数理解析研究所講究録 1329 巻 2003 年 62-68

The order of conformal automorphisms of Riemann surfaces of infinite type —supplement

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1 Introduction

On a compact Riemann surface R of genus $g \geq 2$, the order of a conformal automorphism of R is not greater than $2(2g+1)$ (see [7]). However for a Riemann surface with the infinitely generated fundamental group, the order of a conformal automorphism is not finite, in general. In [3], we showed a necessary and sufficient condition for a conformal automorphism of a Riemann surface to have finite order.

Proposition 1 ([3]) *Let $R = \mathbf{H}/\Gamma$, where Γ is a Fuchsian group which is not necessarily torsion-free. Suppose that R has the non-abelian fundamental group. Then a conformal automorphism f of R has finite order if and only if f fixes either a simple closed geodesic, a puncture, a point or a cone point on R.*

On the basis of Proposition 1, for a Riemann surface R such that the injectivity radius at any point in R is uniformly bounded from above, we estimated the order of conformal automorphisms of R in terms of the injectivity radius. One of the results is the following.

Proposition 2 ([3]) *Let R be a hyperbolic Riemann surface. Suppose that there exists a constant $M > 0$ such that the injectivity radius at any point in R is less than $M/2$. Let f be a conformal automorphism of R such that $f(c) = c$ for a simple closed geodesic c on R whose length is $\ell > 0$. Then the order n of f satisfies*

$$n < (e^M - 1)\cosh(\ell/2).$$

In this note, for a Riemann surface R such that the injectivity radius at any point in R is not necessarily uniformly bounded from above, we prove the same statements.

2 Statements of Theorems

Let $\mathbf{H}$ be the upper-half plane equipped with the hyperbolic metric $|dz|/\mathrm{Im}z$. We say that a Riemann surface R is *hyperbolic* if it is represented by $\mathbf{H}/\Gamma$ for a torsion-free Fuchsian group Γ acting on $\mathbf{H}$. The hyperbolic distance on $\mathbf{H}$ or on R is denoted by $d(\cdot,\cdot)$, and the hyperbolic length of a curve c on R is denoted by $\ell(c)$.

Definition For a constant $M > 0$, we define R_M to be the subset of points $p \in R$ such that there exists a non-trivial simple closed curve c_p passing through p with $\ell(c_p) < M$.

Remark The *injectivity radius* at a point $p \in R$ is the supremum of radii of embedded hyperbolic discs centered at p. The R_M is nothing but the set of points in R where the injectivity radius is less than $M/2$.

We consider the following condition in terms of hyperbolic geometry on Riemann surfaces R.

Definition We say that R satisfies the *upper bound condition* if there exist a constant $M > 0$ and a connected component R_M^* of R_M such that a homomorphism of $\pi_1(R_M^*)$ to $\pi_1(R)$ that is induced by the inclusion map of R_M^* into R is surjective.

Remark (i) If the injectivity radius at any point in R is uniformly bounded from above, then R clearly satisfies the upper bound condition. (ii) If R ($\neq \mathbf{H}$) is a normal covering surface of an analytically finite Riemann surface, then R satisfies the upper bound condition (see [4]).

We state our theorems.

Theorem 1 (hyperbolic case) *Let R be a hyperbolic Riemann surface with the non-abelian fundamental group. Suppose that R satisfies the upper bound condition for a constant $M > 0$ and a connected component R_M^* of R_M. Let f be a conformal automorphism of R such that $f(c) = c$ for a simple closed geodesic c on R with $c \subset R_M^*$ and $\ell(c) = \ell > 0$. Then the order n of f satisfies*

$$n < (e^M - 1)\cosh(\ell/2).$$

Theorem 2 (parabolic case) *Let R be a hyperbolic Riemann surface with the non-abelian fundamental group. Suppose that R satisfies the upper bound condition for a constant $M > 0$. Let f be a conformal automorphism of R such that $f(p) = p$ for a puncture p of R. Then the order n of f satisfies*

$$n < e^M - 1.$$

Theorem 3 (elliptic case) (i) *Let R be a hyperbolic Riemann surface with the non-abelian fundamental group, and f a conformal automorphism of R such that $f(p) = p$ for a point p in R at which the injectivity radius is $M > 0$. Then the order n of f satisfies*

$$n < 2\pi \cosh M.$$

(ii) *Let $R = \mathbf{H}/\Gamma$, where Γ is a Fuchsian group which is not torsion-free. Suppose that R has the non-abelian fundamental group and satisfies the upper bound condition for a constant $M > 0$. Let f be a conformal automorphism of R such that $f(p) = p$ for a cone point p in R which is a projection of a fixed point $\tilde{p}$ of an elliptic element of Γ with order $m > 1$. Then the order n of f satisfies*

$$n < (e^M - 1)\frac{\pi}{m}\left(\frac{1}{\sin^2 \frac{\pi}{m}} + \frac{1}{\sinh^2 \frac{M}{2}}\right)^{\frac{1}{2}}.$$

Remark The upper bound of the order of f obtained in Theorem 2 is the limiting case of that in Theorem 1 as $\ell \to 0$. It is also the limiting case of that in Theorem 3 (ii) as $m \to \infty$.

Remark In [5], we obtained a better estimate than that in Theorem 1 in the case where $\ell \geq M$.

3 Proofs of Theorems

We prove Theorem 1 only, for we can prove the other theorems by using the same argument in the proof of Theorem 1 and the proofs of Theorems 2 and 3 in [3].

Definition A subset $S \subset \mathbf{H}$ is said to be *precisely invariant* under a subgroup Γ_S of a Fuchsian group Γ if $\gamma(S) = S$ for all $\gamma \in \Gamma_S$ and $\gamma(S) \cap S = \emptyset$ for all $\gamma \in \Gamma - \Gamma_S$.

Collar Lemma ([6], [8]) Let Γ be a Fuchsian group (which is not necessarily torsion-free) acting on $\mathbf{H}$, and L an axis of a hyperbolic element $\gamma \in \Gamma$ whose translation length is less than ℓ. Assume that there exists no fixed points of elements in Γ on L and that L is precisely invariant under the cyclic subgroup $\langle\gamma\rangle$ generated by γ. Then a collar

$$C(L) = \{z \in \mathbf{H} \mid d(z, L) \leq \omega(\ell)\}$$

is precisely invariant under $\langle\gamma\rangle$, where $\sinh \omega(\ell) = (2\sinh(\ell/2))^{-1}$. Equivalently, the boundaries $\partial C(L)$ of $C(L)$ and the real axis make an angle θ, where $\tan\theta = 2\sinh(\ell/2)$.

The proof of Theorem 1 follows from the fact that there exists a wider collar of the simple closed geodesic c, as the order of a conformal automorphism f fixing c increases.

Proof of Theorem 1: Let Γ be a Fuchsian model of R, and $\tilde{f}$ a lift of f which is a hyperbolic element in $\mathrm{PSL}_2(\mathbf{R})$. Note that $\tilde{f}^n$ is a hyperbolic element in Γ which is corresponding to c. We consider the quotient $\hat{R} = R/\langle f\rangle$ by the cyclic group $\langle f\rangle$ and its Fuchsian model $\hat{\Gamma} = \langle\Gamma, \tilde{f}\rangle$. Then $\hat{c} = c/\langle f\rangle$ is a non-trivial simple closed geodesic on $\hat{R}$ whose length is ℓ/n. Since $\tilde{f}$ is corresponding to $\hat{c}$, we may assume that $\tilde{f}(z) = \exp(\ell/n)z$ with the axis $L = \{iy \mid y > 0\}$. Applying Collar Lemma for $\hat{\Gamma}$ and $\tilde{f}$, we can take a collar

$$\tilde{C}(L) = \{re^{i\theta} \in \mathbf{H} \mid 0 < r,\ \theta_0 < \theta < \pi - \theta_0\}$$

so that it is precisely invariant under $\langle\tilde{f}\rangle \subset \hat{\Gamma}$, where

$$\tan\theta_0 = 2\sinh(\ell/2n).$$

In particular, $\gamma(\tilde{C}(L)) \cap \tilde{C}(L) = \emptyset$ for any $\gamma \in \Gamma - \langle\tilde{f}^n\rangle$. Then we can take a tubular neighborhood $C(c) = \tilde{C}(L)/\langle\tilde{f}^n\rangle$ of c on R whose fundamental region is

$$A = \{re^{i\theta} \in \mathbf{H} \mid 1 < r < e^{\ell},\ \theta_0 < \theta < \pi - \theta_0\}.$$

We may assume that $d(c, \partial C(c)) = \omega(\ell/n) > M/2$. Indeed, suppose that

$$\omega(\ell/n) = \operatorname{arcsinh}\left(\frac{1}{2\sinh\frac{\ell}{2n}}\right) \leq \frac{M}{2}.$$

Using the fact that $x^{-1}\sinh x$ is a monotone increasing function for $x > 0$, we see that

$$\frac{\cosh\frac{\ell}{2}\exp\frac{M}{2}}{n} \geq \frac{\cosh\frac{\ell}{2}}{n} > \frac{\sinh\frac{\ell}{2}}{n} = \frac{\ell}{2n}\frac{\sinh\frac{\ell}{2}}{\frac{\ell}{2}} \geq \frac{\ell}{2n}\frac{\sinh\frac{\ell}{2n}}{\frac{\ell}{2n}} = \sinh\frac{\ell}{2n}$$

for $n > 1$, $\ell > 0$ and $M > 0$. Then

$$\frac{n}{2\cosh\frac{\ell}{2}\exp\frac{M}{2}} < \frac{1}{2\sinh\frac{\ell}{2n}} \leq \sinh\frac{M}{2}.$$

This implies that

$$\begin{aligned} n &< 2\exp(M/2)\sinh(M/2)\cosh(\ell/2) \\ &= (e^M - 1)\cosh(\ell/2), \end{aligned}$$

and we have nothing to prove.

We can take a point p in $C(c)$ which satisfies $d(p, \partial C(c)) = M/2$ and $p \in R^*_M$. Here $\partial C(c)$ is the boundary of $C(c)$. Indeed, if there exist no such points, then any point in two simple closed curves $\{p \in C(c) \mid d(p, \partial C(c)) = M/2\}$ does not belongs to R^*_M. This means that R^*_M is a tubular neighborhood of c, and this contradicts the upper bound condition.

By the definition of R_M, the length r_p of the shortest non-trivial simple closed curve α passing through p is less than M. Since $d(p, \partial C(c)) = M/2$, the curve α is in $C(c)$. Let $\tilde{p} = re^{i\theta} \in A$ $(\theta_0 < \theta < \pi/2)$ be a lift of p. Setting $z_1(t) = re^{it}$ for $t \geq 0$, we have

$$\frac{M}{2} = d(\tilde{p}, \partial\tilde{C}(L)) = \int_{\theta_0}^{\theta} \frac{|z_1'(t)|}{\operatorname{Im} z_1(t)} dt = \int_{\theta_0}^{\theta} \frac{1}{\sin t} dt \geq \int_{\theta_0}^{\theta} \frac{1}{t} dt = \log\frac{\theta}{\theta_0}.$$

Hence $\theta \leq \exp(M/2)\theta_0$. We put $a = \exp(i\theta)$ and $b = \exp(\ell + i\theta)$. Then $r_p = d(a, b)$. From Theorem 7.2.1 in [1], we have

$$\begin{aligned} \sinh\frac{1}{2}d(a,b) &= \frac{|a-b|}{2\,(\operatorname{Im} a \operatorname{Im} b)^{\frac{1}{2}}} = \frac{e^{\ell} - 1}{2\exp\frac{\ell}{2}\sin\theta} = \frac{\sinh\frac{\ell}{2}}{\sin\theta} \geq \frac{\sinh\frac{\ell}{2}}{\theta} \\ &\geq \frac{\sinh\frac{\ell}{2}}{\theta_0\exp\frac{M}{2}} = \frac{\sinh\frac{\ell}{2}}{\arctan(2\sinh\frac{\ell}{2n})\exp\frac{M}{2}} \geq \frac{\sinh\frac{\ell}{2}}{2\sinh\frac{\ell}{2n}\exp\frac{M}{2}} \\ &= \frac{\frac{\ell}{2n}}{\sinh\frac{\ell}{2n}} \cdot \frac{n\sinh\frac{\ell}{2}}{\ell\exp\frac{M}{2}} \geq \frac{\ell}{\sinh\ell} \cdot \frac{n\sinh\frac{\ell}{2}}{\ell\exp\frac{M}{2}} = \frac{n\sinh\frac{\ell}{2}}{\sinh\ell\exp\frac{M}{2}} \\ &= \frac{n}{2\cosh\frac{\ell}{2}\exp\frac{M}{2}}. \end{aligned}$$

For the last inequality, we used the fact that $x(\sinh x)^{-1}$ is a monotone decreasing function for $x > 0$. Since $r_p < M$, this implies that

$$\begin{aligned} n &< 2\exp(M/2)\sinh(M/2)\cosh(\ell/2) \\ &= (e^M - 1)\cosh(\ell/2). \end{aligned}$$

■

4 Application

We apply Theorem 1 to investigating a certain property on hyperbolic geometry on Riemann surfaces. The following proposition is an extension of Proposition 3 in [3].

Definition We say that R satisfies the *lower bound condition* if there exists a constant $\epsilon > 0$ (which is smaller than the Margulis constant) such that R_ϵ consists only of cusp neighborhoods and neighborhoods of geodesics which are homotopic to boundary components.

Proposition 3 *Let R be a hyperbolic Riemann surface, and $\tilde{R}$ a normal covering surface of R. If $\tilde{R}$ satisfies the lower and upper bound conditions, then R also satisfies these conditions.*

Proof. It is clear that R satisfies the upper bound condition. Suppose that R does not satisfy the lower bound condition. Then R has a sequence $\{c_n\}$ of disjoint simple closed geodesics which are not homotopic to boundary components of R with $\ell_n = \ell(c_n) \to 0$ $(n \to \infty)$. Let $\tilde{c}_n \subset \tilde{R}$ be a connected component of the preimage of c_n. Then $\tilde{c}_n$ is not homotopic to a boundary component of $\tilde{R}$. Since $\tilde{R}$ satisfies the lower bound conditions, there exists a constant $\epsilon > 0$ such that $\ell(\tilde{c}_n) > \epsilon$ for all n. We take a constant $M > 0$ so that $\tilde{R}$ satisfies the upper bound condition for M and for a connected component $\tilde{R}_M^*$ of $\tilde{R}_M$. We may assume that $\tilde{c}_n \subset \tilde{R}_M^*$. Assume that $\ell(\tilde{c}_n) \leq M$ for infinitely many n. Then, by Theorem 1, the order of a conformal automorphism $\tilde{f}_n$ of $\tilde{R}$ fixing $\tilde{c}_n$ is less than $N = (e^M - 1)\cosh(M/2)$. Then we have $\ell(c_n) > \epsilon/N$. However, this contradicts $\ell(c_n) \to 0$ $(n \to \infty)$. Next, we assume that $\ell(\tilde{c}_n) > M$ (including the case that $\tilde{c}_n$ is not closed) for infinitely many n. By Collar Lemma, there exists a tubular neighborhood $C(c_n)$ of c_n with width $\omega(\ell_n)$, where $\sinh \omega(\ell_n) = (2\sinh(\ell_n/2))^{-1}$. From the proof of Theorem 1, there exists a (tubular) neighborhood of $\tilde{c}_n$ with width $\omega(\ell_n)$. Since $\tilde{c}_n \subset \tilde{R}_M^*$, there exists a non-trivial simple closed curve passing through $\tilde{p}_n \in \tilde{c}_n$ whose length is less than M. However, since $\ell(\tilde{c}_n) > M$ and since $\omega(\ell_n) \to \infty$ as $n \to \infty$, we have a contradiction. ∎

For applications of Proposition 3 to the action of Teichmüller modular groups, see [2].

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