map not identically zero such that \( \theta(0) = 0 \). Suppose that \( \theta \subset ST_1(L(a); C) \) as germs at \( 0 \in \mathbb{R}^n \). Then \( L(a) \subset ST_1(\theta; C) \) as germs at \( 0 \in \mathbb{R}^n \).

2. **The Briançon-Speder family is not biLipschitz trivial**

Let \( D = \{ x \in \mathbb{R} : |x| < 1 + \epsilon \} \) where \( \epsilon \) is a sufficiently small positive number. Concerning the Briançon-Speder family [2], we have

**Theorem 2.1.** Let \( f_t : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0) \), \( t \in D \), be a family of weighted homogeneous polynomial functions with isolated singularities defined by

\[
f_t(x, y, z) = z^5 + tzy^6 + y^7x + x^{15}.
\]

Then \( (\mathbb{R}^3, f_0^{-1}(0)) \) is not biLipschitz equivalent to \( (\mathbb{R}^3, f_{-1}^{-1}(0)) \).

**Proof.** Put \( f = f_0 \) and \( g = f_{-1} \). Then \( f^{-1}(0) \) is the graph of a function differentiable at \( 0 \in \mathbb{R}^2 \):

\[
(x, y) \rightarrow z = -(y^7x + x^{15})^{\frac{1}{2}}.
\]

Remark that there is a positive number \( C > 0 \) such that

\[
|x| \leq C|(x, y)|^{\frac{1}{2}}
\]

near \( 0 \in \mathbb{R}^2 \). Let \( \Pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be the projection defined by \( \Pi(x, y, z) = (x, y) \). Then \( \Pi \) gives a homeomorphism from \( f^{-1}(0) \) to \( \mathbb{R}^2 \).

Pick a point \( P_0 = (1, y_1, z_1) \) on \( g^{-1}(0) \) with \( y_1 > 0, z_1 > 0 \). Define the \( C^\omega \) arcs \( \lambda_j \in A(\mathbb{R}^3, 0) \), \( 1 \leq j \leq 4 \), as follows:

\[
\lambda_1(s) = (s, 0, -s^3), \quad \lambda_2(s) = (0, s, 0), \quad \lambda_3(s) = (s, y_1s^2, z_1s^3), \quad \lambda_4(s) = (0, -s, 0) \quad (s \geq 0).
\]
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Then $\lambda_1$ and $\lambda_3$ are tangent at $0 \in \mathbb{R}^3$, and they are perpendicular to $\lambda_2, \lambda_4$ at $0 \in \mathbb{R}^3$.

Assume that there is a biLipschitz homeomorphism $\phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $\phi(g^{-1}(0)) = f^{-1}(0)$. It follows from the relations of $\lambda_j$’s that there are positive numbers $C_1, C_2 > 0$ such that

$$\lambda_3 \subset ST_{\frac{3}{2}}(\lambda_1; C_1), \lambda_j \subset \mathbb{R}^3 - ST_1(\lambda_1; C_2), \ j = 2, 4.$$  

By Lemma 1.1, there are positive numbers $C_3, C_4 > 0$ such that

$$\phi(\lambda_3) \subset ST_{\frac{3}{2}}(\phi(\lambda_1); C_3), \phi(\lambda_j) \subset \mathbb{R}^3 - ST_1(\phi(\lambda_1); C_4), \ j = 2, 4.$$  

in a neighbourhood of $0 \in \mathbb{R}^3$. Then, by (2.1), there are positive numbers $C_5, C_6 > 0$ such that

$$(* \) $\Pi(\phi(\lambda_3)) \subset ST_{\frac{3}{2}}(\Pi(\phi(\lambda_1)); C_5), \Pi(\phi(\lambda_j)) \subset \mathbb{R}^2 - ST_1(\Pi(\phi(\lambda_1)); C_6), \ j = 2, 4,$$  

in a neighbourhood of $0 \in \mathbb{R}^2$.

On the other hand, $\Pi(\phi(\lambda_3)) - \{0\}$ and $\Pi(\phi(\lambda_4)) - \{0\}$ are contained in different components of $\mathbb{R}^2 - \Pi(\phi(\lambda_1)) \cup \Pi(\phi(\lambda_3))$. This contradicts ($*$). Thus $(\mathbb{R}^3, g^{-1}(0))$ is not biLipschitz equivalent to $(\mathbb{R}^3, f^{-1}(0))$. 

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3. The Oka family is not biLipschitz trivial

Let $D = (-1 - \epsilon, 1 + \epsilon)$ be an open interval as in §2. Concerning the Oka family [20], we have

**Theorem 3.1.** Let $f_t : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$, $t \in D$, be a family of polynomial functions with isolated singularities defined by

$$f_t(x, y, z) = x^8 + y^{16} + z^{16} + tx^5z^2 + x^3yz^3.$$ 

Then $(\mathbb{R}^3, f_0^{-1}(0))$ is not biLipschitz equivalent to $(\mathbb{R}^3, f_1^{-1}(0))$.

**Proof.** We first recall some facts which we have observed in the proof of Theorem B in [10]. Put

$$f(x, y, z) = f_0(x, y, z) = x^8 + y^{16} + z^{16} + x^3yz^3.$$ 

In each coordinate plane, $f^{-1}(0) - \{0\} = \emptyset$. Here we put

$$A_1 = \{ x > 0, \ y > 0, \ z < 0 \}, \quad A_2 = \{ x > 0, \ y < 0, \ z > 0 \},$$

$$A_3 = \{ x < 0, \ y > 0, \ z > 0 \}, \quad A_4 = \{ x < 0, \ y < 0, \ z < 0 \}.$$ 

Set $S_i = f^{-1}(0) \cap A_i$, $1 \leq i \leq 4$. Then $f^{-1}(0) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{0\}$ and each $\overline{S_i} = S_i \cup \{0\}$ is homeomorphic to $S^2$. As seen in [10], $D(f) = 1$, $i \neq j$.

Next put

$$g(x, y, z) = f_1(x, y, z) = x^8 + y^{16} + z^{16} + x^5z^2 + x^3yz^3.$$ 

In $(x, y)$-plane or $(y, z)$-plane, $g^{-1}(0) - \{0\} = \emptyset$. Here we put

$$B_1 = \{ x > 0, \ y > 0, \ z < 0 \}, \quad B_2 = \{ x > 0, \ y < 0, \ z > 0 \},$$

$$B_3 = \{ x < 0, \ y > 0, \ z > 0 \}, \quad B_4 = \{ x < 0, \ y < 0, \ z < 0 \}.$$ 

Set $P_i = g^{-1}(0) \cap B_i$, $1 \leq i \leq 4$. Then $g^{-1}(0) = P_1 \cup P_2 \cup P_3 \cup P_4 \cup \{0\}$ and each $\overline{P_i} = P_i \cup \{0\}$ is homeomorphic to $S^2$. By the observations seen in [10], there are $\lambda_1, \lambda_2 \in A(\mathbb{R}^3, 0)$ with $\lambda_1 \subset \overline{P_3} \cap \{y = 0\}$ and $\lambda_2 \subset \overline{P_4} \cap \{y = 0\}$ such that $T(\lambda_1) = T(\lambda_2) = L((-1, 0, 0))$, and there are $\mu_1, \mu_2 \in A(\mathbb{R}^3, 0)$ with $\mu_1 \subset \overline{P_3} \cap \{y = x\}$ and $\mu_2 \subset \overline{P_4} \cap \{y = x\}$ such that $T(\mu_1) = T(\mu_2) = L((-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0))$. Therefore there are $C, D > 0$ and $d > 1$ such that $\lambda_1 \subset ST_d(\lambda_2; D) \subset ST_1(\lambda_1; C)$, $\mu_1 \subset ST_d(\mu_2; D) \subset ST_1(\mu_1; C)$ and $\{0\} \subset ST_1(\mu_1; C)$ and $\{0\}$ in a small neighbourhood of 0 in $\mathbb{R}^3$.

Suppose that there is a biLipschitz homeomorphism germ $\phi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ with Lipschitz constants $K_1, K_2 > 0$ in (1.1) such that $\phi(g^{-1}(0)) = f^{-1}(0)$. Set $\gamma_i = \phi(\lambda_i)$ and $\nu_i = \phi(\mu_i)$, $i = 1, 2$. Here we put $E = \frac{CK_1}{K_2} > 0$. Then it follows from Lemma 1.1 that $ST_1(\gamma_1; E) \subset \phi(ST_1(\lambda_1; C))$ and $ST_1(\nu_1; E) \subset \phi(ST_1(\mu_1; C))$ in a small neighbourhood of 0 of $\mathbb{R}^3$. Remark that

$$ST_1(\gamma_1; E) \cap ST_1(\nu_1; E) = \{0\}$$

in a neighbourhood of 0 of $\mathbb{R}^3$. Remark that

(3.1) $ST_1(\gamma_1; E) \cap ST_1(\nu_1; E) = \{0\}$ in a neighbourhood of 0 of $\mathbb{R}^3$. 


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Each $P_i$, $1 \leq i \leq 4$, is mapped to some $S_j$ by $\phi$ as set-germs at $0 \in \mathbb{R}^3$. Let $\phi(P_3) = S_{j(3)}$ and $\phi(P_4) = S_{j(4)}$. Then $D_{j(3)}(f) \cap D_{j(4)}(f) = \{a\}$ where $a$ is a member of $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. We consider the sea-tangle neighbourhood $V = ST_1(L(a); E)$ of $L(a)$. If there is a neighbourhood $U$ of $0 \in \mathbb{R}^3$ such that $\gamma_1, \nu_1 \subset V$ in $U$, then it follows from Lemma 1.2 that $L(a) \subset ST_1(\gamma_1; E)$ and $L(a) \subset ST_1(\nu_1; E)$ in a neighbourhood of $0 \in \mathbb{R}^3$. This contradicts (3.1). Thus

(3.2) $\gamma_1$ or $\nu_1$ is not contained in $V \cap U$ for any neighbourhood $U$ of $0 \in \mathbb{R}^3$.

Put $H = \frac{Df_{k2}}{\kappa^2}$. By Lemma 1.1, $\gamma_1 \subset ST_d(\gamma_2; H)$ and $\nu_1 \subset ST_d(\nu_2; H)$ in a neighbourhood of $0 \in \mathbb{R}^3$. This contradicts (3.2) because $\gamma_1, \nu_1 \subset A_j(3) \cup \{0\}$ and $\gamma_2, \nu_2 \subset A_{j(4)} \cup \{0\}$ near $0 \in \mathbb{R}^3$. Therefore, $(\mathbb{R}^3, f^{-1}(0))$ and $(\mathbb{R}^3, g^{-1}(0))$ are not biLipschitz equivalent as germs at $0 \in \mathbb{R}^3$.

Remark 3.2. We drew the pictures of the zero-sets of the Briançon-Speder family and the Oka family in [10] using the analysis at high school mathematics level. In [27] T. Fukui is explaining in detail how to draw and understand the pictures of their zero-sets using the Newton Polyhedra determined by the defining functions.

4. PROBLEM

In the previous sections, we discussed the Lipschitz equisingularity of the Briançon-Speder family and the Oka family. In the complex case they are well-known as examples of families of functions which are $\mu$-constant but not $\mu^*$-constant. From our observations, it gives rise to the following problem:

Problem 4.1. Let $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, $t \in J$, be a family of analytic functions, and let $f_t^c : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ denote the complexification of $f_t$. Assume that $\mu(f_t^c)$ ($< \infty$) is constant but $\mu^*(f_t^c)$ is not constant. Then is the family of zero-sets $\{(\mathbb{R}^n, f_t^{-1}(0))\}_{t \in J}$ not biLipschitz trivial?

Remark 4.2. In the case $n = 2$, there is no family which satisfies the assumption of the above problem.

The problem whether the biLipschitz triviality implies the Kuo-Verdier $(w)$-regularity in the complex case has been worked on by J.-P. Henry, T. Mostowski and M. Merle. The case $n = 3$ was shown affirmatively ([19]). See [12, 26] for the $(w)$-regularity which is equivalent to the Whitney $(b)$-regularity in the complex case (J.-P. Henry and M. Merle [7]). This condition is also equivalent to the $\mu^*$-constancy (B. Teissier [25], J. Briançon and J.P. Speder [3]). Therefore, "not $\mu^*$-constancy" implies "not biLipschitz triviality" in the complex 3 variables case. But this does
not hold in the real case, namely, we cannot omit the assumption of $\mu$-constancy from our problem.

**Example 4.3.** Let $f_t: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}, 0), t \in \mathbb{R}$, be a family of polynomial functions defined by

$$ f_t(x, y, z) = x^2 + y^2 + t^2 z^2 + z^4. $$

Then $\mu(f_t^C) < \infty$, $t \in \mathbb{C}$, and $\{f_t^C\}$ is not $\mu$-constant. As a result, it is not $\mu^*$-constant, either. On the other hand, $\{(\mathbb{R}^3, f_t^{-1}(0))\}_{t \in \mathbb{R}}$ is bi-Lipschitz trivial.

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