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Core equivalence in economy under generalized information*

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Abstract. We consider a pure exchange atomless economy under asymmetric information with emphasis on an epistemic point of view, where the traders are assumed to have a non-partitional information structure. We propose a generalized notion of rational expectations equilibrium for the economy and we show the core equivalence theorem: The ex-post core for the economy coincides with the set of all its rational expectations equilibria.

Keywords: Pure exchange economy under reflexive information structure, Ex-post core, Rational expectations equilibrium, Core equivalence theorem.
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1 Introduction

This article relates economies and traders' knowledge. We consider a pure exchange atomless economy under uncertainty where the traders are assumed to have a non-partitional information structure. The purpose is to propose the extended notion of rational expectations equilibrium for the economy, and we investigate the relationship between the ex-post core and the rational expectations equilibrium allocations with emphasis on epistemic point of view. It is shown that

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Main Theorem (Core equivalence theorem). *In a pure exchange atomless economy under generalized information, assume that the traders have a reflexive information structure and they are risk averse. Then the ex-post core coincides with the set of all rational expectations equilibrium allocations for the economy.*

Many authors have investigated several notions of core in an economy under asymmetric information (e.g., Wilson (1978), Volij (2000), Einy et al (2000) and others). The serious limitations of the analysis in these researches are its use of the 'partition' structure by which the traders receive information. The structure is obtained if each trader t's possibility operator $P_t : \Omega \to 2^{\Omega}$ assigning to each state ω in a state space Ω the information set $P_t(\omega)$ that t possesses in ω is reflexive, transitive and symmetric. From the epistemic point of view, this entails t's knowledge operator $K_t : 2^{\Omega} \to 2^{\Omega}$ that satisfies 'Truth' axiom **T**: $K_t(E) \subseteqq E$ (what is known is true), the 'positive introspection' axiom **4**: $K_t(E) \subseteqq K_t(K_t(E))$ (we know what we do) and the 'negative introspection' axiom **5**: $\Omega \setminus K_t(E) \subseteqq K_t(\Omega \setminus K_t(E))$ (we know what we do not know).

One of these requirements, symmetry (or the equivalent axiom **5**), is indeed so strong that describes the hyper-rationality of traders, and thus it is particularly objectionable. The recent idea of 'bounded rationality' suggests dropping such assumption since real people are not complete reasoners. In this article we weaken both transitivity and symmetry imposing only reflexivity. As has already been pointed out in the literature, this relaxation can potentially yield important results in a world with imperfectly Bayesian agents (e.g. Geanakoplos, 1989).

The idea has been performed in different settings. Among other things Geanakoplos (1989) showed the no speculation theorem in the extended rational expectations equilibrium under the assumption that the information structure is reflexive, transitive and *nested* (Corollary 3.2 in Geanakoplos [1989]). The condition 'nestedness' is interpreted as a requisite on the 'memory' of the trader.

Recently, Matsuhisa and Ishikawa (2002) introduced the notion 'rationality about expectations' with respect to a price system p. This is that each trader who learns from the price knows his/her expected utility. They showed the existence theorem of generalized rational expectations equilibrium for an economy under reflexive and transitive information structure; in particular, the existence of the equilibria under the further assumption that all trader are rational everywhere about expectations.

This article is in the line of Geanakoplos (1989) and Matsuhisa and Ishikawa (2002). We shall relax transitivity in an economy under generalized information structure, and we extend the ex-post core equivalence theorem of Einy et al (2000) into an economy under reflexive information structure with removing out transitivity and symmetry.

This article is organized as follows: Section 2 gives an illustration of Main theorem by a simple example of an economy under non-nested reflexive information structure. In Section 3 we present our model: An economy under reflexive information structure, a generalized notion of rational expectations equilibrium and ex-post core for the economy. Section 4 gives the existence theorem of rational expectations equilibrium. In Section 5 we give the proof of Main theorem.

Section 6 presents the fundamental theorem of welfare economics in an economy under reflexive information structure. Finally we conclude by giving some remarks about the assumptions of the theorem.

2 Illustrative example

Let us consider the following situation: Two traders 1 and 2 are willing to buy and sell the tradeable emissions permits with each other. Trader 1 is interested in the global warming problem, but trader 2 is not at all. There is one commodity, and only unused allowances are transferable between two traders 1 and 2.

We shall illustrate the situation as follows: Let Ω be the state space consisting of the three states $\{\omega_1, \omega_2, \omega_3\}$: The state ω_1 represents that the temperature is higher than the normal one, the state ω_2 represents that it is the normal temperature and finally the state ω_3 represents that the temperature is lower than the normal one.

Trader 1 is sensitive to the environmental change that the temperature becomes higher or lower, and so she can know which of either ω_1, ω_2 or ω_3 is the true state when each of them occurs. Hence trader 1 has her information structure $P_1(\omega) = \{\omega\}$ for any $\omega \in \Omega$.

Trader 2 is less sensitive than trader 1. He is ignorant of the environmental change, and so he cannot know which is a true state among ω_1, ω_2 and ω_3 when ω_2 occurs. When the temperature becomes higher or lower he cannot understand it, so he cannot know which of either ω_2 or ω_3 is a true state when ω_3 occurs, and he cannot know which of either ω_1 or ω_2 is a true state when ω_1 occurs. Hence trader 2 has his information structure $P_2(\omega_1) = \{\omega_1, \omega_2\}$, $P_2(\omega_2) = \Omega$ and $P_2(\omega_3) = \{\omega_2, \omega_3\}$.

Suppose that traders 1 and 2 have the initial endowments $e_1(\omega) = e_2(\omega) = 1$ ton for every $\omega \in \Omega$ and they have the risk averse utilities: $U_1(x, \omega) = U_2(x, \omega) = \sqrt{x+4}$ for every $\omega \in \Omega$. Their common prior π is given by $\pi(\omega) = \frac{3}{7}$ for $\omega = \omega_1, \omega_3$ and $\pi(\omega_2) = \frac{1}{7}$.

Then it can be plainly observed that the traders' initial endowments allocation is

- ex-ante Pareto optimal,
- the unique rational expectations equilibrium allocation (Corollary ??), and
- ex-post core allocation. (Main theorem)

It should be noted that P_2 satisfies the reflexivity: For any $\omega \in \Omega$, $\omega \in P_2(\omega)$, however it does not satisfy the transitivity: $P_2(\xi) \subseteq P_2(\omega)$ whenever $\xi \in P_2(\omega)$. Moreover P_2 is not *nested*.[4]

In this article we shall investigate the pure exchange economies under generalized information structure as like this example.

[4] An information structure $(P_i)_{i \in N}$ is said to be *nested* if for each $i \in N$ and for all states ω and ξ in Ω, either $P_i(\omega) \cap P_i(\xi) = \emptyset$, or else $P_i(\omega) \subseteq P_i(\xi)$ or $P_i(\omega) \supseteq P_i(\xi)$.

3 The Model

Let Ω be a non-empty *finite* set called a *state space*, and let 2^{Ω} denote the field of all subsets of Ω. Each member of 2^{Ω} is called an *event* and each element of Ω a *state*. The space of the traders is a measurable space (T, Σ, μ) in which T is a set of *traders*, Σ is a σ-field of subsets of T whose elements are called *coalitions*, and μ is a measure on Σ.

3.1 Information and Knowledge[5]

An *information structure* $(P_t)_{t \in T}$ is a class of mappings P_t of Ω into 2^{Ω}. It is said to be *reflexive* if

Ref $\quad \omega \in P_t(\omega) \quad$ for every $\omega \in \Omega$,

and it is said to be *transitive* if

Trn $\quad \xi \in P_t(\omega)$ implies $P_t(\xi) \subseteqq P_t(\omega)$ for any $\xi, \omega \in \Omega$.

An information structure $(P_i)_{i \in N}$ is called an *RT-information structure* if it is reflexive and transitive.[6]

Given our interpretation, a trader t for whom $P_t(\omega) \subseteqq E$ knows, in the state ω, that some state in the event E has occurred. In this case we say that at the state ω the trader t knows E. i's *knowledge operator* K_t on 2^{Ω} is defined by $K_t(E) = \{\omega \in \Omega | P_t(\omega) \subseteqq E\}$. The set $P_t(\omega)$ will be interpreted as the set of all the states of nature that t knows to be possible at ω, and $K_t E$ will be interpreted as the set of states of nature for which t knows E to be possible. We will therefore call P_t t's *possibility operator* on Ω and also will call $P_t(\omega)$ t's *possibility set* at ω. A possibility operator P_t is determined by the knowledge operator K_t such as $P_t(\omega) = \bigcap_{K_t E \ni \omega} E$. However it is also noted that the operator P_t cannot be uniquely determined by the knowledge operator K_t when P_t does not satisfy the both conditions **Ref** and **Trn**.

A *partitional* information structure is an RT-information structure $(P_t)_{t \in T}$ with the additional condition: For each $t \in T$ and every $\omega \in \Omega$,

Sym $\quad \xi \in P_t(\omega)$ implies $P_t(\xi) \ni \omega$.

3.2 Economy under reflexive information structure

A *pure exchange economy under uncertainty* is a tuple $\langle T, \Sigma, \mu, \Omega, e, (U_t)_{t \in T}, (\pi_t)_{t \in T} \rangle$ consisting of the following structure and interpretations: There are l commodities in each state of the state space Ω , and it is assumed that Ω is *finite* and that the consumption set of trader t is $\mathbb{R}^l_+$;

[5] See Bacharach (1985), Binmore (1992).

[6] An RT-information structure stands for a reflexive and transitive information struc-

- (T, Σ, μ) is the measure space of the traders;
- $e : T \times \Omega \to \mathbb{R}^l_+$ is t's *initial endowment* such that $e(\cdot, \omega)$ is μ-measurable for each $\omega \in \Omega$;
- $U_t : \mathbb{R}^l_+ \times \Omega \to \mathbb{R}$ is t's von-Neumann and Morgenstern utility function;
- π_t is a subjective prior on Ω for a trader $t \in T$.

For simplicity it is assumed that (Ω, π_t) is a finite probability space with π_t *full support*[7] for almost all $t \in T$.

Definition 1. An *economy under reflexive information structure* $\mathcal{E}^K$ is a structure $\langle \mathcal{E}, (P_t)_{t \in T} \rangle$, in which $\mathcal{E}$ is a pure exchange economy under uncertainty with a state-space Ω finite and $(P_t)_{t\in T}$ a reflexive information structure on Ω. Furthermore it is called an economy *under RT-information structure* if $(P_t)_{t\in T}$ is a reflexive and transitive information structure.

Remark 1. An economy under asymmetric information is an economy $\mathcal{E}^K$ under partitional information structure (i.e., $(P_t)_{t\in T}$ satisfies the three conditions **Ref**, **Trn** and **Sym**.)

Let $\mathcal{E}^K$ be an economy under reflexive information structure. We denote by $\mathcal{F}_t$ the field generated by $\{P_t(\omega) \mid \omega \in \Omega\}$ and by $\mathcal{F}$ the join of all $\mathcal{F}_t (t \in T)$; i.e. $\mathcal{F} = \vee_{t\in T}\mathcal{F}_t$. We denote by $\{A(\omega) \mid \omega \in \Omega\}$ the set of all atoms $A(\omega)$ containing ω of the field $\mathcal{F} = \vee_{t\in T}\mathcal{F}_t$.

Remark 2. The set of atoms $\{A_t(\omega) \mid \omega \in \Omega\}$ of $\mathcal{F}_t$ does not necessarily coincide with the partition induced from P_t.

We shall often refer to the following conditions: For every $t \in T$,

A-1 For every $\omega \in \Omega$, $\int_T e(t, \omega) d\mu \gneq 0$ for all $\omega \in \Omega$.
A-2 $e(t, \cdot)$ is $\mathcal{F}_t$-measurable
A-3 For each $x \in \mathbb{R}^l_+$, the function $U_t(x, \cdot)$ is $\mathcal{F}_t$-measurable, and the function: $T \times \mathbb{R}^l_+ \to \mathbb{R}, (t, x) \mapsto U_t(x, \omega)$ is $\Sigma \times \mathcal{B}$-measurable where $\mathcal{B}$ is the σ-field of all Borel subsets of $\mathbb{R}^l_+$.
A-4 For each $\omega \in \Omega$, the function $U_t(\cdot, \omega)$ is continuous, strictly increasing on $\mathbb{R}^l_+$.
A-5 For each $\omega \in \Omega$, the function $U_t(\cdot, \omega)$ is continuous, increasing, strictly quasi-concave and *non-saturated*[8] on $\mathbb{R}^l_+$.

Remark 3. It is plainly observed that **A-5** implies **A-4**. We note also that **A-3** does not mean that trader t knows his/her utility function $U_t(\cdot, \omega)$.[9]

[7] I.e., $\pi_t(\omega) \gneq 0$ for every $\omega \in \Omega$.
[8] I.e.; For any $x \in \mathbb{R}^l_+$ there exists an $x' \in \mathbb{R}^l_+$ such that $U_t(x', \omega) \gneq U_t(x, \omega)$.
[9] That is, $\omega \notin K_t([U_t(\cdot, \omega)])$ for some $\omega \in \Omega$, where $[U_t(\cdot, \omega)] := \{ \xi \in \Omega \mid U_t(\cdot, \xi) = U_t(\cdot, \omega)\}$. This is because the information structure is not a partitional structure.

3.3 Ex-post core

An *assignment* $\boldsymbol{x}$ is a mapping from $T \times \Omega$ into $\mathbb{R}^l_+$ such that for every $\omega \in \Omega$, the function $\boldsymbol{x}(\cdot, \omega)$ is μ-measurable, and for each $t \in T$, the function $\boldsymbol{x}(t, \cdot)$ is at most $\mathcal{F}$-measurable. We denote by $\mathcal{A}ss(\mathcal{E}^K)$ the set of all assignments for the economy $\mathcal{E}^K$.

By an *allocation* we mean an assignment $\boldsymbol{a}$ such that for every $\omega \in \Omega$,

$$\int_T \boldsymbol{a}(t, \omega) d\mu \leqq \int_T \boldsymbol{e}(t, \omega) d\mu.$$

We denote by $\mathcal{A}lc(\mathcal{E}^K)$ the set of all allocations, and for each $t \in T$ we denote by $\mathcal{A}lc(\mathcal{E}^K)_t$ the set of all the functions $\boldsymbol{a}(t, \cdot)$ for $\boldsymbol{a} \in \mathcal{A}lc(\mathcal{E}^K)$.

An assignment $\boldsymbol{y}$ is called an *ex-post improvement* of a coalition $S \in \Sigma$ on an assignment $\boldsymbol{x}$ at a state $\omega \in \Omega$ if

Imp1 $\mu(S) \gneqq 0$;
Imp2 $\int_S \boldsymbol{y}(t, \omega) d\mu \leqq \int_S \boldsymbol{e}(t, \omega) d\mu$; and
Imp3 $U_t(\boldsymbol{y}(t, \omega), \omega) \gneqq U_t(\boldsymbol{x}(t, \omega), \omega)$ for almost all $t \in S$.

We shall present the notion of core in an economy under reflexive information structure $\mathcal{E}^K$.

Definition 2. *An allocation $\boldsymbol{x}$ is said to be an* ex-post core *allocation of an economy under reflexive information structure $\mathcal{E}^K$ if there is no coalition having an ex-post improvement on $\boldsymbol{x}$ at any state $\omega \in \Omega$. The* ex-post core *denoted by $C^{ExP}(\mathcal{E}^K)$ is the set of all the ex-post core allocations of $\mathcal{E}^K$.*

Let $\mathcal{E}^K$ be the economy under reflexive information structure and $\mathcal{E}^K(\omega)$ the economy with complete information $\langle T, \Sigma, \mu, \boldsymbol{e}(\cdot, \omega), (U_t(\cdot, \omega))_{t \in T} \rangle$ for each $\omega \in \Omega$. We denote by $C(\mathcal{E}^K(\omega))$ the set of all core allocations for $\mathcal{E}^K(\omega)$.

Proposition 1. *Let $\mathcal{E}^K$ be a pure exchange economy under reflexive information structure satisfying the conditions* **A-1, A-2** *and* **A-3**. *Suppose that the economy is atomless (that is, (T, Σ, μ) is non-atomic measurable space.) The ex-post core of $\mathcal{E}^K$ is non-empty (i.e., $C^{ExP}(\mathcal{E}^K) \neq \emptyset$). Moreover, $C^{ExP}(\mathcal{E}^K)$ coincides with the set of all assignments $\boldsymbol{x}$ such that $\boldsymbol{x}(\cdot, \omega)$ is a core allocation for the economy $\mathcal{E}^K(\omega)$ for all $\omega \in \Omega$: i.e.,*

$$C^{ExP}(\mathcal{E}^K) = \{\boldsymbol{x} \in \mathcal{A}lc(\mathcal{E}^K) \mid \boldsymbol{x}(\cdot, \omega)) \in C(\mathcal{E}^K(\omega)) \textit{ for all } \omega \in \Omega\}.$$

Proof. Is given by the same way of the proof in Theorem 3.1 in Einy et al (2000). We shall give it in Appendix for readers' convenience.

3.4 Expectation and Pareto optimality

Let $\mathcal{E}^K$ be the economy under *reflexive* information structure. We denote by $\mathbf{E}_t[U_t(\boldsymbol{x}(t, \cdot)]$ the *ex-ante* expectation defined by

$$\mathbf{E}_t[U_t(\boldsymbol{x}(t, \cdot)] := \sum_{\omega \in \Omega} U_t(\boldsymbol{x}(t, \omega), \omega) \pi_t(\omega)$$

for each $x \in \mathcal{A}ss(\mathcal{E}^K)$. We denote by $\mathbf{E}_t[U_t(x(t,\cdot))|P_t](\omega)$ the *interim* expectation defined by

$$\mathbf{E}_t[U_t(x(t,\cdot)|P_t](\omega) := \sum_{\xi \in \Omega} U_t(x(t,\xi),\xi)\pi_t(\xi|P_t(\omega)).$$

Definition 3. *An allocation x in an economy $\mathcal{E}^K$ is said to be ex-ante Pareto-optimal if there is no allocation a with the two properties as follows:*

PO-1 *For almost all $t \in T$,*

$$\mathbf{E}_t[U_t(a(t,\cdot)] \geqq \mathbf{E}_t[U_t(x(t,\cdot)].$$

PO-2 *The set of all the traders $s \in T$ such that*

$$\mathbf{E}_s[U_s(a(t,\cdot)] \gneqq \mathbf{E}_s[U_s(x(t,\cdot)].$$

is not a μ-null set.

3.5 Rational expectations equilibrium

Let $\mathcal{E}^K = \langle N, \Omega, (e_t)_{t\in T}, (U_t)_{t\in T}, (\pi_t)_{t\in T}, (P_t)_{t\in T}\rangle$ be an economy under reflexive information structure. A *price system* is a non-zero $\mathcal{F}$-measurable function $p : \Omega \to \mathbb{R}^l_+$. We denote by $\sigma(p)$ the smallest σ-field that p is measurable, and by $\Delta(p)(\omega)$ the atom containing ω of the field $\sigma(p)$. The *budget set* of a trader t at a state ω for a price system p is defined by

$$B_t(\omega, p) := \{\ x \in \mathbb{R}^l_+ \mid p(\omega)\cdot x \leqq p(\omega)\cdot e(t,\omega)\ \}.$$

Let $\Delta(p) \cap P_t : \Omega \to 2^\Omega$ be defined by $(\Delta(p) \cap P_t)(\omega) := \Delta(p)(\omega) \cap P_t(\omega)$; it is plainly observed that the mapping $\Delta(p) \cap P_t$ satisfies **Ref**. We denote by $\sigma(p) \vee \mathcal{F}_t$ the smallest σ-field containing both the fields $\sigma(p)$ and $\mathcal{F}_t$, and by $A_t(p)(\omega)$ the atom containing ω. It is noted that

$$A_t(p)(\omega) = (\Delta(p) \cap A_t)(\omega).$$

Remark 4. If P_t satisfies **Ref** and **Trn** then $\sigma(p) \vee \mathcal{F}_t$ coincides with the field generated by $\Delta(p) \cap P_t$.

We shall give the extended notion of rational expectations equilibrium for an economy $\mathcal{E}^K$.

Definition 4. *A* rational expectations equilibrium *for an economy $\mathcal{E}^K$ under reflexive information structure is a pair (p, x), in which p is a price system and x is an allocation satisfying the following conditions:*

RE 1 *For every $t \in T$, $x(t,\cdot)$ is $\sigma(p) \vee \mathcal{F}_t$-measurable.*
RE 2 *For almost all $t \in T$ and for every $\omega \in \Omega$, $x(t,\omega) \in B_t(\omega, p)$.*

RE 3 *For almost all $t \in T$, if $y(t,\cdot) : \Omega \to \mathbb{R}^l_+$ is $\sigma(p) \vee \mathcal{F}_t$-measurable with $y(t,\omega) \in B_t(\omega, p)$ for all $\omega \in \Omega$, then*

$$\mathbf{E}_t[U_t(x(t,\cdot))|\Delta(p) \cap P_t](\omega) \geqq \mathbf{E}_t[U_t(y(t,\cdot))|\Delta(p) \cap P_t](\omega)$$

pointwise on Ω.

RE 4 *For every $\omega \in \Omega$, $\int_T x(t,\omega)d\mu = \int_T e(t,\omega)d\mu$.*

The allocation x in $\mathcal{E}^K$ is called a rational expectations equilibrium allocation.

We denote by $RE(\mathcal{E}^K)$ the set of all the rational expectations equilibria of an economy under reflexive information structure $\mathcal{E}^K$, and denote by $\mathcal{R}(\mathcal{E}^K)$ the set of all the rational expectations equilibrium allocations for the economy

4 Existence theorem

We shall prove the existence theorem of the generalized rational expectations equilibrium for an economy under reflexive information structure $\mathcal{E}^K$. Let $\mathcal{E}^K(\omega)$ be the economy with complete information for each $\omega \in \Omega$. We set by $W(\mathcal{E}^K(\omega))$ the set of all the competitive equilibria for $\mathcal{E}^K(\omega)$, and we denote by $\mathcal{W}(\mathcal{E}^K(\omega))$ the set of all the competitive equilibrium allocations for $\mathcal{E}^K(\omega)$.

Theorem 1. *Let $\mathcal{E}^K$ be a pure exchange economy under reflexive information structure*

satisfying the conditions **A-1, A-2, A-3** *and* **A-4**. *Suppose that the economy is atomless (that is, (T, Σ, μ) is non-atomic measurable space.) Then there exists a rational expectations equilibrium for the economy; i.e., $\mathcal{R}(\mathcal{E}^K) \neq \emptyset$.*

Proof. In view of the conditions **A-1**,**A-2**, **A-3** and **A-4**, it follows from the existence theorem of a competitive equilibrium for an atomless economy with complete information (c.f.: Theorem 9 in Debreu (1982)) that for each $\omega \in \Omega$, there exists a competitive equilibrium $(p^*(\omega), x^*(\cdot,\omega)) \in W(\mathcal{E}^K(\omega))$. We take a sequence of strictly positive numbers $\{k_\omega\}_{\omega\in\Omega}$ such that $k_\omega p^*(\omega) \neq k_\xi p^*(\xi)$ for any $\omega \neq \xi$. We define the pair (p, x) as follows: For each $\omega \in \Omega$ and for all $\xi \in A(\omega)$, $p(\xi) := k_\omega p^*(\omega)$ and $x(t,\xi) := x^*(t,\omega)$. It is noted that $x(\cdot,\xi) \in W(\mathcal{E}^K(\omega))$ because $\mathcal{E}^K(\xi) = \mathcal{E}^K(\omega)$, and we note that $\Delta(p)(\omega) = A(\omega)$.

We shall verify that (p, x) is a rational expectations equilibrium for $\mathcal{E}^K$: In fact, it is easily seen that p is $\mathcal{F}$-measurable with $\Delta(p)(\omega) = A(\omega)$ and that $x(t,\cdot)$ is $\sigma(p) \vee \mathcal{F}_t$-measurable, so **RE 1** is valid. Because $(\Delta(p) \cap P_t)(\omega) = A(\omega)$ for every $\omega \in \Omega$, it can be plainly observed that $x(t,\cdot)$ satisfies **RE 2**, and it follows from **A-3** that for almost all $t \in T$,

$$\mathbf{E}_t[U_t(x(t,\cdot))|\Delta(p) \cap P_t](\omega) = U_t(x(t,\omega),\omega) \tag{1}$$

On noting that $\mathcal{E}^K(\xi) = \mathcal{E}^K(\omega)$ for any $\xi \in A(\omega)$, it is plainly observed that $(p(\omega), x(t,\omega)) = (k_\omega p^*(\omega), x^*(t,\omega))$ is also a competitive equilibrium for $\mathcal{E}^K(\omega)$ for every $\omega \in \Omega$, and it can be observed by Eq (1) that **RE 3** is valid for (p, x), in completing the proof. □

Remark 5. Matsuhisa and Ishikawa (2002) shows Theorem 1 for an economy under *RT*-information structure.

5 Proof of Main theorem

We can now state explicitly Main theorem in Section 1 as follows:

Theorem 2. *Let $\mathcal{E}^K$ be a pure exchange economy under reflexive information structure satisfying the conditions* **A-1, A-2, A-3** *and* **A-4**. *Suppose that the economy is atomless (that is, (T, Σ, μ) is non-atomic measurable space.) Then the ex-post core coincides with the set of all rational expectations equilibrium allocations; i.e., $\mathcal{C}^{ExP}(\mathcal{E}^K) = \mathcal{R}(\mathcal{E}^K)$.*

In view of Theorem 1 it is first noted that $\mathcal{R}(\mathcal{E}^K) \neq \emptyset$. Because $\mathcal{E}^K(\omega)$ is an atomless economy for each $\omega \in \Omega$, it follows from the core equivalence theorem of Aumann (1964) that $\mathcal{C}(\mathcal{E}^K(\omega)) = \mathcal{W}(\mathcal{E}^K(\omega))$ for any $\omega \in \Omega$. We shall observe that Main theorem immediately follows from the above Proposition 1 together with the below Proposition 2:

Proposition 2. *Let $\mathcal{E}^K$ be an economy under reflexive information structure satisfying the conditions* **A-1, A-2, A-3** *and* **A-4**. *Then the set of all rational expectations equilibrium allocations $\mathcal{R}(\mathcal{E}^K)$ coincides with the set of all the assignments x such that $x(\cdot, \omega)$ is a competitive equilibrium allocation for the economy with complete information $\mathcal{E}^K(\omega)$ for all $\omega \in \Omega$; i.e.,*

$$\mathcal{R}(\mathcal{E}^K) = \{x \in \mathcal{A}lc(\mathcal{E}^K) \mid \textit{There is a price system } p \textit{ such that } (p(\omega), x(\cdot, \omega)) \in W(\mathcal{E}^K(\omega)) \textit{ for all } \omega \in \Omega\}.$$

Proof of Theorem 2:

Let $x \in \mathcal{R}(\mathcal{E}^K)$. By Proposition 2 we obtain that for each $\omega \in \Omega$, $(p(\omega), x(\cdot, \omega)) \in W(\mathcal{E}^K(\omega))$, and thus it follows from the theorem of Aumann (1964) that $x(\cdot, \omega)) \in \mathcal{C}(\mathcal{E}^K(\omega))$ for any $\omega \in \Omega$. By Propositions 1 it has been verified that $\mathcal{C}^{ExP}(\mathcal{E}^K) \supseteqq \mathcal{R}(\mathcal{E}^K)$.

The converse shall be shown as follows: Let $x \in \mathcal{C}^{ExP}(\mathcal{E}^K)$. It follows from Proposition 2 that for every $\omega \in \Omega$, $x(\cdot, \omega) \in \mathcal{C}(\mathcal{E}^K(\omega))$. By the theorem of Aumann (1964) there is $p^*(\omega) \in \mathbb{R}^l_+$ such that $(p^*(\omega), x(\cdot, \omega)) \in W(\mathcal{E}^K(\omega))$. We take a sequence of strictly positive numbers $\{k_\omega\}_{\omega \in \Omega}$ such that $k_\omega p^*(\omega) \neq k_\xi p^*(\xi)$ for any $\omega \neq \xi$. We define the prime system p as follows: For each $\omega \in \Omega$ and for all $\xi \in A(\omega)$, $p(\xi) := k_\omega p^*(\omega)$. Because $\mathcal{E}^K(\xi) = \mathcal{E}^K(\omega)$ for each $\omega \in \Omega$ and for all $\xi \in A(\omega)$, it can be observed that for every $\omega \in \Omega$, $(p(\omega), x(\cdot, \omega)) \in W(\mathcal{E}^K(\omega))$. By Proposition 2, we have observed that $\mathcal{C}^{ExP}(\mathcal{E}^K) \subseteqq \mathcal{R}(\mathcal{E}^K)$. □

Proof of Proposition 2

Let $x \in \mathcal{R}(\mathcal{E}^K)$ and (p, x) a rational expectations equilibrium for $\mathcal{E}^K$. We shall show that $(p(\omega), x(\cdot, \omega)) \in W(\mathcal{E}^K(\omega))$ for any $\omega \in \Omega$.

Suppose to the contrary that there exist a state $\omega_0 \in \Omega$ and non-null set $S \subseteqq T$ with the property: For each $s \in S$ there is an $a(s, \omega_0) \in B_s(\omega_0, p)$ such that $U_s(a(\omega_0), \omega_0) \gneqq U_s(\boldsymbol{x}(s, \omega_0), \omega_0)$. Define the function $\boldsymbol{y} : T \times \Omega \to \mathbb{R}^l_+$ by

$$\boldsymbol{y}(t, \xi) := \begin{cases} a(t, \omega_0) & \text{for } \xi \in A_t(p)(\omega_0) \text{ and } t \in S; \\ \boldsymbol{x}(t, \xi) & \text{otherwise.} \end{cases}$$

It is easily observed that $\boldsymbol{y}(t, \cdot)$ is $\sigma(p) \vee \mathcal{F}_t$-measurable for every $t \in T$. On noting that $\mathcal{E}^K(\xi) = \mathcal{E}^K(\omega)$ for any $\xi \in A_t(p)(\omega)$, it immediately follows that $B_t(\xi, p) = B_t(\omega, p)$ for every $\xi \in A_t(p)(\omega)$, so $\boldsymbol{y}(t, \omega) \in B_t(\omega, p)$ for almost all $t \in T$ and any $\omega \in \Omega$. Therefore it can be obtained that for all $s \in S$,

$$\mathbf{E}_s[U_s(\boldsymbol{x}(s, \cdot))|\Delta(p) \cap P_s](\omega_0) \lneqq \mathbf{E}_s[U_s(\boldsymbol{y}(s, \cdot))|\Delta(p) \cap P_s](\omega_0),$$

in contradiction for $(p, \boldsymbol{x}) \in \mathcal{R}(\mathcal{E}^K)$.

The converse will be shown as follows: Let $\boldsymbol{x}$ be an assignment with $(p(\omega), \boldsymbol{x}(\cdot, \omega)) \in W(\mathcal{E}^K(\omega))$ for any $\omega \in \Omega$. We take a sequence of strictly positive numbers $\{k_\omega\}_{\omega \in \Omega}$ such that $k_\omega p(\omega) \neq k_\xi p(\xi)$ for any $\omega \neq \xi$. We define the price system $p^* : \Omega \to \mathbb{R}^l_+$ such that for each $\omega \in \Omega$ and for all $\xi \in A(\omega)$, $p^*(\xi) := k_\omega p(\omega)$. We shall show that $(p^*, \boldsymbol{x}) \in RE(\mathcal{E}^K)$: In fact, it is first noted that $\Delta(p^*)(\omega) = A(\omega)$ and that $(p^*(\xi), \boldsymbol{x}(\cdot, \xi)) \in W(\mathcal{E}^K(\omega))$ for every $\xi \in A(p^*)(\omega)$ because $\mathcal{E}^K(\xi) = \mathcal{E}^K(\omega)$. Therefore $\boldsymbol{x}(t, \cdot)$ is $\sigma(p) \vee \mathcal{F}_t$-measurable for every $t \in T$, and $\boldsymbol{x}(t, \omega) \in B_t(\omega, p^*)$ for almost all $t \in T$. Let $\boldsymbol{y}(t, \cdot) : \Omega \to \mathbb{R}^l_+$ be a $\sigma(p^*) \vee \mathcal{F}_t$-measurable function with $\boldsymbol{y}(t, \omega) \in B_t(\omega, p^*)$ for all $\omega \in \Omega$. In viewing that $(\Delta(p^*) \cap P_t)(\omega) = A(\omega)$ it can be obtained from **A-3** that

$$\mathbf{E}_t[U_t(\boldsymbol{x}(t, \cdot))|\Delta(p^*) \cap P_t](\omega) = U_t(\boldsymbol{x}(t, \omega), \omega)$$

and

$$\mathbf{E}_t[U_t(\boldsymbol{y}(t, \cdot))|\Delta(p^*) \cap P_t](\omega) = U_t(\boldsymbol{y}(t, \omega), \omega).$$

Since $(p^*(\omega), \boldsymbol{x}(\cdot, \omega)) \in W(\mathcal{E}^K(\omega))$ it can be observed that $U_t(\boldsymbol{x}(t, \omega), \omega) \geqq U_t(\boldsymbol{y}(t, \omega), \omega)$ for almost all $t \in T$ and for each $\omega \in \Omega$, from which it follows from **A-3** that

$$\mathbf{E}_t[U_t(\boldsymbol{x}(t, \cdot))|\Delta(p^*) \cap P_t](\omega) \geqq \mathbf{E}_t[U_t(\boldsymbol{y}(t, \cdot))|\Delta(p^*) \cap P_t](\omega).$$

Therefore $(p^*, \boldsymbol{x}) \in RE(\mathcal{E}^K)$ and $\boldsymbol{x} \in \mathcal{R}(\mathcal{E}^K)$, in completing the proof. □

6 Fundamental theorem for welfare economics

We shall characterize welfare under the generalized rational expectations equilibrium for an economy under reflexive information structure $\mathcal{E}^K$. Here an economy is not assumed to be atomless.

Theorem 3. *Let $\mathcal{E}^K$ be an economy under reflexive information structure satisfying the conditions* **A-1,A-2, A-3** *and* **A-5**. *An allocation is ex-ante Pareto optimal if and only if it is a rational expectations equilibrium allocation relative to some price system.*

Proof. Follows immediately from Propositions 3 and 4 below. □

Proposition 3. *Let $\mathcal{E}^K$ be an economy under reflexive information structure satisfying the conditions* **A-1,A-2, A-3** *and* **A-5**. *If an allocation x is ex-ante Pareto optimal then it is a rational expectations equilibrium allocation relative to some price system.*

Proof. Is given by the same way in the proof of Proposition 4 in Matsuhisa and Ishikawa (2002). We shall give it in Appendix for readers' convenience. □

Proposition 4. *Let $\mathcal{E}^K$ be an economy under reflexive information structure satisfying the conditions* **A-1,A-2, A-3** *and* **A-5**. *Then an allocation x is ex-ante Pareto optimal if it is a rational expectations equilibrium allocation relative to a price system.*

Proof. It follows from Proposition 2 that $(p(\omega), x(\cdot,\omega))$ is a competitive equilibrium for the economy with complete information $\mathcal{E}^K(\omega)$ at each $\omega \in \Omega$. Therefore in viewing the fundamental theorem of welfare in the economy $\mathcal{E}^K(\omega)$, we can plainly observe that for all $\omega \in \Omega$, $x(\cdot,\omega)$ is Pareto optimal in $\mathcal{E}^K(\omega)$, and thus x is ex-ante Pareto optimal. □

7 Concluding remarks

We shall give a remark about the ancillary assumptions in results in this article. Could we prove the theorems under the generalized information structure removing out the reflexivity? The answer is no vein. If trader t's possibility operator does not satisfy **Ref** then his/her expectation with respect to a price cannot be defined at a state because it is possible that $\Delta(p)(\omega) \cap P_t(\omega) = \emptyset$ for some state ω.

Could we prove the theorems without four conditions **A-1, A-2, A-3** and **A-4** together with **A-5**. The answer is no again. The suppression of any of these assumptions renders the existence theorem of rational expectations equilibrium (Theorem 1) vulnerable to the discussion and the example proposed in Remarks 4.6 of Matsuhisa and Ishikawa (2002).

Appendix

Proof of Proposition 1

First we shall show the first half part of the proposition that $C^{ExP}(\mathcal{E}^K) \neq \emptyset$: In fact, it is noted that for every $\omega \in \Omega$, $C(\mathcal{E}^K(\omega)) \neq \emptyset$.[10] Take $(x_t(\omega))_{t\in T} \in C(\mathcal{E}^K(\omega))$ for each $\omega \in \Omega$. Let $x : T \times \Omega \to \mathbb{R}^l_+$ be the the mapping defined by $x(t,\omega) = x_t(\omega)$. Viewing the assumptions **A-2** and **A-3** we can observe that for each $\omega \in \Omega$, $\mathcal{E}^K(\xi) = \mathcal{E}^K(\omega)$ for all $\xi \in A(\omega)$, from which it immediately follows

[10] C.f. Aumann (1964).

that x is an assignment for $\mathcal{E}^K$. It can be plainly observed that $x \in \mathcal{C}^{ExP}(\mathcal{E}^K)$ as required.

Secondly we shall prove the last half part of the proposition. It can be plainly observed that $x \in \mathcal{C}^{ExP}(\mathcal{E}^K)$ for each assignment $x \in \mathcal{A}ss(\mathcal{E}^K)$ with $x(\cdot,\omega) \in \mathcal{C}(\mathcal{E}^K(\omega))$. The converse will be shown as follows. Suppose to the contrary that there exists a core x for $\mathcal{C}^{ExP}(\mathcal{E}^K)$, and there is a state $\omega_0 \in \Omega$ such that $x(\cdot,\omega_0) \notin \mathcal{C}(\mathcal{E}^K(\omega_0))$. Then there is a coalition $S \in \Sigma$ with $\mu(S) \gneqq 0$ and there is a μ-measurable function $y : T \to \mathbb{R}^l_+$ such that $\int_S y(t)d\mu \leqq \int_S e(t,\omega_0)d\mu$ and $U_s(y(s),\omega_0) \gneqq U_s(x(t,\omega_0),\omega_0)$ for almost all $s \in S$. We set by z the assignment for $\mathcal{E}^K$ defined by

$$z(t,\xi) := \begin{cases} y(t) & \text{if } \xi \in A(\omega_0), \\ e(t,\xi) & \text{if not.} \end{cases}$$

It is easily seen that z is an ex-post improvement of S on x at ω_0 in contradiction. This completes the proof.

Proof of Proposition 3

For each $\omega \in \Omega$ we denote by $G(\omega)$ the set of all the vectors $\int_T x(t,\omega)d\mu - \int_T y(t,\omega)d\mu$ with an assignment $y : T \times \Omega \to \mathbb{R}^l_+$ such that $U_t(y(t,\omega),\omega) \geqq U_t(x(t,\omega),\omega)$ for almost all $t \in T$; i.e.,

$$G(\omega) = \{ \int_T x(t,\omega)d\mu - \int_T y(t,\omega)d\mu \in \mathbb{R}^l \mid y \in \mathcal{A}ss(\mathcal{E}^K) \text{ and } U_t(y(t,\omega),\omega) \geqq U_t(x(t,\omega),\omega) \text{ for almost all } t \in T\}.$$

First, we note that that $G(\omega)$ is convex and closed in $\mathbb{R}^l_+$ by the conditions **A-1, A-2, A-3** and **A-5**. It can be shown that

Claim 1: For each $\omega \in \Omega$ there exists $p^*(\omega) \in \mathbb{R}^l_+$ such that $p^*(\omega) \cdot v \leqq 0$ for all $v \in G(\omega)$.

Proof of Claim 1: By the separation theorem,[11] we can plainly observe that the assertion immediately follows from that $v \leqq 0$ for all $v \in G(\omega)$: Suppose to the contrary that there exist $\omega_0 \in \Omega$ and $v_0 \in G(\omega_0)$ with $v_0 \gneqq 0$. Take an assignment y^0 for $\mathcal{E}^K$ such that for almost all t, $U_t(y^0(t,\omega),\omega_0) \geqq U_t(x(t,\omega_0),\omega_0)$ and $v_0 = \int_T x(t,\omega_0)d\mu - \int_T y^0(t,\omega_0)d\mu$. Consider the allocation z defined by

$$z(t,\xi) := \begin{cases} y^0(t,\omega_0) + \frac{v_0}{\mu(T)} & \text{if } \xi \in A(\omega_0), \\ x(t,\xi) & \text{if not.} \end{cases}$$

[11] See Lemma 8, Chapter 4 in Arrow and Hahn (1971, pp.92.)

It follows that for almost all $t \in T$,

$$\begin{aligned}
\mathbf{E}_t[U_t(z)] &= \sum_{\xi \in A(\omega_0)} U_t(y^0(t,\omega_0) + \frac{v_0}{\mu(T)}, \xi)\pi_t(\xi) \\
&\quad + \sum_{\xi \in \Omega \setminus A(\omega_0)} U_t(x(t,\xi),\xi)\pi_t(\xi) \\
&\gneqq \sum_{\xi \in A(\omega_0)} U_t(y^0(t,\omega_0),\xi)\pi_t(\xi) \\
&\quad + \sum_{\xi \in \Omega \setminus A(\omega_0)} U_t(x(t,\xi),\xi)\pi_t(\xi) \quad \text{because of } \mathbf{A\text{-}4} \\
&\geqq \mathbf{E}_t[U_t(x)].
\end{aligned}$$

This is in contradiction to which x is ex-ante Pareto optimal as required.

Secondly, let p be the price system defined as follows: We take a sequence of strictly positive numbers $\{k_\omega\}_{\omega \in \Omega}$ such that $k_\omega p^*(\omega) \neq k_\xi p^*(\xi)$ for any $\omega \neq \xi$. We define the price system p such that for each $\omega \in \Omega$ and for all $\xi \in A(\omega)$, $p(\xi) := k_\omega p^*(\omega)$. It can be observed that $\Delta(p)(\omega) = A(\omega)$. To conclude the proof we shall show

Claim 2: The pair (p, x) is a rational expectations equilibrium for $\mathcal{E}^K$.

Proof of Claim 2: We first note that for every $t \in T$ and for every $\omega \in \Omega$,

$$(\Delta(p) \cap P_t)(\omega) = \Delta(p)(\omega) = A(\omega),$$

Therefore it follows from **A-3** that for every allocation x,

$$\mathbf{E}_t[U_t(x(t,\cdot))|(\Delta(p) \cap P_t)](\omega) = U_t(x(t,\omega),\omega) \tag{2}$$

To prove Claim 2 it suffices to verify that x satisfies **RE 3**. Suppose to the contrary that there exists a non-null set $S \in \Sigma$ with the two properties:

1. For almost all $s \in S$, there is a $\sigma(p) \vee \mathcal{F}_s$-measurable function $y(s,\cdot) : \Omega \to \mathbb{R}^l_+$ such that $y(s,\omega) \in B_s(\omega,p)$ for all $\omega \in \Omega$;
2. $\mathbf{E}_s[U_s(y(s,\cdot))|(\Delta(p) \cap P_s)](\omega_0) \gneqq \mathbf{E}_s[U_s(x(s,\cdot)|(\Delta(p) \cap P_s)](\omega_0)$ for some $\omega_0 \in \Omega$.

In view of Eq (2) it immediately follows from Property 2 that $U_s(y(s,\omega_0),\omega_0) \gneqq U_s(x(s,\omega_0),\omega_0)$, and thus $y(s,\omega_0) \gneqq x(s,\omega_0)$ by **A-5**. Therefore we obtain that for all $s \in S$, $p(\omega_0) \cdot y(s,\omega_0) \gneqq p(\omega_0) \cdot x(s,\omega_0)$, in contradiction. This completes the proof. □

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