

MULTIPLICITIES OF CUSP FORMS

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1. Introduction

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Let G be a connected simple linear algebraic group defined over a number field F . It is a basic problem in the theory of automorphic forms to describe the spectral decomposition of the unitary representation $L^2(G(F)\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$. Such a unitary representation possesses an orthogonal decomposition

$$L^2(G(F)\backslash G(\mathbb{A})) = L^2_{disc} \oplus L^2_{cont}$$

into the direct sum of its discrete spectrum and its continuous spectrum. Let us write:

$$L^2_{disc} = \bigoplus_{\pi} m_{disc}(\pi) \cdot \pi.$$

It is known that the discrete multiplicities $m_{disc}(\pi)$ are finite. The discrete spectrum has a further orthogonal decomposition

$$L^2_d(G(F)\backslash G(\mathbb{A})) = L^2_{cusp} \oplus L^2_{res}$$

where L^2_{cusp} is the subspace of cusp forms, and L^2_{res} is the so-called residual spectrum. Let us write:

$$L^2_{cusp} = \hat{\oplus}_{\pi} m_{cusp}(\pi) \cdot \pi \quad \text{and} \quad L^2_{res} = \hat{\oplus}_{\pi} m_{res}(\pi) \cdot \pi.$$

In this talk, we consider the following two simple-minded questions:

- (A) Does there exist π such that $m_{cusp}(\pi) \cdot m_{res}(\pi) \neq 0$?
- (B) Can the collection of non-negative integers $\{m_{cusp}(\pi)\}$ be unbounded?

Here are some prior results on these questions:

- (i) When $G = PGL_n$, the results of Jacquet-Shalika [JS] and the multiplicity one theorem imply that $m_{disc}(\pi) \leq 1$ and thus the answers are negative for both questions.
- (ii) When $G = SL_2$, it is a recent result of Ramakrishnan [R] that $m_{disc}(\pi) \leq 1$.
- (iii) For a more general classical group G , it is known that $m_{cusp}(\pi)$ can be > 1 . Examples of such failure of multiplicity one were constructed by Labesse-Langlands [LL] for the inner forms of SL_2 , by Blasius [B] for SL_n (with $n \geq 3$) and by Li [L]

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for quaternionic unitary groups. However, in these examples, the multiplicities are bounded above by a number depending only on the given G .

In this talk, I will discuss the following theorem, which was obtained jointly with N. Gurevich and D.-H. Jiang in [GGJ]:

Theorem 1.1. *When $G = G_2$, both questions A and B have positive answers. More precisely, for each finite set S of places of F , with $\#S \geq 2$, there is an irreducible unitary representation π_S of $G_2(\mathbb{A})$ with*

$$\begin{cases} m_{\text{res}}(\pi_S) = 1, \\ m_{\text{disc}}(\pi_S) \geq \frac{1}{6}(2^{\#S} + (-1)^{\#S}2). \end{cases}$$

The representations π_S of the theorem are very degenerate: their local components are non-tempered and non-generic. They are the so-called unipotent representations. This may lead one to think that the phenomenon of unbounded cuspidal multiplicities only happens for very degenerate representations. However, as we explain in Section 3, it should already occur for representations in tempered L -packets. We shall discuss in Section 5 how we intend to construct these tempered representations of arbitrarily high cuspidal multiplicities.

In fact, the unboundedness of discrete multiplicities for G_2 is a consequence of a famous conjecture of J. Arthur (see [A1] and [A2]). Hence, we shall begin by reviewing his conjecture in the following section.

2. My Understanding of Arthur's Conjecture

In this section, we shall briefly discuss Arthur's conjecture on $L_{\text{disc}}^2(G(F)\backslash G(\mathbb{A}))$. For simplicity, we assume that G is split, simple and simply-connected, so that the dual group \widehat{G} is adjoint. We begin by introducing some notations.

Let L_F denote the Langlands group of F (whose existence is still conjectural). For the purpose of understanding Arthur's conjecture, there is no loss in pretending that L_F is the absolute Galois group of F . For each place v of F , one also has a local group L_{F_v} , and there should be a natural class of embeddings $L_{F_v} \hookrightarrow L_F$. The group L_{F_v} is actually known to exist: it is the Weil group if v is archimedean and the Weil-Deligne group if v is finite.

By an Arthur parameter for G , we mean a \widehat{G} -conjugacy class of homomorphisms

$$\psi : L_F \times SL_2(\mathbb{C}) \longrightarrow \widehat{G}$$

so that the following conditions hold:

- $\psi(L_F)$ is bounded in \widehat{G} ;
- the centralizer \mathcal{S}_ψ of the image of ψ is finite.

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Given ψ , Arthur defined a quadratic character ϵ_ψ of \mathcal{S}_ψ . In the examples we will look at later, ϵ_ψ turns out to be the trivial character. Hence we will not bother to go into the general definition here.

We will describe the conjecture in the statements A, B and C below.

(A) There is a decomposition:

$$L^2_{\text{disc}}(G(F)\backslash G(\mathbb{A})) = \bigoplus_{\psi} L^2[\psi],$$

indexed by the Arthur parameters for G .

Fix a parameter ψ . We must now describe the $G(\mathbb{A})$ -module $L^2[\psi]$. Using the embedding $L_{F_v} \hookrightarrow L_F$, we obtain local parameters

$$\psi_v : L_{F_v} \times SL_2(\mathbb{C}) \hookrightarrow \widehat{G}.$$

Let us set:

- \mathcal{S}_{ψ_v} = the finite group of components of the centralizer of the image of ψ_v .
- $\mathcal{S}_{\psi, \mathbb{A}} = \prod_v \mathcal{S}_{\psi_v}$, a compact group.
- $\Delta : \mathcal{S}_\psi \longrightarrow \mathcal{S}_{\psi, \mathbb{A}}$, the natural diagonal map.

(B) For each place v of F , there is a finite subset A_{ψ_v} of unitary representations of $G(F_v)$ associated to ψ_v ; this is the so-called local Arthur packet. This finite set is indexed by the irreducible characters of \mathcal{S}_{ψ_v} :

$$A_{\psi_v} = \{\pi_{\eta_v} : \eta_v \in \widehat{\mathcal{S}_{\psi_v}}\}.$$

Moreover, it should satisfy the following conditions:

- for almost all v where $\psi_v|_{L_{F_v}}$ is unramified, $\pi_{\mathbf{1}_v}$ is the irreducible unramified representation with Satake parameter

$$s_{\psi_v} := \psi_v \left(\text{Frob}_v \times \begin{pmatrix} q_v^{1/2} & \\ & q_v^{-1/2} \end{pmatrix} \right).$$

- a particular linear combination of the characters of the π_{η_v} 's is a stable distribution.
- certain identities involving transfer to endoscopic groups hold.

Here we have not described the last two conditions precisely as they will not be relevant for us in this talk.

If $\eta = \bigotimes_v \eta_v$ is an irreducible character of $\mathcal{S}_{\psi, \mathbb{A}}$, then we may set

$$\pi_\eta = \bigotimes_v \pi_{\eta_v}.$$

This is possible because for almost all v , $\eta_v = \mathbf{1}_v$ and $\pi_{\mathbf{1}_v}$ is required to be unramified by the above. We can now state the last statement of Arthur's conjecture:

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(C) The $G(\mathbb{A})$ -submodule $L^2[\psi]$ has a decomposition given by:

$$L^2[\psi] = \bigoplus_{\eta \in \widehat{\mathcal{S}_{\psi, \mathbb{A}}}} m_\eta \cdot \pi_\eta$$

where

$$m_\eta = \langle \epsilon, \Delta^*(\eta) \rangle_{\mathcal{S}_\psi}$$

is the multiplicity of ϵ in the representation $\Delta^*(\eta)$ of \mathcal{S}_ψ .

This concludes our discussion of Arthur's conjecture.

3. The Example of G_2

Now we examine the special case when $G = G_2$ so that $\widehat{G} = G_2(\mathbb{C})$. We shall write down some Arthur parameters for G_2 and see what Arthur's conjecture says for them. Essentially, the only fact we need to know about G_2 is the following:

Lemma 3.1. $G_2(\mathbb{C})$ contains a subgroup isomorphic to $SO_3(\mathbb{C}) \times S_3$, where S_3 is the symmetric group on 3 letters. Moreover, the centralizer of $SO_3(\mathbb{C})$ is precisely S_3 .

The map $SL_2(\mathbb{C}) \rightarrow SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$ corresponds via the Jacobson-Morozov theorem to the subregular unipotent orbit in $G_2(\mathbb{C})$. With this lemma in hand, we can now write down our first family of Arthur parameters.

3.1. Cubic unipotent parameters. Let E be an étale cubic F -algebra. Then E corresponds to a conjugacy class of maps

$$\rho_E : L_F \longrightarrow \text{Gal}(\overline{F}/F) \longrightarrow S_3.$$

Using ρ_E and the natural projection map from $SL_2(\mathbb{C})$ to $SO_3(\mathbb{C})$, we set:

$$\psi_E : L_F \times SL_2(\mathbb{C}) \longrightarrow S_3 \times SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

The maps ψ_E are the cubic unipotent Arthur parameters.

For simplicity, we shall only consider the case when $E = E_0$ is the split algebra $F \times F \times F$. In this case, ρ_{E_0} is the trivial map, and so we have:

$$\begin{cases} \mathcal{S}_{\psi_{E_0}} = \mathcal{S}_{\psi_{E_0}, v} = S_3 \\ \mathcal{S}_{\psi_{E_0, \mathbb{A}}} = S_3(\mathbb{A}). \end{cases}$$

The map $\mathcal{S}_\psi \rightarrow \mathcal{S}_{\psi, \mathbb{A}}$ is simply the natural embedding $S_3(F) \hookrightarrow S_3(\mathbb{A})$.

What does Arthur's conjecture say for the parameter ψ_{E_0} ? Well, statement B predicts that for each place v , the corresponding local Arthur packet has 3 members indexed by the irreducible characters of S_3 . So we have:

$$A_{\psi_{E_0}} = \{\pi_{1_v}, \pi_{r_v}, \pi_{\epsilon_v}\}$$

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where ϵ_v is the sign character of S_3 and r_v is the 2-dimensional one. Further, for S a finite set of places of F , let

$$\eta_S = (\otimes_{v \in S} r_v) \otimes (\otimes_{v \notin S} \mathbf{1}_v).$$

Then statement C predicts that the representation

$$\pi_S := \pi_{\eta_S} = (\otimes_{v \in S} \pi_{r_v}) \otimes (\otimes_{v \notin S} \pi_{\mathbf{1}_v})$$

occurs in $L^2[\psi_{E_0}]$ with multiplicity equal to the multiplicity of the trivial representation in $r \otimes r \otimes \dots \otimes r$ ($\#S$ times). A quick computation gives:

$$m_{disc}(\pi_S) \geq \frac{1}{6} \cdot (2^{\#S} + (-1)^{\#S} 2),$$

which is one of the main claims of Theorem 1.1. Thus Arthur's conjecture predicts the existence of a family of representations $\{\pi_S\}$ whose discrete multiplicities are unbounded as $\#S \rightarrow \infty$.

3.2. Some Tempered Parameters. Now we consider some tempered Arthur parameters, i.e. those for which ψ is trivial on $SL_2(\mathbb{C})$. Let us start with a cuspidal representation τ of PGL_2 such that

$$\tau_v = \begin{cases} \text{Steinberg representation for } v \in S_\tau; \\ \text{an unramified representation for } v \notin S_\tau \end{cases}$$

for some finite set S_τ of finite places of F . Conjecturally, τ corresponds to a map $\phi_\tau : L_F \rightarrow SL_2(\mathbb{C})$. Because of our assumptions, the map ϕ_τ is surjective; in fact, for $v \in S_\tau$, the local parameter ϕ_{τ_v} is already surjective, since it corresponds to the Steinberg representation.

Now we construct an Arthur parameter for G_2 using ϕ_τ as follows:

$$\psi_\tau : L_F \rightarrow SL_2(\mathbb{C}) \rightarrow SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

Then we have:

$$\begin{cases} \mathcal{S}_{\psi_\tau} = \mathcal{S}_{\psi_\tau, v} = S_3 & \text{for all } v \in S_\tau. \\ \mathcal{S}_{\psi_\tau, v} = \{1\} & \text{for all } v \notin S_\tau. \end{cases}$$

In particular, statement B in Arthur's conjecture predicts that the local packets have the following form:

$$A_{\psi_\tau, v} = \begin{cases} \{\pi'_{\mathbf{1}_v}, \pi'_{r_v}, \pi'_{\epsilon_v}\} & \text{if } v \in S_\tau; \\ \{\pi'_{\mathbf{1}_v}\} & \text{if } v \notin S_\tau. \end{cases}$$

Moreover, the representations in the local packets should be tempered.

In fact, the parameter ψ_τ is an example of Langlands parameter considered by Lusztig. Hence, in this case, the local packet $A_{\psi_\tau, v}$ has already been defined, and it does consist of 3 discrete series representations (see [GrS]).

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Finally, if we set

$$\pi_\tau = \left(\otimes_{v \in S_\tau} \pi'_{\tau_v} \right) \otimes \left(\otimes_{v \notin S_\tau} \pi'_{\mathbf{1}_v} \right),$$

then statement C in Arthur's conjecture implies that

$$m_{disc}(\pi_\tau) \geq \frac{1}{6} \cdot (2^{\#S_\tau} + (-1)^{\#S_\tau} 2).$$

In fact, since the representation π'_τ is tempered, it cannot occur in the residual spectrum, and so we have

$$m_{cusp}(\pi_\tau) \geq \frac{1}{6} \cdot (2^{\#S_\tau} + (-1)^{\#S_\tau} 2).$$

Now one can find cuspidal representations τ of PGL_2 of the above type and with S_τ as big as one wishes (using the trace formula for example). Hence, Arthur's conjecture predicts that one can find a family of tempered representations of $G_2(\mathbb{A})$ whose cuspidal multiplicities are unbounded.

4. Construction of Unipotent Cusp Forms

In this section, we explain how one constructs the unipotent representation π_S and demonstrates Theorem 1.1.

Let H be the disconnected linear algebraic group $Spin_8 \rtimes S_3$. For each place v of F , the group $H(F_v)$ has a distinguished representation Π_v known as the minimal representation. To be more precise, Π_v is a particular extension to $H(F_v)$ of the unramified representation of $Spin_8(F_v)$ whose Satake parameter is

$$\iota \left(\begin{array}{cc} q_v^{1/2} & \\ & q_v^{-1/2} \end{array} \right)$$

where $\iota : SL_2(\mathbb{C}) \rightarrow PGSO_8(\mathbb{C})$ is the map associated to the subregular unipotent orbit of the dual group $PGSO_8(\mathbb{C})$.

Now H contains the subgroup $S_3 \times G_2$, and one may restrict the representation Π_v to the subgroup $S_3(F_v) \times G_2(F_v)$ to get:

$$\Pi_v = \bigoplus_{\eta_v \in \widehat{S_3(F_v)}} \eta_v \otimes \pi_{\eta_v}.$$

In the beautiful papers [HMS] and [V], Huang-Magaard-Savin (for non-archimedean v) and Vogan (for archimedean v) showed that each π_{η_v} is a non-zero irreducible unitarizable representation and the π_{η_v} 's are mutually distinct. Moreover, the representations π_{η_v} can be completely determined, and $\pi_{\mathbf{1}_v}$ is unramified with Satake parameter $s_{\psi_{E_0, v}}$. In view of these results, it seems natural to take the set of representations π_{η_v} as the elements of the local Arthur packet $A_{\psi_{E_0, v}}$.

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Consider now the global situation. If $\Pi = \otimes_v \Pi_v$, then as an abstract representation of $S_3(\mathbb{A}) \times G_2(\mathbb{A})$, we have:

$$\Pi = \bigoplus_{\eta} \eta \otimes \pi_{\eta}$$

as $\eta = \otimes_v \eta_v$ ranges over the irreducible representations of $S_3(\mathbb{A})$. In particular, for each η , we have an embedding

$$\iota_{\eta} : \eta \otimes \pi_{\eta} \hookrightarrow \Pi.$$

Using residues of Eisenstein series, one can construct a $Spin_8(\mathbb{A})$ -equivariant embedding

$$\Theta : \Pi \hookrightarrow \mathcal{A}^2(Spin_8)$$

of Π into the space of square-integrable automorphic forms of $Spin_8$. For each η , we may now define a $G_2(\mathbb{A})$ -equivariant map Θ_{η} as follows:

$$\Theta_{\eta} : \eta \otimes \pi_{\eta} \xrightarrow{\iota_{\eta}} \Pi \xrightarrow{\Theta} \mathcal{A}^2(Spin_8) \xrightarrow{\text{restriction}} \{\text{functions on } G_2(F) \backslash G_2(\mathbb{A})\}.$$

Then the following was proved in [GGJ]:

Theorem 4.1. (i) *The image of Θ_{η} is contained in $\mathcal{A}^2(G_2)$.*

(ii) *The restriction of Θ_{η} to the subspace $\eta^{S_3(F)} \otimes \pi_{\eta}$ is injective.*

The proof of the theorem is not difficult; it involves showing the non-vanishing of certain Fourier coefficients. Also, it is easy to see that the restriction of Θ_{η} to $(\eta^{S_3(F)})^{\perp} \otimes \pi_{\eta}$ is identically zero. In any case, the theorem immediately implies that

$$m_{disc}(\pi_S) \geq \frac{1}{6} \cdot (2^{\#S} + (-1)^{\#S} 2).$$

In fact, in [G], we show that equality holds when F is totally real.

To complete the proof of Theorem 1.1, one may appeal to the determination of the residual spectrum of G_2 by H. Kim [K] and S. Zampera [Z]. Their results show that L_{res}^2 has the multiplicity one property, and further that $m_{res}(\pi_S) = 1$. This concludes the proof of Theorem 1.1.

5. Potential Construction of some Tempered Cusp Forms

Finally, we would like to explain how we expect to show that the tempered representation π_{τ} discussed in Section 3 has cuspidal multiplicity at least that predicted by Arthur's conjecture.

The parameter

$$\psi_{\tau} : L_F \longrightarrow SO_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$$

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actually factors as:

$$\psi_\tau : L_F \longrightarrow SO_3(\mathbf{C}) \hookrightarrow SL_3(\mathbf{C}) \hookrightarrow G_2(\mathbf{C}).$$

Hence, instead of lifting the cuspidal representation τ of PGL_2 directly to G_2 , one may first lift it to a cuspidal representation of PGL_3 . This is precisely the Gelbart-Jacquet lift, and we denote this cuspidal representation of PGL_3 by $GJ(\tau)$. Note that

$$GJ(\tau)_v = \begin{cases} \text{the Steinberg representation } St_v & \text{if } v \in S_\tau; \\ \text{a specific unramified representation} & \text{if } v \notin S_\tau. \end{cases}$$

Now it turns out that $PGL_3 \times G_2$ is a dual pair in the split (adjoint) exceptional group of type E_6 . This suggests that we may use exceptional theta correspondence to lift $GJ(\tau)$ from PGL_3 to G_2 : hopefully we will get the representation π_τ . For this to work out, one should first verify that under local theta correspondence, the Steinberg representation St_v of $PGL_3(F_v)$ lifts to the representation π'_τ of $G_2(F_v)$. However, it was shown in [GS] that the theta lift of St_v is equal to $\pi'_1 \oplus \pi'_\epsilon$. So this doesn't work out as planned.

Thankfully, a homomorphism $L_F \longrightarrow SL_3(\mathbf{C})$ is not just a Langlands parameter for PGL_3 ; it is also a parameter for any inner form of PGL_3 . Such an inner form is of the form PD^\times where D is a degree 3 division algebra. Over a p -adic field F_v , there are two such division algebras: D_v and its opposite D_v^{opp} . Being opposite algebras, their groups of invertible elements define isomorphic algebraic groups. Thus, locally, PGL_3 has precisely one inner form PD^\times .

Now under the local Jacquet-Langlands correspondence, the Steinberg representation St_v corresponds to the trivial representation $\mathbf{1}_v$ of $PD^\times(F_v)$. Moreover, $PD^\times \times G_2$ is a dual pair in an inner form of E_6 . It was shown in [S] that the local theta lift of $\mathbf{1}_v$ is indeed equal to π'_τ .

Hence we are led to the following strategy for embedding π_τ into L_{cusp}^2 . Choose a global division algebra D of degree 3 which is ramified precisely at the set S_τ . Then one lifts τ from PGL_2 to G_2 as follows:

$$\begin{array}{ccccccc} PGL_2 & \xrightarrow{\text{Gelbart-Jacquet}} & PGL_3 & \xrightarrow{\text{Jacquet-Langlands}} & PD^\times & \xrightarrow{\text{theta lift}} & G_2 \\ \tau & \longrightarrow & GJ(\tau) & \longrightarrow & JL_D(GJ(\tau)) & \longrightarrow & \Theta(JL_D(GJ(\tau))). \end{array}$$

As an abstract representation, $\Theta(JL_D(GJ(\tau)))$ is indeed isomorphic to π_τ (if it is non-zero).

How does the multiplicity $\frac{1}{6} \cdot (2^{\#S_\tau} + (-1)^{\#S_\tau} 2)$ arise in this case? The answer lies in the following lemma:

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Lemma 5.1. *The number of global division algebras of degree 3 ramified precisely at a set S is equal to*

$$\frac{1}{3} \cdot (2^{\#S} + (-1)^{\#S} 2).$$

In particular, the number of inner forms of PGL_3 which are ramified at the set S is half of the above number.

Note that the various inner forms of the lemma are non-isomorphic as algebraic groups, but their groups of adelic points are abstractly isomorphic. Thus the reason for the high multiplicity here is the failure of Hasse principle for the inner forms of PGL_3 !

In order for the above strategy to work, it remains to show:

- the non-vanishing of the theta lift $\Theta(JL_D(GJ(\tau)))$;
- the various $\Theta(JL_D(GJ(\tau)))$'s generate linearly independent copies of π_τ in L^2_{cusp} .

At the moment, we are still trying to resolve these questions.

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