数理解析研究所講究録 1331 巻 2003 年 67-70

Appearance of singularities for a system of nonlinear wave equations with different speeds

東京理科大・理学部　加藤 圭一 (Keiichi Kato)
Faculty of Science, Tokyo University of Science

1. Introduction

We consider the following system of wave equations

$$(1.1)\qquad \begin{cases} \Box_{c_1} u &= f(u,v) \\ \Box_{c_2} v &= g(u,v) \end{cases}$$

where $\Box_c = (1/c^2)\partial^2/\partial t^2 - \sum_{j=1}^n \partial^2/\partial x_j^2$ and c_1 and c_2 are positive constants. We assume that $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are in C^∞. In what follows, we shall study the singularities of the solutions to (1.1) when the solutions are 'conormal distributions' to some hyperplanes.

In 1979, J. Rauch[5] has shown that H^s-singularities with $n/2 < s < 2s - 2/n$ for semilinear wave equations propagate along the null bicharacteristic curves for the linear part of the equation. Rauch-Reed[4] has given an example which illustrates the occurance of new singularities by nonlinear interaction. J. M. Bony[1] and Melrose-Ritter[3] has shown that no new singularities occur except on the light cone from the interaction point if singularities are conormal.

In this section, we introduce the result of the author's paper[2], which shows the same result as J. M. Bony[1] and Melrose-Ritter[3] for a system of wave equations with different speeds. In next section, we give an example which illustrates the appearance of new singularities.

Definition 1 (Conormal distributions). *Let Ω be a domain in $\mathbb{R}^n$. Let L be a C^∞- manifold in Ω. We call that u is in $H^s(L,\infty)$ in Ω if*

$$M_1 \circ M_2 \circ \cdots \circ M_l u \in H^s_{loc}(\Omega) \quad \text{for } l = 0, 1, 2, \dots,$$

where each M_j is a C^∞ vector field which is tangent to L.

We can define the space of conormal distributions not only for a C^∞–manifold but also for a union of two hypersurfaces which intersect each other transversally.

Now we shall state the main results. Let $\omega \in S^{n-1}$ and $L_{ij} = \{(t,x) \in \mathbb{R}^n; c_i t + (-1)^j \omega \cdot x = 0\}$ for i, $j = 1$, 2.

Theorem 1.1. *Let Ω be a neighborhood of the origin of $\mathbb{R}^{n+1}$, $i = 1$ or 2 and $j = 1$ or 2. Suppose that u, v are in $H^s_{loc}(\Omega)$ for $s > (n+1)/2$, u and v are solutions to (1.1) and*

$$u,\, v \in H^s(L_{ij}, \infty) \quad \text{in } \Omega \cap \{t < 0\},$$

then

$$u,\, v \in H^s(L_{ij}, \infty) \quad \text{in } K$$

where K is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

Theorem 1.2. *Let Ω be a neighborhood of the origin of $\mathbb{R}^{n+1}$ and i, i', j, $j' \in \mathbb{N}$ with $i + i' = 3$, $j + j' = 3$. Suppose that $0 < c_1 < c_2$, u, v are in $H^s_{loc}(\Omega)$ for $s > (n+1)/2$, u and v are solutions to (1.1) and*

$$u,\, v \in H^s(L_{ij} \cup L_{i'j}, \infty) \quad \text{in } \Omega \cap \{t < 0\},$$

then

$$u,\, v \in H^s(L_{ij} \cup L_{i'j} \cup L_{ij'} \cup L_{i'j'}, \infty) \quad \text{in } K$$

where K is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

Theorem 1.3. *Let Ω be a neighborhood of the origin of $\mathbb{R}^{n+1}$ and i, i', j, $j' \in \mathbb{N}$ with $i + i' = 3$, $j + j' = 3$. Suppose that $0 < c_1 < c_2$, u, v are in $H^s_{loc}(\Omega)$ for $s > (n+1)/2$, u and v are solutions to (1.1) and*

$$u,\, v \in H^s(L_{ij} \cup L_{ij'}, \infty) \quad \text{in } \Omega \cap \{t < 0\},$$

then

$$u,\, v \in H^s(L_{ij} \cup L_{i'j} \cup L_{ij'} \cup L_{i'j'}, \infty) \quad \text{in } K$$

where K is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

2. Apperance of singularities

In this section, we give an example which shows the appearance of new singularities by nonlinear interaction for the following system of nonlinear wave equations with different speeds;

$$\begin{cases} \Box_{c_1} u = f(u, v), & \text{in } \Omega, \\ \Box_{c_2} v = g(u, v), & \text{in } \Omega, \end{cases} \tag{2.2}$$

where $\Omega \subset \mathbb{R} \times \mathbb{R}^2$, u and v are real valued functions on Ω, $0 < c_1 < c_2$ and f and g are some functions appropriately determined later.

Let $\omega \in S^1$ and we put $u_1 = h(c_j t - \omega \cdot x)$ and $u_2 = h(c_j t + \omega \cdot x)$ where h is the heviside function defined by

$$h(s) = \begin{cases} 0 & \text{for } s < 0 \\ 1 & \text{for } s \geq 0. \end{cases} \tag{2.3}$$

Then u_j solves $\Box_{c_j} u_j = 0$ for $j = 1.2$. We put $\chi(t,x) = u_1 u_2$ and we define V_j as a solution of the equation;

$$\begin{cases} \Box_{c_j} V_j = \chi(t,x), & \text{in } \Omega, \\ V_j \equiv 0 & \text{for } t << 0. \end{cases} \tag{2.4}$$

In the following, we write $\{t \geq 0\} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^2 | t \geq 0\}$ for abreviation.

Proposition 2.1. *For any* $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, *we have* $V_1(t,x), V_2(t,x) \geq 0$ *and*

$$\textit{sing supp } V_1 = \{t \geq 0\} \cup \{c_1 t \pm \omega \cdot x = 0\} \cup \{c_2 t + \omega \cdot x = 0\} \tag{2.5}$$

$$\textit{sing supp } V_2 = \{t \geq 0\} \cup \{c_2 t \pm \omega \cdot x = 0\} \cup \{c_1 t - \omega \cdot x = 0\} \tag{2.6}$$

We construct an example of appearance of new singularities before the proof of this proposition to the end of our notes.

Let $F(s)$ be a C^∞ function defined as

$$F(s) = \begin{cases} 0 & \text{for } s \leq 3/2 \\ 1 & \text{for } s \geq 2, \end{cases} \tag{2.7}$$

and we put $u = u_1 + V_1$ and $v = u_2 + V_2$. Since V_j for $j = 1,2$ is a positive function and $\mathrm{supp} V_j = \{-c_2 t \leq \omega \cdot x \leq c_2 t,\ t \geq 0\}$ for $j = 1,2$, u and v solve

$$\begin{cases} \Box_{c_1} u = \Box V_1 = \chi(t,x) = F(u+v), \\ \Box_{c_2} v = \Box V_2 = \chi(t,x) = F(u+v), \end{cases} \tag{2.8}$$

for $t < \delta$ with $\delta > 0$ sufficiently small and u (v) has new singularities on $\{c_1 t + \omega \cdot x = 0\}$ $(\{c_2 t + \omega \cdot x = 0\})$ respectively.

Proof. We only prove the proposition for V_1. We note that V_1 is expressed as follows:

$$V_1(t,x) = C \int_{C^-_{(t,x)}} \frac{\chi(t',x')}{\sqrt{c_1^2(t-t')^2 - |x-x'|^2}} dx' dt',$$

where $C^-_{(t,x)}$ is backward light cone for $\Box_{c_1}$ starting from (t,x). Evidently V_1 is a positive function. It is easy to see that sing supp V_1 includes $\{c_2t+\omega\cdot x=0\}$ and $\{c_1t-\omega\cdot x=0\}$. So we prove that sing supp V_1 includes $\{c_1t+\omega\cdot x=0\}$.

We put $V_1 = w_1 + w_2$, where

$$w_1 = C\int_{C^-_{(t,x)}} \frac{u_2(t',x')}{\sqrt{c_1^2(t-t')^2-|x-x'|^2}}dx'dt',$$

$$w_2 = C\int_{C^-_{(t,x)}} \frac{\chi(t',x')-u_2(t',x')}{\sqrt{c_1^2(t-t')^2-|x-x'|^2}}dx'dt'.$$

Since sing supp $w_1(t,x) = \{c_2t+\omega\cdot x=0\}$, it suffices to show that sing supp w_2 includes $\{c_1t+\omega\cdot x=0\}$. By using the same argument as in the paper of J. Rauch and M. Reed[4], we have w_2 cannot be twice differetiated on $\{c_1t+\omega\cdot x=0\}$. $\square$

References

[1] J. M. Bony, *Second microlocalization and propagation of singularities for semi-linear hyperbolic equations*, Taniguchi Symp. Katata 1984, 11–49.

[2] K. Kato, *Singularities of solutions to system of wave equations with different speed*, Adv. Stud. Pure Math. **23**(1994), 217–221.

[3] R. Melrose and N. Ritter, *Interaction of nonlinear progressing waves*, Ann. math. **121**(1985), 187–213.

[4] J. Rauch and M. Reed, *Singularities produced by the nonlinear interaction of three progressing waves: examples*, Comm. Partial Differential Equations **7**(1982),1117–1133.

[5] J. Rauch, *Sigularities of solutions to semilinear wave equations*, J. Math. Pures et Appl. **58**(1979),299–308.