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# Certain series attached to an even number of elliptic modular forms

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## 1 Results

Let $n \in \mathbf{Z}_{>0}$, $k := (k_1, \ldots, k_n) \in (\mathbf{Z}_{>0})^n$, $m = (m_1, \ldots, m_n) \in (\mathbf{Z}_{>0})^n$ and $s \in \mathbf{C}$. We put

$$\begin{aligned} Q_k^{(n)}(m, s) \ &:= \ \int_0^\infty t^{s+|k|-n-1} dt \\ &\cdot \ \prod_{j=1}^n \int_0^\infty u_j^{k_j - 2} e^{-4\pi m_j u_j t} (\sqrt{u_j}\theta(iu_j) - 1) du_j \ ; \end{aligned} \tag{1}$$

here $|k| := \sum_{j=1}^n k_j$ and

$$\theta(z) := \sum_{l=-\infty}^{\infty} e^{\pi i l^2 z}$$

is the Jacobi theta function. The right-hand side of (1) converges absolutely and locally uniformly for $\mathrm{Re}(s) > \frac{n}{2}$. It is easy to see

$$Q_k^{(n)}(m, \sigma) > 0 \quad \text{for} \quad \frac{n}{2} < \sigma \in \mathbf{R}.$$

For $w \in \mathbf{Z}$ let $M_w$ be the space of holomorphic modular forms of weight $w$ for $SL_2(\mathbf{Z})$ and $S_w$ be the space of cusp forms in $M_w$. Let $f_j$ and $g_j$ be elements of $M_{k_j}$ such that $f_j(z)g_j(z)$ is a cusp form for each $j = 1, \ldots, n$. Let

$$f_j(z) = \sum_{l=0}^\infty a_j(l) e^{2\pi i l z} \quad \text{and} \quad g_j(z) = \sum_{l=0}^\infty b_j(l) e^{2\pi i l z} \tag{2}$$

be the Fourier expansions. The series we treat here is the following:

$$
\begin{aligned}
&\mathcal{D}(s; f_1, \ldots, f_n; g_1, \ldots, g_n) \\
:= &\sum_{m=(m_1,\ldots,m_n)\in(\mathbf{Z}_{>0})^n} \left( \prod_{j=1}^{n} a_j(m_j)\overline{b_j(m_j)} \right) Q_k^{(n)}(m, s). \qquad (3)
\end{aligned}
$$

The right-hand side of (3) converges absolutely and locally uniformly for

$$
\mathrm{Re}(s) > \frac{n}{2}(\max_{1\le j\le n}(k_j) + 1).
$$

**Theorem 1.**
(i) *The series* (3) *has a meromorphic continuation to the whole s-plane.*
(ii) *Let* $( \, , \, )$ *be the Petersson inner product. Then the function*

$$
\sum_{\nu=1}^{n} \sum_{1\le i_1<\ldots<i_\nu\le n} \left( \prod_{\substack{j\ne i_1,\ldots,i_\nu \\ 1\le j\le n}} (f_j, g_j) \right) \cdot \mathcal{D}(s; f_{i_1}, \ldots, f_{i_\nu}; g_{i_1}, \ldots, g_{i_\nu})
$$

*is invariant under the substitution* $s \mapsto n - s$ ; *it has possible simple poles at* $s = 0$ *and* $s = n$ *with residues* $-\prod_{j=1}^{n}(f_j, g_j)$ *and* $\prod_{j=1}^{n}(f_j, g_j)$ *respectively, and is holomorphic elsewhere.*

In case where every $g_j$ is the Eisenstein series we have

**Corollary.** *Suppose* $f_j \in S_{k_j}$ $(j = 1, \ldots, n)$ *with Fourier expansions as in* (2). *For* $l \in \mathbf{Z}_{>0}$ *put*

$$
\sigma_\nu(l) := \sum_{d|l} d^\nu \quad \text{for} \quad \nu \in \mathbf{C}.
$$

*Then the series*

$$
\mathcal{S}(s; f_1, \ldots, f_n) := \sum_{m=(m_1,\ldots,m_n)\in(\mathbf{Z}_{>0})^n} \left( \prod_{j=1}^{n} a_j(m_j)\sigma_{k_j-1}(m_j) \right) Q_k^{(n)}(m, s)
$$

*has a holomorphic continuation to the whole s-plane and satisfies the functional equation*

$$
\mathcal{S}(s; f_1, \ldots, f_n) = \mathcal{S}(n - s; f_1, \ldots, f_n).
$$

# 2 A key to the proof: an integral of Rankin-Selberg type

We use the following type of Eisenstein series for the Siegel modular group $\Gamma_n := Sp_{2n}(\mathbf{Z})$ whose properties were studied by Kohnen-Skoruppa [2], Yamazaki [5], and Deitmar-Krieg [1]:

$$E_s^{(n)}(Z) := \sum_{M \in \Delta_{n,n-1} \backslash \Gamma_n} \left( \frac{\det(\mathrm{Im}(M\langle Z \rangle))}{\det(\mathrm{Im}(M\langle Z \rangle^*))} \right)^s . \tag{4}$$

Here $s \in \mathbf{C}$, $Z$ is a variable on $H_n$, the Siegel upper half space of degree $n$,

$$\Delta_{n,n-1} := \left\{ \begin{pmatrix} * & * \\ 0^{(1,2n-1)} & * \end{pmatrix} \in \Gamma_n \right\},$$

$M$ runs over a complete set of representatives of $\Delta_{n,n-1} \backslash \Gamma_n$; for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D$ being $n \times n$ blocks ,

$$M\langle Z \rangle := (AZ + D)(CZ + D)^{-1}$$

and $M\langle Z \rangle^*$ is the upper left $(n-1) \times (n-1)$ block of $M\langle Z \rangle$. We understand that

$$\det(\mathrm{Im}(M\langle Z \rangle^*)) = 1$$

if $n = 1$. The right-hand side of (4) converges absolutely and locally uniformly for $\mathrm{Re}(s) > n$. Put

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

By [1][5], the Eisenstein series (4) has meromorphic continuation in $s$ to the whole $s$-plane; the function $\xi(2s)E_s^{(n)}(Z)$ is invariant under the substitution $s \mapsto n - s$ and is holomorphic except for the simple poles at $s = 0$ and $s = n$ with residues $-1/2$ and $1/2$, respectively.

Theorem 1 follows from the following integral representation:

**Theorem 2.** *For*

$$F_j(z) := \overline{f_j(z)} g_j(z) \mathrm{Im}(z)^{k_j}$$

*we have*

$$\left( \left( \cdots \left( E_s^{(n)} \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix}, F_1(z_1) \right), \cdots \right), F_n(z_n) \right)$$

$$= \frac{1}{2\xi(2s)} \sum_{\nu=1}^{n} \sum_{1 \le i_1 < \ldots < i_\nu \le n} \left( \prod_{\substack{j \ne i_1, \ldots, i_\nu \\ 1 \le j \le n}} (f_j, g_j) \right)$$
$$\cdot \mathcal{D}(s; f_{i_1}, \ldots, f_{i_\nu}; g_{i_1}, \ldots, g_{i_\nu}).$$

*Remark.* Define a symmetric positive definite matrix

$$P_Z := \begin{pmatrix} 1_n & {}^tX \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ X & 1_n \end{pmatrix}.$$

Then

$$E_s^{(n)}(Z) = \frac{1}{2\zeta(2s)} \sum_{h \in \mathbf{Z}^{(2n,1)} - \{0\}} ({}^thP_Zh)^{-s} \quad \text{for} \quad \mathrm{Re}(s) > n.$$

# 3 Supplementary remarks

(i) Let

$$\varphi_j(z) = \sum_{l=1}^{\infty} c_j(l) e^{2\pi i l z}$$

be holomorphic primitive cusp forms of weight 1 for $\Gamma_0(N_j)$ with odd characters $\chi_j$ where $N_j \in \mathbf{Z}_{>0}$ and $j = 1, \ldots, n$. Suppose $n \ge 3$. Then by Kurokawa [3, Theorem 5], the Dirichlet series

$$\sum_{l=1}^{\infty} c_1(l) \cdots c_n(l) l^{-s}$$

has meromorphic continuation in the region $\mathrm{Re}(s) > 0$ but has the line $\mathrm{Re}(s) = 0$ as a natural boundary. (Cf. also [4, Theorem 8].) Thus it is a nontrivial problem to find a series associated with more than two elliptic modular forms which has analytic continuation to the whole $s$-plane.
(ii) In case $n = 1$ we have

$$\mathcal{D}(s; f_1; g_1) = 2\xi(2s)(4\pi)^{1-k_1-s}\Gamma(s + k_1 - 1)D(s + k_1 - 1, f_1, g_1)$$

for $\mathrm{Re}(s) > (k_1 + 1)/2$, where

$$D(s, f_1, g_1) := \sum_{m=1}^{\infty} a_1(m)\overline{b_1(m)} m^{-s}.$$

Thus in this case Theorem 1 states nothing but the well-known properties of the Rankin series $D(s, f_1, g_1)$.

(iii) In case $n = 2$ we have

$$\mathcal{D}(s; f_1, f_2; g_1, g_2)$$
$$= 2^{6-2|k|}\pi^{2-|k|}(2\pi)^{-2s}\frac{\Gamma(s)\Gamma(s+|k|-2)\Gamma(s+k_1-1)\Gamma(s+k_2-1)}{\Gamma(2s+|k|-2)}$$
$$\cdot \sum_{m_1,m_2\in\mathbf{Z}_{>0}} a_1(m_1)a_2(m_2)\overline{b_1(m_1)b_2(m_2)}m_1^{1-k_1-s}m_2^{1-k_2}$$
$$\cdot \sum_{\lambda_1,\lambda_2\in\mathbf{Z}_{>0}} \lambda_1^{-2s}F\left(s, s+k_1-1; 2s+|k|-2; 1-\frac{m_2\lambda_2^2}{m_1\lambda_1^2}\right)$$

for $\mathrm{Re}(s) > \max(k_1, k_2) + 1$, where $F = {}_2F_1$ is the hypergeometric function.
(iv) The function $Q_k^{(n)}(m, s)$ has another representation:

$$Q_k^{(n)}(m,s) \quad = 2^{3n-|k|+1}\pi^{\frac{n-|k|}{2}-s}\left(\prod_{j=1}^{n} m_j^{\frac{1-k_j}{2}}\right)\cdot \sum_{\lambda_1,\ldots,\lambda_n\in\mathbf{Z}_{>0}}\left(\prod_{j=1}^{n}\lambda_j^{k_j-1}\right)$$
$$\cdot \int_0^\infty t^{2s-1+|k|-n}\prod_{j=1}^{n}K_{k_j-1}(4\sqrt{\pi m_j}\lambda_j t)dt$$

for $\mathrm{Re}(s) > n/2$, where $K_\nu$ is the modified Bessel function of order $\nu$.

# References


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