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Viscous shock profile for 2×2 systems of hyperbolic conservation laws with an umbilic point

Fumioki ASAKURA*and Mitsuru YAMAZAKI†
浅倉史興（大阪電気通信大学工学部）・山崎満（筑波大学数学系）

1 Introduction

Let us consider a 2×2 system of conservation laws in one space dimension:

$$U_t + F(U)_x = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+ \tag{1}$$

where $U = {}^t(u,v) \in \Omega$ for a domain $\Omega \subsetneqq \mathbf{R}^2$ and $F = {}^t(F_1, F_2) : \Omega \to \mathbf{R}^2$ is a smooth map. We suppose that this system of equations (1) is *hyperbolic*, i.e. the Jacobian matrix $F'(U)$ has *real* eigenvalues $\lambda_1(U), \lambda_2(U)$ for any $U \in \Omega$. If, in particular, these eigenvalues are *distinct*: $\lambda_1(U) < \lambda_2(U)$, the system is called *strictly hyperbolic* at U. A state $U^* \in \Omega$ is called an *umbilic* point, if $\lambda_1(U) = \lambda_2(U)$ and $F'(U)$ is diagonal at $U = U^*$. We suppose that the system of equations (1) is strictly hyperbolic at any $U \in \Omega \setminus \{U^*\}$ and that U^* is a single umbilic point in Ω. Since $U = U^*$ is an isolated umbilic point, we have the Taylor expansion of $F(U)$ near $U = U^*$:

$$F(U) = F(U^*) + \lambda^*(U - U^*) + Q(U - U^*) + O(1)|U - U^*|^3$$

where $\lambda^* = \lambda_1(U^*) = \lambda_2(U^*)$ and $Q : \mathbf{R}^2 \to \mathbf{R}^2$ is a homogeneous quadratic mapping. After the Galilean change of variables: $x \to x - \lambda^* t$ and $U \to U + U^*$, we observe that the system of equations (1) is reduced to

$$U_t + Q(U)_x = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+ \tag{2}$$

*Faculty of Engineering, Osaka Electro-Communication Univ., Neyagawa, Osaka 572-8530, JAPAN, asakura@isc.osakac.ac.jp
†Institute of Mathematics, Univ. of Tsukuba, Tsukuba, Ibaraki 305-8571, JAPAN, yamazaki@math.tsukuba.ac.jp

modulo higher order terms. Now by a change of unknown functions $V = S^{-1}U$ with a regular constant matrix S, we have a new system of equations $V_t + P(V)_x = 0$ where $P(V) = S^{-1}Q(SV)$. Thus we come to

Definition 1.1 *Two quadratic mappings $Q_1(U)$ and $Q_2(U)$ are said to be* equivalent, *if there is a constant matrix $S \in GL_2(\mathbf{R})$ such that*

$$Q_2(U) = S^{-1}Q_1(SU) \quad \text{for all} \quad U \in \mathbf{R}^2. \tag{3}$$

A general quadratic mapping $Q(U)$ has six coefficients and $GL_2(\mathbf{R})$ is a four dimensional group. Thus by the above equivalence transformations, we can eliminate four parameters. These procedures are successfully carried out by Schaeffer-Shearer [25] and they obtained the following *normal forms*.

Let $Q(U)$ be a hyperbolic quadratic mapping with an isolated umbilic point $U = 0$, then there exist two real parameters a and b with $a \neq 1 + b^2$ such that $Q(U)$ is equivalent to $\frac{1}{2}\nabla C$ where $\nabla = {}^t(\partial_u, \partial_v)$ and

$$C(U) = \tfrac{1}{3}au^3 + bu^2v + uv^2. \tag{4}$$

Moreover, if $(a, b) \neq (a', b')$, then the corresponding quadratic mappings: $\frac{1}{2}\nabla C$ and $\frac{1}{2}\nabla C'$ are not equivalent.

In the following argument, we shall confine ourselves to the quadratic mapping:

$$F(U) = Q(U) = \frac{1}{2}\nabla C(U) = \frac{1}{2}\begin{pmatrix} au^2 + 2buv + v^2 \\ bu^2 + 2uv \end{pmatrix} \quad (a \neq 1 + b^2). \tag{5}$$

Mathematical properties of the systems of equations (1) depends on (a, b). Schaeffer-Shearer classify in [25] ab-plane into four cases: Case I is $a < \frac{3}{4}b^2$; Case II is $\frac{3}{4}b^2 < a < 1+b^2$; for $a > 1+b^2$, the boundary between Case III and Case IV is $4\{4b^2 - 3(a-2)\}^3 - \{16b^3 + 9(1-2a)b\}^2 = 0$. We notice that these 2×2 system of hyperbolic conservation laws with an isolated umbilic point is a generalization of a three phase Buckley-Leverett model for oil reservoir flow where the flux functions are represented by a quotient of polynomials of degree two. In Appendix of [25]: in collaboration with Marchesin and Paes-Leme, they show that the quadratic approximation of the flux functions is either Case I or Case II.

The Riemann problem for (1) is the Cauchy problem with initial data of the form

$$U(x, 0) = \begin{cases} U_L & \text{for} \quad x < 0, \\ U_R & \text{for} \quad x > 0 \end{cases} \tag{6}$$

where U_L, U_R are constant states in Ω. A jump discontinuity defined by

$$U(x,t) = \begin{cases} U_L & \text{for} \quad x < st, \\ U_R & \text{for} \quad x > st \end{cases} \tag{7}$$

is a piecewise constant weak solution to the Riemann problem, provided these quantities satisfy the *Rankine-Hugoniot condition*:

$$s(U_R - U_L) = F(U_R) - F(U_L). \tag{8}$$

We say that the above discontinuity is a *j-compressive* shock wave ($j = 1, 2$) if it satisfies the *Lax entropy conditions* :

$$\lambda_j(U_R) < s < \lambda_j(U_L), \quad \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R) \tag{9}$$

(Lax [16], [17]). Here we adopt the convention $\lambda_0 = -\infty$ and $\lambda_3 = \infty$. The presence of an umbilic point bring us to face with non-classical: overcompressive shocks and crossing shocks. We say that a piecewise constant weak solution (7) is a *overcompressive shock* if it satisfies

$$\lambda_1(U_R) < s < \lambda_1(U_L), \quad \lambda_2(U_R) < s < \lambda_2(U_L). \tag{10}$$

We say also that a piecewise constant weak solution (7) is a *crossing shock* if it satisfies

$$\lambda_1(U_R) < s < \lambda_2(U_R), \quad \lambda_1(U_L) < s < \lambda_2(U_L). \tag{11}$$

In this note, we shall confine ourselves to Case II of the representative quadratic mapping $F(U) = Q(U)$ defined by (5). Our aim is to show that there is no crossing shock with viscous profile on the complement of medians $M_1 \cup M_3$ hence the associated vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ in Case II. In Section 2, we introduce the vector field $X_s(U, U_L)$ which allows us to determine the existence of a viscous profile to the shock wave solutions. Then we classify the character of critical points for the vector field $X_s(U_L, U)$. In Section 3, we show that there is no crossing shock with viscous profile on the complement of $M_1 \cup M_3$. In Section 4, as conclusion, we show that the vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ in Case II.

2 Viscous Shock Profiles

One admissibility condition for shock wave solutions (7) to the Riemann problem (6) for a hyperbolic system of conservation laws (1) is to obtain these solutions as limits of travelling wave solutions to an associated parabolic equation:

$$U_t + F(U)_x = \epsilon(B(U)U_x)_x, \epsilon > 0 \quad (12)$$

with an admissible matrix $B(U)$ in [4, 8, 9, 21, 28, 31]. More precisely, let U_L and U_R be two constant states to Riemann problem (1), (6). If there exists a shock $U(x,t)$ (7) with speed s to this Riemann problem and the two constant states U_L and U_R are connected through a travelling wave solution $U_\epsilon(x,t) = U\left(\frac{x-st}{\epsilon}\right)$ to (12) with shock speed s which converges to the shock wave $U(x,t)$ (7) as ϵ tends to 0, then we say that this shock (7) satisfies the *viscosity admissibility criterion* and that it has a *viscous shock profile* $U_\epsilon(x,t) = U\left(\frac{x-st}{\epsilon}\right)$. The travelling wave $U_\epsilon(x,t) = U\left(\frac{x-st}{\epsilon}\right)$ should satisfy, by integrating (12), the 2×2 system of nonlinear ordinary equations:

$$B(U)U_\xi = -s(U - U_L) + f(U) - f(U_L) \quad (13)$$

with $\xi = \dfrac{x-st}{\epsilon}$ and the boundary conditions at the infinity

$$\lim_{\xi \to -\infty} U(\xi) = U_L, \lim_{\xi \to \infty} U(\xi) = U_R. \quad (14)$$

The conditions (13), (14) required for the travelling wave solution imply automatically the Rankine-Hugoniot condition (8) for the Riemann problem. The existence of shock with a viscous profile is equivalent to the system of (13) with the boundary condition (14).

Let $X_s(U, U_L)$ be the vector field

$$X_s(U, U_L) = -s(U - U_L) + F(U) - F(U_L). \quad (15)$$

The shock wave solution (7) has a viscous shock profile if and only if there exists an orbit along the vector-field $X_s(U, U_L)$ from the critical point U_L to the critical point U_R of this vector-field.

Let p be a critical point of a vector field X. We say that p is hyperbolic if dX has two eigenvalues with non-zero real part at p. Clearly the eigenvalues of $dX_s(U, U_L)$ are $-s + \lambda_j(U)$. In particular, $dX_s(U, U_L)$ has real eigenvalues.

The critical point U of X_s is not hyperbolic if and only if $s = \lambda_j(U)$ ($j =$ 1 or 2).

Proposition 2.1 *The shock wave (7) is*

- *1-compressive shock if and only if U_L is repeller and U_R is saddle.*
- *2-compressive shock if and only if U_L is saddle and U_R is attractor.*
- *overcompressive shock if and only if U_L is repeller and U_R is attractor.*
- *crossing shock if and only if U_L and U_R are saddles.*

For all above shocks, both critical point U_L and U_R are hyperbolic.. Moreover there exists a shock wave (7) with a viscous profile if and only if there exists an orbit connecting two critical points of the vector field X_s.

We say, for example, *repeller-saddle connection* or simply *R-S connection* an orbit from a repeller point to a saddle point.

In Case II, we investigate the critical points of the vector-field $X_s(U, U_L)$ in the finite part of the $U-$plane and at the infinity. The Poincaré transformation [2, 9] enables us to make a one-to-one correspondence from $U-$plane including the infinity to the sphere S^2 by identifying two antipodal points. The line joining two antipodal points of $S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3; x_1^2+x_2^2+x_3^2 = 1\}$ intercepts the plane $P_1 = \{(u, v, -1); (u, v) \in \mathbf{R}^2\} \simeq U-$ plane at one point. This mapping induces the vector field $X_s(U, U_L)$ on $U-$plane to the vector field $X_s^{S^2}(U, U_L)$ on the sphere S^2 minus the equator $\{x_3 = 0\}$. The equator $\{x_3 = 0\}$ corresponds to $\infty \times S^1$ of $U-$plane. Similarly the line joining the origin and a point on $P_2 = \{(1, w, -z); (w, z) \in \mathbf{R}^2\}$ intercepts S^2 at two antipodal points. By this mapping, a vector field on P_2 is induced to a vector field on the sphere S^2 minus the equator $\{x_1 = 0\}$. Therefore the composition of two mappings above transforms a point $(1, w, -z) \in P_2$ to a point $(u, v, 1) \in P_1$:

$$u = 1/z,\ v = w/z \text{ if } z \neq 0,$$

or equivalently

$$w = v/u,\ z = 1/u \text{ if } u \neq 0.$$

For $u = 0$, we take instead of the plane P_2 the plane $P_2' = \{(w, 1, -z); (w, z) \in \mathbf{R}^2\}$. Similarly a point $(w, 1, -z) \in P_2'$ corresponds to a point $(u, v, 1) \in P_1$:

$$w = u/v,\ z = 1/v \text{ if } v \neq 0.$$

By the mapping from P_2 to P_1, the differential equation $\dfrac{dv}{du} = \dfrac{-sv + F_2(U)}{-su + F_1(U)}$ of the vector field $X_s(U, U_L)$ is induced to the differential equation

$$\frac{dz}{dw} = \frac{\Psi}{\Xi} \tag{16}$$

where

$$\begin{array}{rcl} \Psi & = & -z\{-sz(1-zu_L)+F_1(1,w)-z^2F_1(U_L)\}, \\ \Xi & = & -w\{-sz(1-zu_L)+F_1(1,w)-z^2F_1(U_L)\}+F_2(1,w) \\ & & -z^2F_2(U_L)-sz(w-zv_L). \end{array}$$

The right-hand side of the differential equation (16) is well-defined also for $\{z=0\}$ which corresponds to the equator $\{x_3=0\}$ of S^2 then to the infinity of $U-$plane.

We consider the critical points of $X_s(U,U_L)$ at the infinity. They satisfy $z=0$ then

$$-wF_1(1,w)+F_2(1,w)=-\Phi(w)=-(w^3+2bw^2+(a-2)w-b)=0$$

which has three distinct real roots μ_1,μ_2,μ_3 for $a<1+b^2$. The corresponding vector field of (16) is $\dot{w}=\Xi$, $\dot{z}=\Psi$ and its Jacobian matrix at $z=0$ is

$$\begin{pmatrix} -F_1(1,w)-wF_1'(1,w)+F_2'(1,w) & 0 \\ 0 & -F_1(1,w) \end{pmatrix}. \tag{17}$$

We have already known [3] the configuration of the roots μ_i of $\Phi(w)=0$. For $b>0$,

$$\text{in Case II, } \mu_1<-b<\mu_2<-b/2<0<\mu_3. \tag{18}$$

Then we have

$$-F_1(1,w)-wF_1'(1,w)+F_2'(1,w)=-\Phi'(w)\left\{\begin{array}{ll} <0 & \text{for } w=\mu_1,\mu_3, \\ >0 & \text{for } w=\mu_2 \end{array}\right. \tag{19}$$

and

$$-F_1(1,w)=-\frac{1}{w}(\Phi(w)+2w+b)\left\{\begin{array}{ll} <0 & \text{for } \mu_1,\mu_2, \\ >0 & \text{for } \mu_3. \end{array}\right. \tag{20}$$

Therefore in Case II, μ_1 is a attractor, μ_2 is a saddle and μ_3 is a repeller. On account of the fact that, at the antipodal point, the character of a critical point is the inverse, we have

Theorem 2.1 *The vector field $X_s(U,U_L)$ has six singularities at infinity. In Case II, two are repellers, two are attractors and two are saddles.*

We investigate critical points of $X_s(U,U_L)$ in the bounded region of $U-$plane. Owing to the Poincaré-Hopf theorem, we can show

Theorem 2.2 *The vector field $X_s(U, U_L)$ has two, three or four critical points in the bounded region of U−plane. In Case II,*

(i) if the vector field $X_s(U, U_L)$ has four critical points in the bounded region of U−plane, then the critical points are two nodes and two saddles.

(ii) if the vector field $X_s(U, U_L)$ has three critical points in the bounded region of U−plane, then the critical points are one node, one saddle and one saddle-node.

(iii) if the vector field $X_s(U, U_L)$ has two critical points in the bounded region of U−plane, then the critical points are one node and one saddle or two saddle-nodes.

Let us recall the notion of structurally stable vector fields. Let $\chi(M^2)$ be the space of all vector fields of C^1 class on a 2-dimensional compact manifold M^2 with the C^1-topology.

Definition 2.1 *A vector field $X \in \chi(M^2)$ is said to be structurally stable if there exists a neighborhood N of X in $\chi(M^2)$ such that for any $Y \in N$, there exists a homeomorphism $\rho : M^2 \to M^2$ which maps any orbit of X to an orbit Y.*

The following theorem due to Peixoto [24] gives a characterization of structurally stable vector fields.

Theorem 2.3 *A vector field $X \in \chi(M^2)$ is structurally stable if and only if it satisfies the following conditions:*

- *there are only a finite number of critical points and all are hyperbolic,*
- *there are only a finite number of closed orbits and all are hyperbolic,*
- *the ω-limit sets and α-limit sets of any orbit consist only of critical points or closed orbits,*
- *there are no saddle-saddle connections.*

Since both eigenvalues of $X_s(U_L, U)$ are real, we have

Proposition 2.2 *The vector field $X_s(U_L, U)$ has no closed orbits, nor singular closed orbit, nor ω-limit sets, nor α-limit sets.*

The most unstable connection is clearly saddle-saddle connection. We will show in the next section that there are no saddle-saddle connections on the complement of $M_1 \cup M_3$ in Case II.

3 Saddle-Saddle Connections

The aim of this section is to show that there is no crossing shock on the complement of $M_1 \cup M_3$ in the Case II.

Theorem 3.1 *A crossing shock has a viscous profile if and only if this profile comes from a saddle-saddle connection which is a straight line on the median* $M_j = \{U = {}^t(u, v); v = \mu_j u\}$ $(j = 1, 2, 3)$.

Proof. Suppose that there is a crossing shock. It is obvious, from Proposition 2.1 and its following remark, that the existence of a crossing shock is equivalent to the existence of a S-S connection. The next lemma is due to Chicone [6].

Lemma 3.1 *Let* $X = {}^t(\Psi, \Xi)$ *be a quadratic vector field on the plane where* Ψ *and* Ξ *are relatively prime polynomials. Then every saddle-saddle connection lies on a straight line.*

To accomplish the proof of the theorem, we make of a use of a strategy of Gomes [9]. Let U_L and U_R be two saddle points connected by an straight orbit $L : U = {}^t(1, k)t + U_L$. Owing to the fact that the segment $\tilde{L}$ from U_L to U_R is invariant under the vector field X_s, we have $(X_s|_{\tilde{L}}, {}^t(-k, 1)) = 0$.

Denoting $U = {}^t(u, v)$ and $U_L = {}^t(u_L, v_L)$, we have, from the above equation,

$$F_2(U) - F_2(U_L) = k\,(F_1(U) - F_1(U_L))\,, \tag{21}$$

i.e. $(kF_1(1, k) - F_2(1, k))\,u^2 = 0$ modulo polynomial of u of degree ≤ 1. It implies that

$$kF_1(1, k) - F_2(1, k)(= \Phi(k)) = 0, \tag{22}$$

then $k = \mu_j$ $(j = 1, 2$ or $3)$. Substituting $k = \mu_j$ into (21), we obtain

$$k^2(bu_L + v_L) + k\,((a - 1)u_L + bv_L) - (bu_L + v_L) = 0. \tag{23}$$

(22) $\times\, u_L$ $-$ (23) gives us $(k^2 + bk - 1)(ku_L - v_L) = 0$. Because clearly $k^2 + bk - 1 \neq 0$, we have $ku_L = v_L$. Then L is on a median.

Therefore the straight orbit lies on the medians and every median is invariant of the vector field X_s, which proves the assertion. The converse is quite clear.

In the context of the above proof, we showed

Corollary 3.1 *i) Every median M_j is invariant under the vector field X_s and every straight line orbit lies on a median. ii) The orbit of any saddle-saddle connection lies on a median.*

Let us investigate the structure of orbits on the medians. Let $U_L = {}^t(u_L, v_L)$ be a point on a median $M = \{U = {}^t(u,v); v = \mu u\}$ where $\mu = \mu_j\,(1 \leq j \leq 3)$. Owing to Corollary 3.1, the orbit through U_L lies on the median M. Then we have

$$X_s(U, U_L) = \{(a + 2b\mu + \mu^2)(u^2 - u_L^2) - s(u - u_L)\} \begin{pmatrix} 1 \\ \mu \end{pmatrix}. \tag{24}$$

Let $U_1 = {}^t(u_1, v_1)$ be a point $X_s(U_1, U_L) = 0\,(U_1 \neq U_L)$. Then we have $v_1 = \mu u_1$ and

$$u_1 = -u_L + \frac{\mu}{b + 2\mu} s. \tag{25}$$

If $u_1 < u_L$ i.e. $u_L > \dfrac{\mu}{2(b + 2\mu)} s$, then both components of $X_s(U, U_L)$ are negative for $u_1 < u < u_L$ and positive for $u < u_1$ and for $u > u_L$. Hence there is an orbit from U_L to U_1.

If $u_1 > u_L$ i.e. $u_L < \dfrac{\mu}{2(b + 2\mu)} s$, then both components of $X_s(U, U_L)$ are negative for $u_L < u < u_1$ and positive for $u < u_L$ and for $u > u_1$. Hence there is an orbit from U_1 to U_L.

In any case, there is an orbit between U_L and U_1. Therefore we have

Theorem 3.2 *Any point U_L on a median M_j $(1 \leq j \leq 3)$ can be connected via one shock to a point U_1 on the common median M_j and this shock has a viscous profile.*

Furthermore the character of shock waves on the median $M_j\,(1 \leq j \leq 3)$ can be determined in Case II by the following two propositions

Proposition 3.1 *Let $b \geq 0$. On the median M_2, there is no crossing shock in Case II.*

Proof. On the median $M_2 = \{{}^t(u,v); v = \mu_2 u\}$, the system (1) is reduced to the equation

$$v_t + \left(\frac{b}{\mu_2^2} + \frac{2}{\mu_2}\right)\left(\frac{v^2}{2}\right)_x = 0. \tag{26}$$

Then the speed of shock wave joining $U_+ = {}^t(u_+.v_+)$ and $U_- = {}^t(u_-.v_-)$ is $s(U_+,U_-) = \dfrac{b+2\mu_2}{2\mu_2^2}(v_+ + v_-)$. The Jacobian matrix $F'(U)$ on the median M_2 is

$$F'(U) = \begin{pmatrix} au+bv & bu+v \\ bu+v & u \end{pmatrix} = \frac{1}{\mu_2}\begin{pmatrix} a+b\mu_2 & b+\mu_2 \\ b+\mu_2 & 1 \end{pmatrix} v.$$

As we have already seen in Proposition 5.1 [3], the eigenvalues of $F'(U)$ are

$$\lambda(U) = \left(\frac{a}{\mu_2} + 2b + \mu_2\right) v = \frac{b+2\mu_2}{\mu_2^2} v \text{ and } \lambda^{\perp}(U) = \left(\frac{1}{\mu_2} - b - \mu_2\right) v$$

and its eigenvectors are ${}^t(v, \mu_2 v)$ and ${}^t(-\mu_2 v, v)$ respectively. We can determine $\lambda_1(U)$ and $\lambda_2(U)$ according to the sign of v (or u). In fact, we have

$$\lambda(U) - \lambda^{\perp}(U) = \frac{v}{\mu_2^2}(1+\mu_2^2)(\mu_2 + b). \tag{27}$$

On the median M_2, taking into account of (18), for $v > 0$, $\lambda_1(U) = \lambda^{\perp}(U)$, $\lambda_2(U) = \lambda(U)$ and, for $v < 0$, $\lambda_1(U) = \lambda(U)$, $\lambda_2(U) = \lambda^{\perp}(U)$.

Suppose that there is a crossing shock on the median M_2. We have four cases: $i) v_+ \geq 0, v_- > 0$, $ii) v_+ > 0, v_- \leq 0$, $iii) v_+ < 0, v_- \geq 0$, $iv) v_+ \leq 0, v_- < 0$. In case $i)$, we would have

$$\begin{aligned} s(U_+,U_-) - \lambda_2(U_+) &= \frac{2\mu_j + b}{\mu_j^2}(v_- - v_+) < 0, \\ s(U_+,U_-) - \lambda_2(U_-) &= \frac{2\mu_j + b}{\mu_j^2}(v_+ - v_-) < 0 \end{aligned}$$

which is not possible to realize. In case $ii)$, we would have

$$s(U_+,U_-) - \lambda_1(U_-) = \frac{2\mu_j + b}{2\mu_j^2}(v_+ - v_-) > 0 \text{ then } v_+ < v_-$$

which is not possible to realize. In case $iii)$, we would have

$$s(U_+,U_-) - \lambda_1(U_+) = \frac{2\mu_j + b}{2\mu_j^2}(v_- - v_+) > 0 \text{ then } v_- < v_+$$

which is not possible to realize. In case $iv)$, we would have

$$\begin{aligned} s(U_+,U_-) - \lambda_1(U_+) &= \frac{2\mu_j + b}{\mu_j^2}(v_- - v_+) < 0, \\ s(U_+,U_-) - \lambda_1(U_-) &= \frac{2\mu_j + b}{\mu_j^2}(v_+ - v_-) < 0 \end{aligned}$$

which is not possible to realize.

Therefore there is no crossing shock on the median M_2.

Proposition 3.2 *Let $b \geq 0$. Suppose that (a, b) belongs to Case II. On the median M_1, there is a saddle-saddle connection from U_- to U_+ if and only if $v_- < 0 < v_+$. On the median M_3, there is a saddle-saddle connection from U_- to U_+ if and only if $v_+ < 0 < v_-$.*

We can prove this proposition using a similar strategy as Proposition 3.1. Combining Corollary 3.1, Proposition 3.1 and Proposition 3.2, we have

Theorem 3.3 *There is no saddle-saddle connection nor crossing shock with viscous profile on the complement of $M_1 \cup M_3$ in Case II.*

The relation $X_s(U, U_L) = 0$ is the intersection of two quadratic equations $F_1(U) - F_1(U_L) - s(u - u_L) = 0$ and $F_2(U) - F_2(U_L) - s(v - v_L) = 0$. Then it consists of at most four points including U_L and U_1. In fact, the others are two saddle points. More precisely

Proposition 3.3 *Let U_L be a point on a median M_j $(1 \leq j \leq 3)$. The set $X_s(U, U_L) = 0$ consists of at most four points. The others critical points than U_L and U_1 consist only of saddle points.*

Proof. Let U_L be a point on a median M_j : $v_L = \mu_j u_L$. The equation $X_s(U, U_L) = 0$ implies that

$$\begin{aligned} F_1(U) - F_1(U_L) - s(u - u_L) &= 0, & (28)\\ F_2(U) - F_2(U_L) - s(v - v_L) &= 0. & (29) \end{aligned}$$

(29) $-$ (28) $\times\, \mu_j$ implies that

$$(a\mu_j - b)u^2 + 2(b\mu_j - 1)uv + \mu_j v^2 - s\mu_j u + sv + \{F_2(U_L) - \mu_j F_1(U_L)\} = 0.$$

Here

$$\begin{aligned} F_2(U_L) - \mu_j F_1(U_L) &= (b - a\mu_j)u_L^2 + 2(1 - b\mu_j)u_L v_L - \mu_j v_L^2 \\ &= u_L^2\{(b - a\mu_j) + 2\mu_j(1 - b\mu_j) - \mu_j^3\} \\ &= -u_L^2\{\mu_j^3 + 2b\mu_j^2 + (a - 2)\mu_j - b\} \\ &= 0. \end{aligned}$$

Hence we have

$$\begin{aligned} 0 &= (a\mu_j - b)u^2 + 2(b\mu_j - 1)uv + \mu_j v^2 - s\mu_j u + sv \\ &= (v - \mu_j u)\{\mu_j v - \tfrac{1}{\mu_j}(a\mu_j - b)u + s\} \\ &= (v - \mu_j u)\{\mu_j v + (\mu_j^2 + 2b\mu_j - 2)u + s\}. \end{aligned}$$

Therefore we have $v = \mu_j u$ and

$$\begin{aligned} v &= \frac{1}{\mu_j^2}(a\mu_j - b)u - \frac{s}{\mu_j} && (30) \\ \text{or equivalently } v &= \left(-\mu_j - 2b + \frac{2}{\mu_j}\right) u - \frac{s}{\mu_j}. && (31) \end{aligned}$$

Substituting $v = \mu_j u$ into $X_s(U, U_L) = 0$, we obtain as above $U = U_L, U_1$. Similarly substituting $v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right) u - \frac{s}{\mu_j}$ into $X_s(U, U_L)$, we obtain

$$\begin{aligned} X_s(U, U_L) &= x_s^1(U, U_L) \begin{pmatrix} 1 \\ \mu_j \end{pmatrix} && (32) \\ \text{where } x_s^1(U, U_L) &= \left(-3b - 2\mu_j + \frac{4}{\mu_j}\right) u^2 + s\left(2b + \mu_j - \frac{4}{\mu_j}\right) u && (33) \\ &\quad + \frac{s^2}{\mu_j} - (b + 2\mu_j)u_L^2 + s\mu_j u_L. && (34) \end{aligned}$$

Therefore on the line $v = \left(-\mu_j - 2b + \frac{2}{\mu_j}\right) u - \frac{s}{\mu_j}$, the vector field $X_s(U, U_L)$ has the constant direction $\pm{}^t(1, \mu_j)$ and passing through the critical point, $X_s(U, U_L)$ changes the sign. It occurs only in the case of saddle points, which proves the proposition.

4 Structural Stability

Applying Theorem 3.3 and Proposition 2.2 to Theorem 2.3, a vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ if and only if there are only a finite number of singularities and all are hyperbolic. Even if there are many variations of critical points as stated in Theorem 2.2, in any case, a vector field $X_s(U_L, U)$ has at most four critical points in bounded region and six critical points at infinity of U-plane and all of these are hyperbolic. Therefore we have

Theorem 4.1 *A vector field $X_s(U_L, U)$ is structurally stable on the complement of $M_1 \cup M_3$ in Case II.*

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