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Stock Price Process and Statistical Mechanics

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Abstract

A model of an auction for a stock market is introduced to derive a discrete time stock price process. A probability distribution of trader's positions configuration is given by a Gibbs distribution in which trading strategies of traders and an effect from market sentiments are contained. To derive a scaling limit the method of statistical mechanics is applied to this process , and a volatility function and a drift function are described in terms of polymer weight functions.

Keywords: Auction model,Gibbs distribution, Stock price process, Polymer expansion

1 Introduction

To describe a stock price process many stochastic processes such as Black Sholes model[1], ICR model [2] and Vasicek model [9] are investigated, particularly in an option price theory.

In this article we introduce an auction model for a stock market composed of N traders and a single stock. We derive a discrete time stochastic process describing a stock price process from this auction model, in which each trader can trade unit number of stocks at each time $t \in \Lambda_n = \{1, 2, \cdots, n\}$. It is also assumed that each trader can take one of the following three positions, buying position, selling position and neutral position at each time $t \in \Lambda_n = \{1, 2, \cdots, n\}$.

Probability distribution for trader's positions is given by a Gibbs measure, in which trading strategies of traders and an effect from market sentiments are contained. Gibbs measures is introduced by Dobrushin [4], Lanford and Ruelle [8] to investigate a statistical mechanics from mathematical point of view. It is characterized by an interaction energy. In our model this interaction energy is determined from trading strategies of traders and market sentiments.

Applying a method of polymer expansion developed in a theory of statisitical mechanics [3],[6] we derive a continuous time stock price process,

$$S_t = S_0 \exp\{\int_0^t \sigma(s)dB(s) + \int_0^t \mu(s)ds\}$$

by scaling the discrete time process, where $B(t)$ is a standard Brownian motion.

The function $\sigma(s)$ is called a volatility function which describes the strength of the variation of the stock price, and the function $\mu(s)$ is called a drift function which describes the trend of the stock price process.

We describe these functions in terms of polymer weight functions defined by the interaction energies which determined by trading strategies and market sentiments.

2 Model

As stated in the introduction each trader can take one of the following three positions, buying position denoted by $+$, selling position denoted by $-$ and neutral position denoted by 0.

Denote a positions configuration of N traders at time $t \in \Lambda_n$ by

$$\omega_t = (\omega_t(1), \omega_t(2), \cdots, \omega_t(N))$$

a sequence of positions configuration from time 0 to $n-1$ by

$$\omega = (\omega_0, \cdots, \omega_{n-1}).$$

and the totality of ω by Ω_n.

For any ω_t the number of traders with buying position, selling position, and neutral position are denoted by ω_t^+, ω_t^- and ω_t^0 respectively. The number of market participants is given by $\omega_t^+ + \omega_t^-$. We introduce a notion of modified number of market participants by

$$|\omega_t| = \begin{cases} \omega_t^+ - \omega_t^- - d_0 & \text{if } \omega_t \text{ is active} \\ 0 & \text{otherwise} \end{cases},$$

where d_0 is a positive constant.

We say ω_t(or time t) is *active* if $|\omega_t| > 0$ and *static* if $|\omega_t| = 0$.

We also put

$$< \omega_t >= \begin{cases} \omega_t^+ - \omega_t^- - d_0 & \text{if } \omega_t^+ - \omega_t^- > d_0 \\ 0 & \text{if } |\omega_t^+ - \omega_t^-| \leq d_0 \\ -(\omega_t^- \omega_t^+ - d_0) & \text{if } \omega_t^- - \omega_t^+ > d_0 \end{cases}$$

If $< \omega_{t-1} >> 0 (< 0)$, then the number of traders with buying (selling) position is greater than the number of traders with selling(buying) position and the stock price goes up(down) at time t. Let S_t be a stock price at time t. When ω is given we define the stock price process from ω by

$$\frac{S_t}{S_{t-1}} = e^{\alpha < \omega_{t-1} >}.$$

It is easily seen that

$$S_t = S_0 \exp\{\sum_{u=0}^{t-1} < \omega_u > \}.$$

Gibbs measure

We introduce a Gibbs measure as a probability distribution of positions configuration ω.

First we define an interaction energy $H(\omega)$ by

$$\begin{aligned} H(\omega) = {} & \beta_1 \sum_{t=0}^{n-1} |\omega_t|^2 + \beta_2 \sum_{t=0}^{n-1} \Phi(\omega_t | \omega_{t-a}, \cdots, \omega_{t-1}) \\ & - \beta_3 \sum_{t=0}^{n-1} c(t) f_1(|\omega|_{t,a}) |\omega_t| - \beta_4 \sum_{t=0}^{n-1} d(t) f_2(< \omega >_{t,a}) < \omega_t > \end{aligned}$$

where $\beta_1, \beta_2, \beta_3$, and β_4 are positive numbers which control the strength of interactions and $c(t)$ and $d(t)$ are positive functions defined on Λ_n.

Now we define a Gibbs distribution of ω by

$$P_n(\omega) = \frac{1}{Z_n} \exp\{-H(\omega)\},$$

where Z_n is a normalization constant called a partition function.

For this Gibbs distribution , the probability $P(\omega)$ is high(low) if the energy $H(\omega)$ is low(high).

The first term of $H(\omega)$ is a term controling the number of market participants. The second term corresponds to trading strategies of N traders and it is assumed that

(A-1) $\Phi(\omega_t|\omega_{t-a}, \cdots, \omega_{t-1}) = 0$ if ω_t is static

(A-2) $\Phi(\bar{\omega}_t|\bar{\omega}_{t-a}, \cdots, \bar{\omega}_{t-1}) = \Phi(\omega_t|\omega_{t-a}, \cdots, \omega_{t-1})$

(A-3) $\Phi(\omega_t|\omega_{t-a}, \cdots, \omega_{t-1}) > -c|\omega_t|$

where $\bar{\omega}_t$ is a reflection image of ω_t defined by $\bar{\omega}_t = -\omega_t$.

Traders look at a history $(\omega_{t-a}, \cdots, \omega_{t-1})$ of positions configurations and determine a present position ω_t.

The third term describes an effect on the market from a total number of market participants from time $t-a$ to $t-1$ and $|\omega|_{t,a}$ is given by $|\omega|_{t,a} = |\omega_{t-a}| + \cdots + |\omega_{t-1}|$.

If $f_1(|\omega|_{t,a}) > 0$, this term works for increasing a number of market participants.

We assume that $f_1(x) \leq c_1 x \quad (x > 0)$ for some $c_1 > 0$.

The firth term describes an effect on the market from a chenge of stock price from time $t-a$ to $t-1$ and $< \omega >_{t,a}$ is given by $< \omega >_{t,a} = < \omega_{t-a} > + \cdots + < \omega_{t-1} >$. We assume that

(1) $f_2(x) \leq c_2|x|$

(2) $f_2(0) = 0$

(3) $f_2(x) + f_2(-x) \geq 0$ for all $x \in \mathbf{R}$

(4) $f_2(x) + f_2(-x) > \epsilon_1$ unless $|x - b_3| < \delta_0$ or $|x + b_3| < \delta_0$ or $|x| < \delta_0$ for some $\epsilon_1 > 0, b_3 > 0, \delta_0 > 0$.

If $f_2(< \omega >_{t,a}) > 0$, this term works for increasing a number of traders with buying positions.

3 Method of cluster expansion

The method of polymer expansion has been developed by mainly Gallavotti[6], and Del Grosso[3], for rigorous investigations of phase transitions in lattice spin systems. (See also Phister[5] for a nice summary of this method.)

In this section we summarize a method of polymer expansion to state our results.

For a given configuration ω we denote by $\{t_1, \cdots, t_k\}$ a set of all active times. Decompose a set

$$\alpha(t_1) \cup \alpha(t_2) \cup \cdots \cup \alpha(t_k)$$

into a set of connected components $W_1, \cdots, W_m$, where $\alpha(t_i) = \{t_i - a, \cdots, t_i - 1\}$. We call a couple ξ^i of $W_i = \{t_f, \cdots, t_q\}$ and the configuration $\{\omega_{t_f}, \cdots, \omega_{t_q}\}$ on W_i a "*cluster*" and denote it by

$$\xi^i = \begin{pmatrix} \omega_f & \cdots & \omega_q \\ f & \cdots & q \end{pmatrix}$$

For any cluster ξ^i given in (3.1) put

$$p(\xi^i) = \{f, \cdots, q\}.$$

When a configuration $\omega \in \Omega_n$ is given , a family of clusters $\{\xi^1, \cdots, \xi^m\}$ is determined uniquely, but this correspondence is not one-to-one.

To define a probability distribution of $\{\xi^1, \cdots, \xi^m\}$ we first introduce a set $A(\xi^1, \cdots, \xi^m)$ by

$$A(\xi^1, \cdots, \xi^m) = \{\omega \in \Omega_n; \{\xi^1, \cdots, \xi^m\} \text{ is obtained from } \omega \text{ as a set of clusters } \}.$$

Then a probability distribution $P(\xi^1, \cdots, \xi^m)$ for a set of clusters is defined by

$$P(\xi^1, \cdots, \xi^m) = \sum_{\omega \in A(\xi^1, \cdots, \xi^m)} P(\omega).$$

It follows from the definition of interaction energies that the occurrence of clusters becomes independent and

$$P(\xi^1, \cdots, \xi^m) = \prod_{i=1}^{m} \mathcal{W}(\xi^i)$$

,where $\mathcal{W}(\xi)$ is given by

$$\mathcal{W}(\xi) = e^{-|p(\xi)|\log m_0} \prod_{t \in \hat{p}(\xi)} \exp\{-\beta_1|\xi_t|^2 - \beta_2\Phi(\xi_t|\xi_{t-a}, \cdots, \xi_{t-1}) \\ + \beta_3 c(t) f_1(|\xi|_{t,a})|\xi_t| + \beta_4 d(t) f_2(<\xi>_{t,a}) <\xi_t>\}$$

Algebraic formalism of cluster expansion

Let $\mathfrak{C}$ be a set of all clusters in $\mathbf{Z}$, and $\mathfrak{A}$ be a set of mappings given by

$$\mathfrak{A} = \{A : \mathfrak{C} \to \mathbf{N}; |A| < \infty \},$$

where

$$|A| = \sum_{\xi \in \mathfrak{C}} A(\xi).$$

Each $A \in \mathfrak{A}$ stands for a configuraion of finite number of clusters with multiplicity.

Furthermore, we define a functional space $\mathfrak{L}$ by

$$\mathfrak{L} = \{\varphi : \mathfrak{A} \to \mathbf{R}; \sup_{|A|=n}|\varphi(A) < \infty \text{ for any } n \}.$$

For any $\varphi_1, \varphi_2 \in \mathfrak{L}$ define a product $\varphi_1 * \varphi_2$ by

$$\varphi_1 * \varphi_2(A) = \sum_{A_1+A_2=A} \frac{A!}{A_1!} A_2! \varphi_1(A_1)\varphi_2(A_2)$$

where the sum runs over all ordered pairs (A_1, A_2) such that $A_1 + A_2 = A$.

It is easily seen that

$$\varphi_1 * (\varphi_2 * \varphi_3) = (\varphi_1 * \varphi_2) * \varphi_3.$$

This functional space $\mathfrak{L}$ becomes a commutative algebra with a unit element $\mathbf{1}$ given by

$$\mathbf{1}(A) = \begin{cases} 1 & \text{if } A = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

We define subspaces $\mathfrak{L}_0$ and $\mathfrak{L}_1$ given by

$$\mathfrak{L}_0 = \{\varphi \in \mathfrak{L}; \varphi(\emptyset) = 0 \}, \qquad \mathfrak{L}_1 = \{\varphi \in \mathfrak{L}; \varphi(\emptyset) = 1 \},$$

and a mapping Exp: $\mathfrak{L}_0 \to \mathfrak{L}_1$ by

$$\mathrm{Exp}\varphi(A) = \sum_{k=0}^{\infty} \frac{1}{k!}\varphi * \cdots * \varphi(A).$$

As an inverse mapping of Exp we define a mapping Log: $\mathfrak{L}_1 \to \mathfrak{L}_0$ by

$$\mathrm{Log}\varphi(A) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}\varphi_0 * \cdots \varphi_0(A)$$

where

$$\varphi_0 = \varphi - 1.$$

Then we have the followig relation between two mappings

$$\mathrm{Log}(\mathrm{Exp}\varphi) = \varphi \ \text{ for any } \ \varphi \in \mathfrak{L}_0$$
$$\mathrm{Exp}(\mathrm{Log}\varphi) = \varphi \ \text{ for any } \ \varphi \in \mathfrak{L}_1.$$

We say $\chi \in \mathfrak{L}$ is *multiplicative* if

$$\chi(A_1 + A_2) \ = \ \chi(A_1) \cdot \chi(A_2) \text{for any } A_1, A_2 \in \mathfrak{A}.$$

The following Lemma is a fundamental Lemma for cluster expansion.

Lemma 3.2([3])

If χ is multiplicative and

$$\sum_{A \in \mathfrak{A}} \frac{\mathrm{Log}\varphi(A)\chi(A)}{A!} < \infty$$

then

$$\sum_{A \in \mathfrak{A}} \frac{|\varphi(A)\chi(A)|}{A!} < \infty$$

and

$$\sum_{A \in \mathfrak{A}} \frac{\varphi(A)\chi(A)}{A!} \ = \ \exp\left\{\sum_{A \in \mathfrak{A}} \frac{\mathrm{Log}\varphi(A)\chi(A)}{A!}\right\}.$$

Application of cluster expansion to the probability distribution of clusters

Now we apply the method of cluster expansion to the probability distribution $P(\xi_1, \cdots, \xi_m)$ and derive a scaling limit of $W_t(\cdot)$.

First we define functionals $\phi_0(\xi), \phi_1(\xi), \phi_2(\xi)$ by

$$\phi_0(\xi) = \exp\{-\beta_1|\xi|^2 - \beta_2\Phi(\xi_t|\xi_{t-a}, \cdots, \xi_{t-1}) - \log m_0|p(\xi)|\}$$

$$\phi_1(\xi) = \exp\{\beta_3 \sum_{t\in p(\xi)} c(t)f_1(|\xi|_{t,a}>)|\xi_t|\}$$

$$\phi_2(\xi) = \exp\{\beta_4 \sum_{t\in p(\xi)} d(t)f_2(<\xi>_{t,a}) <\xi_t>\}$$

and we define $\phi_0(A)$,$\phi_1(A)$, $\phi_2(A)$, $\alpha(A)$ by

$$\phi_0(A) = \prod_{\xi\in\mathfrak{C}} \phi_0(\xi)^{A(\xi)} \qquad \phi_1(A) = \prod_{\xi\in\mathfrak{C}} \phi_1(\xi)^{A(\xi)} \qquad \phi_2(A) = \prod_{\xi\in\mathfrak{C}} \phi_1(\xi)^{A(\xi)}$$

$$\alpha(A) = \begin{cases} 1 & \text{if } A! = 1, p(\xi^i)\cap p(\xi^j) = \emptyset \text{ for all } \xi^i \neq \xi^j \in \operatorname{supp} A \\ 0 & \text{otherwise.} \end{cases}$$

Also we put

$$<A> = \sum_{\xi\in C} <\xi> A(\xi)$$

$$f_1(A) = \beta_3 \sum_{\xi\in C}\sum_{t\in p(\xi)} c(t)f_1(|\xi_0|_{t,a})|\xi_t|A(\xi)$$

$$f_2(A) = \beta_4 \sum_{\xi\in C}\sum_{t\in p(\xi)} d(t)f_2(<\xi>_{t,a}) <\xi_t> A(\xi),$$

and define a refletion image $\bar{A}$ of A by $\bar{A}(\xi) = A(\bar{\xi})$.

The following Lemma is obtained in the similar way developed in [].

Lemma 3.3

For any $\beta_2 > 0$ and $\beta_3 > 0$ we have

$$\sum_{\xi\in\mathfrak{C};O\in p(\xi)} W(\xi) < 1$$

for sufficiently large $\beta_1 > 0$.

Put

$$\beta_0 = \inf\{\beta_1 > 0; \sum_{\xi\ni 0} \exp\{-\beta_1|\xi|^2 - c_3|\xi|) + \log m_0|p(\xi)|\} < 1\} < \infty.$$

For any $A \in \mathcal{A}$ we call A a *polymer* if

$$\bigcup_{\xi A} p(\xi)$$

is connected.

Lemma 3.4
$\alpha^T(A) = 0$ unless A is a polymer.

Lemma 3.5
(1) If $\beta_1 > \beta_0$, then we have

$$\sum_{A \ni 0} \frac{\phi_0(A)\phi_1(A)\phi_2(A)}{A!} |\alpha^T(A)| \geq g(\beta_1)$$

where $\alpha^T(A) = \log \alpha(A)$ and $g(\beta_1) \to 0$ as $n \to \infty$.
(2) For any $0 < c < 1$ we have

$$\sum_{A \ni 0, |A|^2 \geq k} \frac{\phi_0(A)\phi_1(A)\phi_2(A)}{A!} |\alpha^T(A)| \geq g((1-c)\beta_1) e^{-c\beta_1 k}$$

if $(1-c)\beta_1 > \beta_0$, where

$$|A|^2 = \sum_{\xi} |\xi|^2 A(\xi).$$

As $\phi_0(A), \phi_1(A)$ and $\phi_2(A)$ are multiplicative, by applying the method of cluster expansion we have

$$Z_n = \exp\{ \sum_{A \subset \Lambda_n} \frac{\phi_0(A)\phi_1(A)\phi_2(A)\alpha^T(A)}{A!} \}.$$

4 Main Result

We define a scaled process $W_t^{(n)}(\cdot)$ by

$$W_t^{(n)}(\cdot) = \frac{1}{\sqrt{n}} W_{[nt]}(\cdot).$$

Decompose Λ_n into a set of intervals with length n^α:

$$\Lambda_n = B_n(1) \cup \cdots \cup B_n(n^{1-\alpha})$$

where $B_n(k) = ((k-1)n^\alpha, kn^\alpha]$.

Furthermore we assume that

$$c(t) = h(\frac{k}{n^{1-\alpha}}), \qquad d(t) = \frac{1}{\sqrt{n}} g(\frac{k}{n^{1-\alpha}}) \qquad (t \in B_n(k))$$

for continuous functions $f(x)$ and $g(x)$,
and put

$$\beta_{00} = \inf\{\beta_1 > \beta_0; \sum_{i(A)=0} |A|^4 e^{-(\beta_1|A|^2 - c_3|A|)} \frac{|\alpha^T(A)|}{A!} < \infty \}.$$

Theorem 4.1

If $\beta_1 > \beta_{00}$, then a finite dimensional distribution of $W_t^{(n)}(\cdot)$ converges to the correspond finite dimensional distribution of

$$\int_0^t \sigma(s) dB(s) + \int_0^t \mu(s) ds$$

where

$$\sigma^2(s) = \sum_{i(A)=0} < A >^2 e^{\beta_3 h(s) f_1(A)} \frac{\phi_0(A)\alpha^T(A)}{A!}$$

$$\mu(s) = \sum_{i(A)=0} g(s) e^{\beta_3 h(s) f_1(A)} f_2(A) < A > \frac{\phi_0(A)\alpha^T(A)}{A!}$$

Outline of the proof of Theorem 4.1

We state an outline of the proof of Theorem 4.1. (See [7] for detail.) The main tool for the proof is the method of polymer expansion developed in the previous section.

Applying Lemma 3.2 for $\varphi(A) = \exp\{iz\frac{1}{\sqrt{n}}W_{[nt]}(A)\}\phi_0(A)\phi_1(A)\phi_2(A)\alpha(A)$ we have the following description of a characteristic function $\varphi_t^{(n)}(z)$ of one dimensional distribution of $W_t^{(n)}$,

$$\begin{aligned}\varphi_t^{(n)}(z) &= E[\exp\{iz\frac{1}{\sqrt{n}}\sum_{u\le[nt]} <\xi_u>\}] \\ &= \exp\{\sum_{A\subset\Lambda_n}(e^{iz\frac{1}{\sqrt{n}}W_{[nt]}(A)}-1)\phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A!}\}.\end{aligned}$$

Using the Taylor's expansion we have

$$\begin{aligned}&\sum_{A\subset\Lambda_n}(e^{iz\frac{1}{\sqrt{n}}W_{[nt]}(A)}-1)\phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A!}\} \\ &= izI_1(n)-\frac{1}{2}z^2I_2(n)-\frac{iz^3}{6}I_3(n),\end{aligned}$$

where

$$\begin{aligned}I_1(n) &= \frac{1}{\sqrt{n}}\sum_{A\subset\Lambda_n}W_{[nt]}(A)\phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A!} \\ I_2(n) &= \frac{1}{n}\sum_{A\subset\Lambda_n}W_{[nt]}(A)^2\phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A!} \\ I_3(n) &= \frac{1}{n\sqrt{n}}\sum_{A\subset\Lambda_n}W_{[nt]}(A)^3e^{\frac{iz\theta_1}{\sqrt{n}}W_{[nt]}(A)}\phi_0(A)\phi_1(A)\phi_2(A)\frac{\alpha^T(A)}{A!}\end{aligned}$$

for some $\theta_1 \in (0,1)$.

Employing the method of poymer expansion developed in the theory of statistical mechanics we have the following results.

When $\beta_1 > \beta_{00}$ we prove that

$$\begin{aligned}\lim_{n\to\infty} I_1(n) &= \int_0^t \mu(s)ds \\ \lim_{n\to\infty} I_2(n) &= \int_0^t \sigma^2(s)ds \\ \lim_{n\to\infty} I_3(n) &= 0\end{aligned}$$

It follows from this proposition that

$$\varphi_t^{(n)}(z) \rightarrow \exp\{iz\int_0^t \mu(s)ds - \frac{1}{2}z^2\int_0^t \sigma^2(s)ds\}$$

Hence the one dimensional distribution of $W^{(n)}(t)$ converges to the corresponding distribution of

$$\int_0^t \mu(s)ds + \int_0^t \sigma(s)dB(s).$$

A characteristic function of a finite dimensional distribution $\varphi_{t_1,\cdots,t_k}(z_1,\cdots,z_k)$ of $W^{(n)}(t)$ is given by

$$\varphi_{t_1,\cdots,t_k}(z_1,\cdots,z_k) = E[\exp\{\frac{1}{\sqrt{n}}\sum_{k=1}^{m} z_k W_{[nt_k]}\}].$$

Using the result in one dimensional case we have

$$\varphi_{t_1,\cdots,t_k}(z_1,\cdots,z_k) \rightarrow \exp\{i\sum_{j=1}^{k} z_j\int_0^{t_j} \mu(s)ds - \frac{1}{2}\sum_{p=1}^{k}\sum_{q=1}^{k} z_p z_q\int_0^{t_p\wedge t_q} \sigma^2(s)ds\}$$

as $n \rightarrow \infty$. This implies that the finite dimensional distribution of $W^{(n)}(t)$ converges to the corresponding distribution of

$$\int_0^t \mu(s)ds + \int_0^t \sigma(s)dB(s).$$

Remark on trading volume

In this auction model a trading volume v_t at time t is given by

$$v_t = \mathrm{Min}\{\omega_t^+, \omega_t^-\}.$$

Let us remark that a trading volume is not always zero when t is static and a probability distribution of a trading volume v_t for static t is a uniform distribution. So, the expectation value of v_t for static t is given by

$$e(v) = \frac{e(v)}{n_0}$$

$$e(v) = \sum_{\omega_t:\text{static}} v(\omega_t) \qquad \text{and} \quad n_0 = \sum_{\omega_t:\text{static}} 1.$$

We denote by $V_{[nt]}$ a total trading volume traded in time interval $[0,[nt]]$ and consider a asymptotic behavior of $E[V_{[nt]}]$ and $V[V_{[nt]}]$, where $E[V_{[nt]}]$ and $V[V_{[nt]}]$ are an expectation and a variance of $V_{[nt]}$, respectively.

Using the method of polymer expansion we describe a characteristic function of $V_{[nt]}$ and obtain the formulas of $E[V_{[nt]}]$ and $V[V_{[nt]}]$ in terms of polymer weight functions.

From these formulas and using the same method obtaining the scaling limit of the characteristic function for a finite dimensional distribution $W_{[nt]}$ we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}E[V_{[nt]}] &= \int_0^t (\sum_{i(A)=O} v(A)e^{h(x)f_1(A)}\frac{\phi_0(A)\alpha^T(A)}{A!}dx \\
\lim_{n\to\infty}\frac{1}{n}V[V_{[nt]}] &= t\left(\frac{e(v^2)}{n_0}-(\frac{e(v)}{n_0})^2\right) \\
&\quad -\left(\frac{e(v^2)}{n_0}-(\frac{e(v)}{n_0})^2\right)\int_0^t \sum_{i(A)=O} |p(A)|e^{h(x)f_1(A)}\frac{\phi_0(A)\alpha^T(A)}{A!}dx \\
&\quad +\int_0^t \sum_{i(A)=O} (v(A)-\frac{e(v)}{n_0}||p(A)|)^2 e^{h(x)f_1(A)}\frac{\phi_0(A)\alpha^T(A)}{A!}dx
\end{aligned}$$

where

$$\begin{aligned}
v(A) &= \sum_{\xi}\sum_{u\in p(\xi)} v(\xi_u)A(\xi) \\
e(v^2) &= \sum_{\omega_t:\text{static}} v(\omega_t)^2
\end{aligned}$$

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