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A volume formula for hyperbolic tetrahedra and a problem of Fenchel

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Abstract

一般化された双曲四面体とは、三次元双曲空間内の(コンパクトで無くても良い)有限体積の多面体で、(頂点が双曲空間の「外」に「少し」出ているかもしれない)双曲四面体に対し、その様な頂点では polar plane によりその周りを切り落とす事で得られるものの事です。通常の意味での双曲四面体に対しては、村上 順 氏と 八野 正和 氏が量子 $6j$-symbol を用いて体積の公式を構成しました。この公式が、一般化された双曲四面体に対しても使える事を紹介します。

証明に際しての要点は、一般化された双曲四面体も、通常のものと同じく面角でその形が定まる事です。この考察から、与えられた数の組に対し、それらを面角とする一般化された双曲四面体が存在するかどうかの必要充分条件も得られました。この事も紹介します。

1 Introduction

Obtaining volume formulae for basic polyhedra is one of the basic and important problems in geometry. In hyperbolic space this problem has been attacked from its birth. An *orthoscheme* is a simplex of a generalization (with respect to dimensions) of a right triangle. Since any hyperbolic polyhedra can be decomposed into finite number of orthoschemes, the first step of solving the problem is to obtain a volume formula for orthoschemes. In three-dimensional case, N. I. Lobachevsky [Lo] found in 1839 a volume formula for orthoschemes. Thus the next step is to find a formula for ordinary tetrahedra.

Although the lengths of six edges of tetrahedra determine their sizes and shapes, in hyperbolic space it is known that the six dihedral angles also plays the same role. Actually the valuables of the Lobachevsky's formula mentioned above are dihedral angles, and following this way we can obtain a volume formula for *generalized* hyperbolic tetrahedra in terms of the six dihedral angles (the meaning of the term "generalized" will be explained later). Here we note that, by the fact mentioned previous paragraph, we can construct such a formula from that of orthoschemes. But in this case we have to calculate the dihedral angles of orthoschemes from those of tetrahedra. It is emphasized that the formula can be calculated directly from the dihedral angles of a given generalized tetrahedron.

The first answer to this problem was (at least the author is aware) given by W.-Y. Hsiang in 1988. In [Hs] the formula was presented through an integral expression. On the other hand, many volume formulae for hyperbolic polyhedra is presented via Lobachevsky function or the dilogarithm function (see, for example, [Ve]). The first such presentation was given by Y. Cho and H. Kim in 1999 (see [CK]). Their formula was derived from that of ideal tetrahedra, and has the following particular property: due to the way of the proof it is not symmetric with respect to the dihedral angles. Thus the next problem is to obtain its essentially symmetrical expression. Such a formula was obtained by J. Murakami and M. Yano in about 2001. In [MY] they derived it from the quantum $6j$-symbol, but the proof is to reduce it to Cho-Kim's one.

In hyperbolic space it is possible to consider that vertices of a polyhedron lie "outside the space and its sphere at infinity." For each such vertex there is a canonical way to cut off the vertex with its neighborhood. This operation is called a *truncation* at the vertex, which will be defined more precisely in Section 3, and we thus obtain convex polyhedron (possibly non-compact) with finite volume. Such a polyhedron appears, for example, as fundamental polyhedra for hyperbolic Coxeter groups or building blocks of three-dimensional hyperbolic manifolds with totally geodesic boundary. Since the volume of hyperbolic manifolds are topological invariants, it is meaningful, also for 3-manifold theory, to obtain a volume formula for such polyhedra.

R. Kellerhals presented in 1989 that the formula for orthoschemes can be applied, without any modification, to "mildly" truncated ones at *principal vertices* (see [Ke] for detail). This result inspires that Murakami-Yano's formula also may be applied to *generalized hyperbolic tetrahedra*, tetrahedra of each vertices being finite, ideal or truncated (see Definition 3.2 for precise definition), and this is the main result of this report (see Theorem 1.1). Here we note that, although the lengths of the edges emanating from a vertex at infinity are infinity, the dihedral angles may be finite. This is another reason why we take not the edge lengths but the dihedral angles as the valuables of our volume formula.

The key tool for the proof is so-called Schläfli's differential formula, a simple description for the volume differential of polyhedra as a function of the dihedral angles and the volume of the apices. Since the volume formula is a function of the dihedral angles, to apply Schläfli's differential formula we have to translate the dihedral angles of a generalized tetrahedron to the lengths of its edges. This requirement yields a necessary and sufficient condition of a set of positive numbers to be the dihedral angles of a generalized simplex (see Theorem 3.3).

W. Fenchel asked in [Fe] a necessary and sufficient conditions for a given set of positive real numbers to be the dihedral angles of a hyperbolic n-simplex. F. Luo answered in [Lu] this question. Theorem 3.3 in this report is a generalization of his result.

1.1 Volume formula for generalized hyperbolic tetrahedra

Let $T = T(A, B, C, D, E, F)$ be a generalized tetrahedron in the three-dimensional hyperbolic space $\mathbb{H}^3$ whose dihedral angles are A, B, C, D, E, F. Here the configuration of the dihedral angles are as follows (see also Figure 1): three edges corresponding to A, B and C arise from a vertex, and the angle D (resp. E, F) is put on the edge opposite to that of A (resp. B, C). Let G be the *Gram matrix* of T defined as follows:

$$G := \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix}.$$

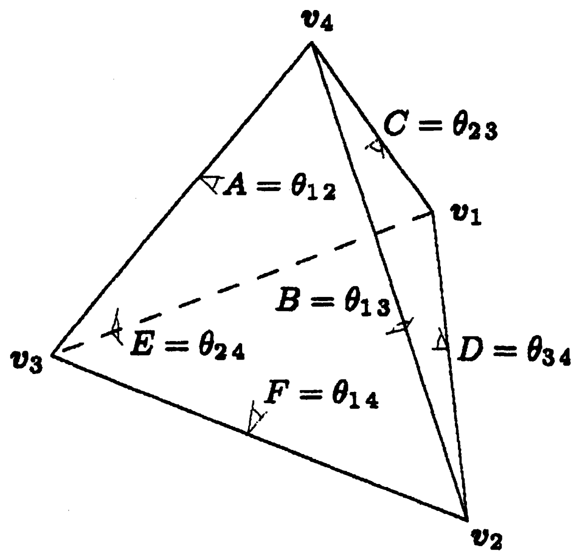

Figure 1: The dihedral angles of T

Let $a := \exp\sqrt{-1}A, b := \exp\sqrt{-1}B, \dots, f := \exp\sqrt{-1}F$, and let $U(z, T)$ be the complex valued function defined as follows:

$$\begin{aligned} U(z,T) \ := \ & \frac{1}{2}\{\mathrm{Li}_2(z) + \mathrm{Li}_2(abdez) + \mathrm{Li}_2(acdfz) + \mathrm{Li}_2(bcefz) \\ & - \mathrm{Li}_2(-abcz) - \mathrm{Li}_2(-aefz) - \mathrm{Li}_2(-bdfz) - \mathrm{Li}_2(-cdez)\}, \end{aligned}$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function defined by the analytic continuation of the following integral:

$$\mathrm{Li}_2(x) := -\int_0^x \frac{\log(1-t)}{t} dt \quad \text{for a positive real number } x.$$

We denote by z_1 and z_2 the two complex numbers defined as follows:

$$\begin{aligned} z_1 &:= -2\frac{\sin A \sin D + \sin B \sin E + \sin C \sin F - \sqrt{\det G}}{ad + be + cf + abf + ace + bcd + def + abcdef}, \\ z_2 &:= -2\frac{\sin A \sin D + \sin B \sin E + \sin C \sin F + \sqrt{\det G}}{ad + be + cf + abf + ace + bcd + def + abcdef}. \end{aligned}$$

Theorem 1.1 (A volume formula for generalized tetrahedra) *The volume* $\mathrm{Vol}(T)$ *of a generalized tetrahedron* T *is given as follows:*

$$\mathrm{Vol}(T) = \frac{1}{2}\Im\left(U(z_1, T) - U(z_2, T)\right), \tag{1.1}$$

where $\Im$ *means the imaginary part.* □

1.2 Other known volume formulae for tetrahedra

In this subsection we recall some known formulae and results for tetrahedra before Cho and Kim's one. The notations are those in Figure 1, and the function $\Lambda(x)$ means the Lobachevsky function defined as follows:

$$\Lambda(x) := -\int_0^x \log|2\sin t|\, dt.$$

The relationship between $\Lambda(x)$ and $\mathrm{Li}_2(x)$ is as follows (see, for example, [Ki, §1.1.4]):

$$\Im\,\mathrm{Li}_2(\exp\sqrt{-1}x) = 2\Lambda(\frac{x}{2}) \quad \text{for any } x \in \mathbb{R}.$$

1.2.1 Volume formulae for orthoschemes

In three-dimensional case, an *orthoscheme* is a tetrahedron with angles B, E and F are $\pi/2$ (i.e., the edge $\boldsymbol{v}_1\boldsymbol{v}_2$ is orthogonal to the face $\boldsymbol{v}_2\boldsymbol{v}_3\boldsymbol{v}_4$, and the face $\boldsymbol{v}_1\boldsymbol{v}_2\boldsymbol{v}_3$ is orthogonal to the edge $\boldsymbol{v}_3\boldsymbol{v}_4$).

It is known that in [Lo] N. I. Lobachevsky found a volume formula for orthoschemes. Later R. Kellerhals proved in [Ke] that his formula can be applied to orthoschemes with truncations at vertices $\boldsymbol{v}_1$ and/or $\boldsymbol{v}_4$ (and furthermore she constructed a formula for Lambert cubes). The formula is as follows:

Theorem 1.2 (The volume formula for orthoschemes) *The volume* $\mathrm{Vol}(T)$ *of an (maybe partially truncated) orthoscheme* T *is given as follows:*

$$\begin{aligned} \mathrm{Vol}(T) = \frac{1}{4}\Big\{&\Lambda(A+\theta) - \Lambda(A-\theta) + \Lambda(\frac{\pi}{2} + C - \theta) + \Lambda(\frac{\pi}{2} - C - \theta) \\ &+\Lambda(D+\theta) - \Lambda(D-\theta) + 2\Lambda(\frac{\pi}{2} - \theta)\Big\}, \end{aligned}$$

where $\theta := \arcsin \dfrac{\sqrt{\cos^2 C - \sin^2 A \sin^2 D}}{\cos A \cos D}$. $\square$

1.2.2 Volume formulae for ideal tetrahedra

An *ideal tetrahedron* is a tetrahedron with all vertices lying at the sphere at infinity. In this case the sum of the dihedral angles of edges emanating from an ideal vertex is π. This implies that the dihedral angles of edges opposite to each other are equal, namely $A = D$, $B = E$ and $C = F$. J. Milnor proved in [Mi] the following volume formula for ideal tetrahedra:

Theorem 1.3 (The volume formula for ideal tetrahedra) *The volume* $\mathrm{Vol}(T)$ *of an ideal tetrahedron* T *is given as follows:*

$$\mathrm{Vol}(T) = \Lambda(A) + \Lambda(B) + \Lambda(C). \qquad \square$$

1.2.3 Maximal volume tetrahedra

U. Haagerup and H. Munkholm proved in [HM] that, in hyperbolic space of dimension greater than one, a simplex is of maximal volume if and only if it is ideal and regular. Once we extend the class of tetrahedra to the generalized ones in this report, the maximal volume generalized tetrahedron is the truncated tetrahedron of all dihedral angles are 0, namely the ideal right-angled octahedron (see Theorem 4.2 of [Us2]).

The following graphs show the relationships between the dihedral angle and the volume or the edge length of regular tetrahedra. The dihedral angle varies from 0 to $\arccos \frac{1}{3} \approx 1.230959$. The angle 0 corresponds to the regular ideal octahedron mentioned as above, and the angle $\arccos \frac{1}{3}$ corresponds to the Euclidean regular tetrahedron, corresponding to infinitesimally small hyperbolic regular tetrahedra. The dihedral angle of the regular ideal tetrahedron is $\pi/3$.

Acknowledgement

The author would like to give his thanks to Jun Murakami and Masakazu Yano for useful discussion about their results.

2 Preliminaries

In this section we review several well-known facts about hyperbolic geometry. See, for example, [Us] for more precise explanation and the proofs of the propositions.

The $n+1$-*dimensional Lorentzian space* $\mathbb{E}^{1,n}$ is the real vector space $\mathbb{R}^{n+1}$ of dimension $n+1$ with the *Lorentzian inner product* $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n$, where $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_n)$. Let $H_T := \{ \boldsymbol{x} \in \mathbb{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1 \}$ be the (standard) hyperboloid of two sheets, and let $H_T^+ := \{ \boldsymbol{x} \in \mathbb{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1 \text{ and } x_0 > 0 \}$ be its upper sheet. The restriction

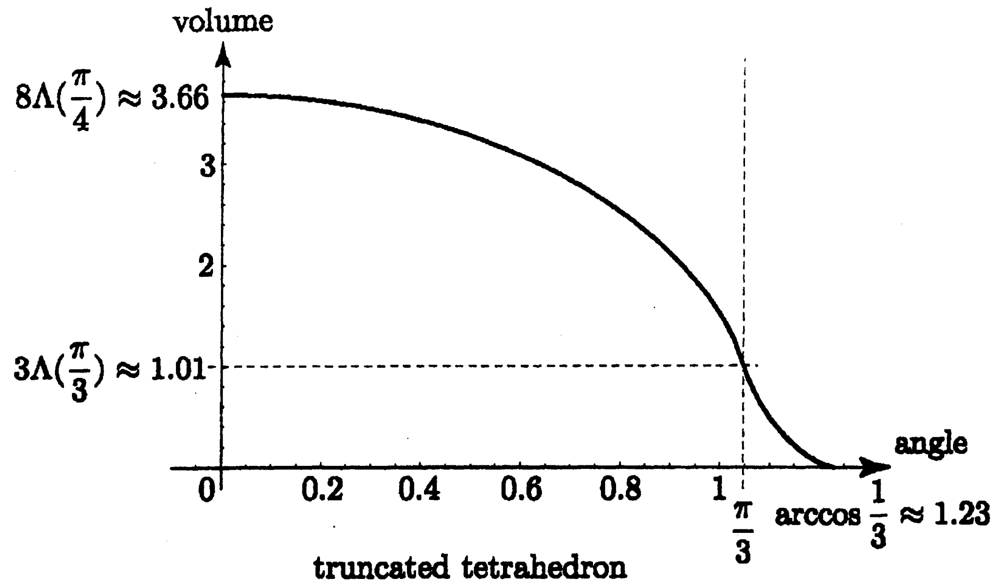

Figure 2: Volume of a regular tetrahedron

of the quadratic form induced by $\langle\cdot,\cdot\rangle$ on $\mathbb{E}^{1,n}$ to the tangent space of H_T^+ is positive definite and gives a Riemannian metric on H_T^+. The space obtained from H_T^+ equipped with the metric above is called the *hyperboloid model* of the n-dimensional hyperbolic space, and we denote it by $\mathbb{H}^n$. Under this metric the hyperbolic distance, say d, between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ can be measured by the following formula:

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -\cosh d. \tag{2.1}$$

Let $L := \{\boldsymbol{x} \in \mathbb{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0\}$ be the (standard) cone and let $L^+ := \{\boldsymbol{x} \in \mathbb{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0 \text{ and } x_0 > 0\}$ be its upper half. Then a ray in L^+ started from the origin $\boldsymbol{o}$ corresponds to a point in the ideal boundary of $\mathbb{H}^n$. The set of such rays forms the sphere at infinity, and we denote it by $\mathbb{S}_\infty^{n-1}$. Then each ray in L^+ becomes a point at infinity of $\mathbb{H}^n$.

Let us denote by $\mathcal{P}$ the radial projection from $\mathbb{E}^{1,n} - \{\boldsymbol{x} \in \mathbb{E}^{1,n} \mid x_0 = 0\}$ to an affine hyperplane $\mathbb{P}_1^n := \{\boldsymbol{x} \in \mathbb{E}^{1,n} \mid x_0 = 1\}$ along the ray from the origin $\boldsymbol{o}$. The projection $\mathcal{P}$ is a homeomorphism on $\mathbb{H}^n$ to the n-dimensional open unit ball $\mathbb{B}^n$ in $\mathbb{P}_1^n$ centered at the origin $(1,0,0,\dots,0)$ of $\mathbb{P}_1^n$, which gives the *projective model* of $\mathbb{H}^n$. The affine hyperplane $\mathbb{P}_1^n$ contains not only $\mathbb{B}^n$ and its set theoretic boundary $\partial\mathbb{B}^n$ in $\mathbb{P}_1^n$, which is canonically identified with $\mathbb{S}_\infty^{n-1}$, but also the outside of the compactified projective model $\overline{\mathbb{B}^n} := \mathbb{B}^n \sqcup \partial\mathbb{B}^n \approx \mathbb{H}^n \sqcup \mathbb{S}_\infty^{n-1}$. So $\mathcal{P}$ can be naturally extended to the mapping from $\mathbb{E}^{1,n} - \{\boldsymbol{o}\}$ to the n-dimensional real projective space $\mathbb{P}^n := \mathbb{P}_1^n \sqcup \mathbb{P}_\infty^n$, where $\mathbb{P}_\infty^n$ is the set of lines in the affine hyperplane $\{\boldsymbol{x} \in \mathbb{E}^{1,n} \mid x_0 = 0\}$ through $\boldsymbol{o}$. We denote by $\operatorname{Ext}\overline{\mathbb{B}^n}$ the exterior of $\overline{\mathbb{B}^n}$ in $\mathbb{P}^n$.

The (standard) *hyperboloid of one sheet* H_S is defined to be $H_S := \{\boldsymbol{x} \in \mathbb{E}^{1,n} \mid \langle \boldsymbol{x}, \boldsymbol{x}\rangle = 1\}$. For an arbitrary point $\boldsymbol{u}$ in H_S, we define a half-space

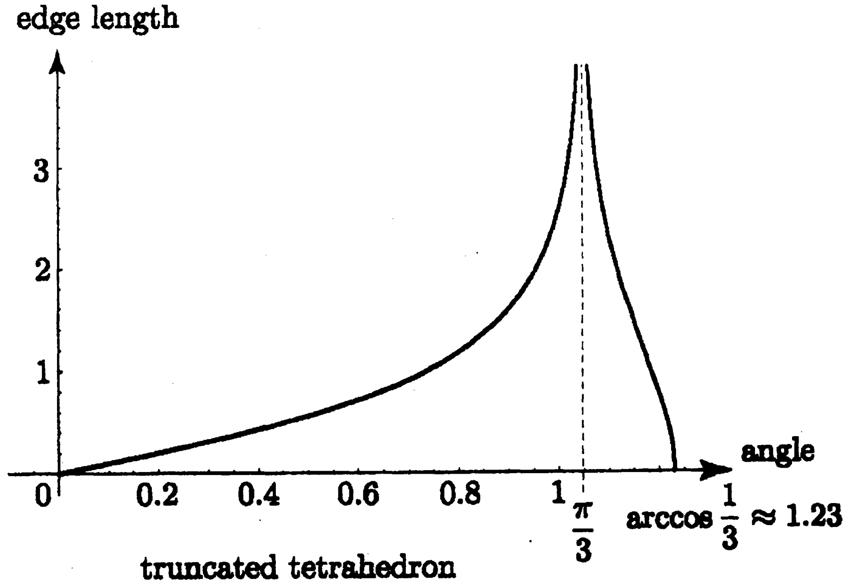

Figure 3: Edge length of a regular tetrahedron

R_u and a hyperplane P_u in $\mathbf{E}^{1,n}$ as follows:

$$R_u := \{ x \in \mathbf{E}^{1,n} \mid \langle x, u \rangle \leq 0 \} ,$$

$$P_u := \{ x \in \mathbf{E}^{1,n} \mid \langle x, u \rangle = 0 \} = \partial R_u .$$

We denote by Γ_u (resp. Π_u) the intersection of R_u (resp. P_u) and $\mathbf{B}^n$. Then Π_u is a geodesic hyperplane in $\mathbb{H}^n$, and the correspondence between the points in H_S and the half-spaces Γ_u in $\mathbb{H}^n$ is bijective. We call u a *normal vector* to P_u (or Π_u). The following two propositions are on relationships between the Lorentzian inner product and geometric objects.

Proposition 2.1 *Let x and y be arbitrary two non-parallel points in H_S. Then one of the followings hold:*

(1) *Two geodesic hyperplanes Π_x and Π_y intersect if and only if $|\langle x, y \rangle| < 1$. In this case the (hyperbolic) angle, say θ, between them measured in Γ_x and Γ_y is calculated by the following formula:*

$$\langle x, y \rangle = -\cos\theta . \tag{2.2}$$

(2) *Two geodesic hyperplanes Π_x and Π_y never intersect in $\overline{\mathbf{B}^n}$, i.e., they intersect in $\mathrm{Ext}\,\overline{\mathbf{B}^n}$, if and only if $|\langle x, y \rangle| > 1$. In this case the (hyperbolic) distance, say d, between them is calculated by*

$$|\langle x, y \rangle| = \cosh d , \tag{2.3}$$

and then Π_x and Π_y are said to be ultraparallel.

(3) *Two geodesic hyperplanes* Π_x *and* Π_y *intersect not in* B^n *but in* $\partial\mathrm{B}^n$ *if and only if* $|\langle x, y\rangle| = 1$. *In this case the angle and the distance between them is* 0, *and then* Π_x *and* Π_y *are said to be* parallel. □

Proposition 2.2 *Let* x *be a point in* B^n *and let* Π_y *be a geodesic hyperplane whose normal vector is* $y \in H_S$ *with* $\langle x, y\rangle < 0$. *Then the distance* d *between* x *and* Π_y *is obtained by the following formula:*

$$\langle x, y\rangle = -\sinh d. \qquad \square \qquad (2.4)$$

Let v be a point in $\mathrm{Ext}\,\overline{\mathrm{B}^n}$. Then $\mathcal{P}^{-1}(v) \cap H_S$ consists of two points and, independent of the choice of $\tilde{v} \in \mathcal{P}^{-1}(v) \cap H_S$, we can define the same hyperplane $\Pi_{\tilde{v}}$. We call $\Pi_{\tilde{v}}$ the *polar geodesic hyperplane to* v and v the *pole* of $\Pi_{\tilde{v}}$. Then the following proposition holds:

Proposition 2.3 *Let* v *be a point in* $\mathrm{Ext}\,\overline{\mathrm{B}^n}$.

(1) *Any hyperplane through* v *with intersecting* B^n *is perpendicular to* $\Pi_{\tilde{v}}$ *in* $\mathbb{H}^n$.

(2) *Let* u *be a limit point of* $\Pi_{\tilde{v}}$, *i.e.,* $u \in P_v \cap \partial\mathrm{B}^n$. *Then the line through* u *and* v *is tangent to* $\partial\mathrm{B}^n$. □

3 A characterization theorem for the dihedral angles of generalized hyperbolic simplices

Unlike spherical or Euclidean one, in hyperbolic geometry we can consider not only points at infinity but also points "beyond infinity." This situation can be easily seen in the projective ball model, and it extends the concept of polyhedra. Such polyhedra is called the *generalized polyhedra.* We start this section with their precise definition.

In this section we assume $n \geq 3$. We also note that any polyhedron in $\mathbb{P}^n_1$ intersecting B^n can be moved to the one containing the origin of $\mathbb{P}^n_1$ by the action of orientation-preserving isomorphisms of $\mathbb{H}^n$.

Definition 3.1 Let Δ be a polyhedron in $\mathbb{P}^n_1$ containing the origin of $\mathbb{P}^n_1$. Suppose each of the interior of its ridge (i.e., $(n-2)$-dimensional face) intersects $\overline{\mathrm{B}^n}$.

1. Let v be a vertex of Δ in $\mathrm{Ext}\,\overline{\mathrm{B}^n}$. The *(polar) truncation* of Δ at v is an operation of omitting the pyramid of apex v with base polyhedron $\Pi_{\tilde{v}} \cap \Delta$, and then capping the open end by $\Pi_{\tilde{v}} \cap \Delta$ (see Figure 4).

2. The *truncated* polyhedron Δ' obtained from Δ is a polyhedron in $\overline{\mathrm{B}^n}$ by the truncation at all vertices in $\mathrm{Ext}\,\overline{\mathrm{B}^n}$. □

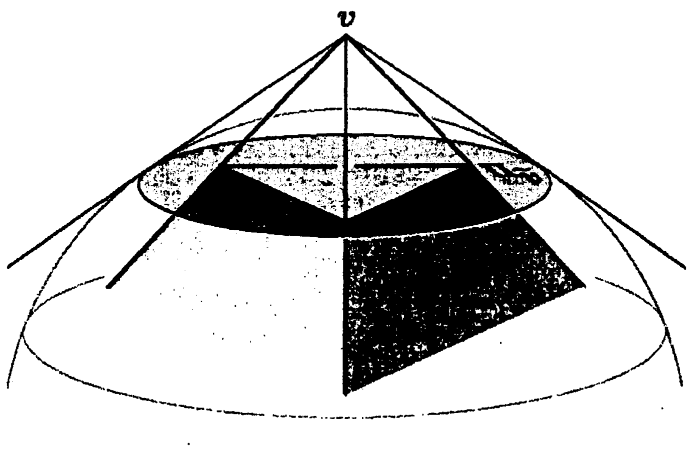

Figure 4: The truncation of Δ at $\boldsymbol{v}$

It should be noted that we regard vertices of Δ as those of Δ', namely we do not call vertices (in the ordinary sense) arose by the truncation "vertices of Δ'." A vertex $\boldsymbol{v}$ of Δ' is called *finite* (resp. *ideal*, *ultraideal*) when $\boldsymbol{v} \in \mathbf{B}^n$ (resp. $\partial\mathbf{B}^n, \operatorname{Ext}\overline{\mathbf{B}^n}$).

Definition 3.2 A *generalized polyhedron* in $\mathbb{H}^n$ is either an ordinary polyhedron or a truncated polyhedron described above. □

We note that, by definition, the generalized polyhedra are of finite volume in $\mathbb{H}^n$.

Let σ^n be an n-dimensional generalized simplex in the projective model with vertices $\{\boldsymbol{v}_i\}_{i\in I}$, where $I := \{1, 2, \ldots, n+1\}$. Then we regard the lift of its vertices to $\mathbb{E}^{1,n}$ as follows: if a vertex $\boldsymbol{v}$ is finite, then the lift is uniquely determined by $\mathcal{P}^{-1}(\boldsymbol{v}) \cap H_T^+$. If a vertex is ultraideal, there are two choice of the lift, and we choose the one defining the half-space containing σ^n. If a vertex is ideal, we do not need at this point to determine the exact lift in L^+. The *i-th facet* of σ^n is the $(n-1)$-dimensional face of σ^n opposite to $\boldsymbol{v}_i$.

Let G be a matrix of order $n+1$. We prepare several notations on matrices which will be used later.

1. We denote by G_{ij} the submatrix of order n obtained from G by removing the ith row and jth column.
2. We denote a cofactor of G by $c_{ij} := (-1)^{i+j} \det G_{ij}$.
3. Suppose G is a real symmetric matrix. Then we denote by $\operatorname{sgn} G$ the signature of G, i.e., if $\operatorname{sgn} G = (a, b)$ then G has a positive and b negative eigenvalues.

The following theorem tells us a necessary and sufficient condition of a set of positive numbers to be the dihedral angles of a generalized simplex. For the proof, see [Us2].

Theorem 3.3 *Suppose the following set of positive numbers is given:*

$$\{\theta_{ij} \in [0, \pi] \mid i, j \in I, \theta_{ij} = \theta_{ji}, \theta_{ij} = \pi \text{ iff } i = j\}.$$

Then the following two conditions are equivalent:

(1) *There exists a generalized hyperbolic simplex in* $\mathbb{H}^n$ *with dihedral angle between i-th facet and j-th facet is* θ_{ij}.

(2) *The real symmetric matrix* $G := (-\cos\theta_{ij})$ *of order* $n+1$ *satisfies the following two conditions:*

  (a) $\operatorname{sgn} G = (n, 1)$,

  (b) $c_{ij} > 0$ *for any* $i, j \in I$ *with* $i \neq j$. □

We call the matrix G appeared in the previous theorem the *Gram matrix* of an n-dimensional generalized simplex σ. Here it should be noted that this definition is slight different from the ordinary one (see, for example, [Vi]), since we do not deal with the normal vectors of the faces obtained by truncation.

The proof of Theorem 3.3 follows that of THEOREM in [Lu], where it is proved a necessary and sufficient condition for a given set of positive real numbers to be the dihedral angles of a hyperbolic simplex in the ordinary sense. Moreover such conditions for spherical and Euclidean simplices are also presented in [Lu].

The proof also tells us the way to calculate the edge lengths from a given set of dihedral angles. The method is as follows:

1. Prepare the Gram matrix, say G, with respect to the dihedral angles.

2. Calculate the cofactor matrix $(c_{ij})_{i,j=1}^{n} = (\det G)\, G^{-1}$ of G. Here each diagonal element c_{ii} represents the type of the vertex v_i, namely if $c_{ii} > 0$ (resp. $c_{ii} = 0, c_{ii} < 0$), then v_i is finite (resp. ideal, ultraideal).

3. Let l_{ij} be the length of the edge joining v_i and v_j.

   (a) When one of them is ideal vertex, then $l_{ij} = \infty$.

   (b) When both of them are finite vertices or ultraideal vertices, then $l_{ij} = \operatorname{arccosh} \dfrac{c_{ij}}{\sqrt{|c_{ii}c_{jj}|}}$ by formulae (2.1) and (2.3).

   (c) When one vertex is finite vertex and the other is ultraideal vertices, then $l_{ij} = \operatorname{arcsinh} \dfrac{c_{ij}}{\sqrt{|c_{ii}c_{jj}|}}$ by the formula (2.4).

Tracing the way conversely, we also obtain the method of calculating the dihedral angles from a given edge lengths. Here we note that, if some vertices are ideal, then we need to give *horospheres* at each ideal vertices for the determination of the dihedral angle. Since we do not deal with horospheres in this report, we do not deal with the case. Thus, for a given set of edge lengths with types of vertices, the following is a way to determine the dihedral angles.

1. Prepare the Gram matrix $G = (g_{ij})_{i,j=1}^{n}$ with respect to the edge lengths. Here each element g_{ij} is determined as follows:
   (a) The element $g_{ii} = -1$ (resp. $g_{ii} = 1$) when the vertex $\boldsymbol{v}_i$ is finite (resp. ultraideal).
   (b) When both $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are finite or ultraideal, then $g_{ij} = -\cosh l_{ij}$.
   (c) When one vertex is finite vertex and the other is ultraideal vertices, then $g_{ij} = -\sinh l_{ij}$.
2. Calculate the cofactor matrix $(c_{ij})_{i,j=1}^{n}$ of G.
3. Then the dihedral angle $\theta_{ij} = \arccos \dfrac{c_{ij}}{\sqrt{|c_{ii}c_{jj}|}}$.

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