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Principal series Whittaker functions on $SL(3, \mathbf{R})$

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This is an extract from a preprint with the same title. The full proofs are contained in that. Here we write only the major results. *The numbering of the statement are the same as the original full paper. Some statements in the original are skipped.*

Introduction

The study of Whittaker models of algebraic groups over local fields has already some history. The Jacquet integral is named after the investigation of H.Jacquet [7]. Multiplicity free theorem by J.Shalika for quasi-split groups, was later enhanced for the case of the real field by N.Wallach. For redutive groups over the real field, this theme was investigated by M.Hashizume [5], B.Kostant, D. Vogan, H.Matsumoto, and the joint work of R.Goodman and N.Wallach [4].

More specifically $GL(n, \mathbf{R})$, explicit expressions for class 1 Whittaker functions are obtained, firstly for $n = 3$ by D.Bump [2]. The main contributor for the case of general n seems to be E.Stade. Other related results will be find in the references of the papers of him ([9],[10]).

Let us explain the outline of this paper. The purpose of the master thesis [1] refered above is to investigate the Whittaker functions belonging to the non-spherical principal series representations of $SL(3, \mathbf{R})$. The minimal K-type of such representations is 3-dimensional. So we have to consider vector-valued functions. The main results are, firstly, to obtain the holonomic system of the A-radial part of such Whittaker functions with minimal K-type explicitly (§4), and secondly to have 6 formal solutions (§5, Theorem (5.5)), which are considered as examples of confluent hypergeometric series of two variables. We also have integral expressions of these 6 solutions(§5, Theorem (5.6)). In the subsequent section, the Jacquet integral (so to say, the primary Whittaker function) is written as a sum of these 6 *secondary* Whittaker functions (§6-8).

1 Preliminaries. Basic terminology

1.1 Whittaker model

Given an irreducible admissible representation (π, H) of $G = SL(3, \mathbf{R})$, we consider its model or realization in the space of Whittaker functions. This means, for a nondegenerate unitary character ψ of a maximal unipotent subgroup $N = \{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G\}$ of G defined by

$$\psi(\begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}) = \exp\{2\pi\sqrt{-1}(c_1 x_{12} + c_2 x_{23})\}$$

with $c_1, c_2 \in \mathbf{R}$ being non-zero, we consider a smooth induction $C^\infty\text{-Ind}_N^G(\psi)$ to G, and the space of intertwining operators of smooth G-modules

$$\mathrm{Hom}_G(H_\infty, C^\infty\text{-Ind}_N^G(\psi))$$

with H_∞ the subspace consisting of C^∞-vectors in H. Or more algebraically speaking, we might consider the corresponding space in the context of $(\mathfrak{g}, K)$-modules (with $\mathfrak{g} = \mathrm{Lie}(G), K = SO(3)$):

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H_\infty, C^\infty\text{-Ind}_N^G(\psi)).$$

1.2 Principal series representations

Let P_0 be a minimal parabolic subgroup of G given by the upper triangular matrices in G, and $P_0 = MAN$ be a Langlands decomposition of P_0 with $M = K \cap \{\text{diagonals in } G\}$, $A = \exp\mathfrak{a}$, with

$$\mathfrak{a} = \{\mathrm{diag}(t_1, t_2, t_3) | t_i \in \mathbf{R}, t_1 + t_2 + t_3 = 0\}.$$

In order to define a principal series representation with respect to the minimal parabolic subgroup P_0 of G, we firstly fix a character σ of the finite abelian group M of type $(2,2)$ and a linear form $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$. For such data, we can define a representation $\sigma \otimes e^\nu$ of MA, and extend this to P_0 by the identification $P_0/N \cong MA$. Then we set

$$\pi_{\sigma,\nu} = L^2\text{-Ind}_{P_0}^G(\sigma \otimes e^{\nu+\rho} \otimes 1_N).$$

Here $\nu(\mathrm{diag}(t_1, t_2, t_3)) = \sum_{i=1}^3 \nu_i t_i$ with $\nu_i \in \mathbf{C}$ and ρ is the half-sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ for P_0, given as follows. For $i < j$ $(1 \le i, j \le 3)$, we put $\eta_{ij}(a) = a_i/a_j$ for $a = \mathrm{diag}(a_1, a_2, a_3)$ $(a_1 a_2 a_3 = 1)$. Then we have $a^{2\rho} = \prod_{i<j} a_i/a_j = a_1^2/a_3^2 = a_1^4 a_2^2$ by definition. Hence $a^\rho = a_1^2 a_2$.

Here the characters σ_j of M are identified as follows. The group M consisting of 4 elements is a finite abelian group of $(2,2)$ type, and its elements except for the unity is given by the matrices

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since M is commutative, all the irreducible unitary representations of it is 1-dimensional. For any $\sigma \in \widehat{M}$, we have $\sigma^2 = 1$. Therefore the set $\widehat{M}$ consisting of

4 characters $\{\sigma_j : j = 0, 1, 2, 3\}$, where each σ_j, except for the trivial character σ_0, is specified by the following table of values at the elements m_i.

	m_1	m_2	m_3
σ_1	1	-1	-1
σ_2	-1	1	-1
σ_3	-1	-1	1

Proposition (1.1) (i) *If σ is the trivial character of M, the representation $\pi_{\sigma,\nu}$ is spherical or class 1, i.e., it has a (unique) K-invariant vector in the representation space $H_{\sigma,\nu}$.*
(ii) *If σ is not trivial, then the minimal K-type of the restriction $\pi_{\sigma,\nu|K}$ to K is a 3-dimensional representation of $K = SO(3)$, which is isomorphic to the unique standard one (τ_2, V_2). The multiplicity of this minimal K-type is one:*

$$\dim_{\mathbf{C}} \operatorname{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1,$$

namely there is a unique non-zero K-homomorphism

$$\iota : (\tau_2, V_2) \to (\pi_{\sigma,\nu|K}, H_{\sigma,\nu})$$

up to constant multiple.

2 Representations of $K = SO(3)$

2.1 The spinor covering

To describe the finite dimensional irreducible representations of $SO(3)$, the simplest way seems to utilize the double covering $s : SU(2) = Spin(3) \to SO(3)$, which is realized as follows.

The Hamilton quaternion algebra $\mathbf{H}$ is realized in $M_2(\mathbf{C})$ by

$$\mathbf{H} = \{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbf{C}) | a, b \in \mathbf{C}\}.$$

Then $SU(2)$ is the subgroup of the multiplicative group consisting of quaternions with reduced norm 1, i.e.,

$$SU(2) = \{x \in \mathbf{H} | \det x = 1\}.$$

Let $\mathbf{P} = \{x \in \mathbf{H} | \mathrm{tr} x = 0\}$ be the 3-dimensional real Euclidean space consisting of pure quaternions. Then for each $x \in SU(2)$, the map

$$p \in \mathbf{P} \mapsto x \cdot p \cdot x^{-1} \in \mathbf{P}$$

preserve the Euclid norm $p \mapsto \det p$ and the orientation, hence we have a homomorphism

$$s : SU(2) \to SO(\mathbf{P}, \det) = SO(3),$$

which is surjective, since the range is a connected group. The kernel of this homomorphism is given by $\{\pm 1_2\}$.

By the derivation of s $ds : \mathfrak{su}(2) \to \mathfrak{so}(3)$, the standard generators:

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

are mapped to $2K_1, 2K_2, 2K_3$ with

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{k},$$

respectively. Here $\mathfrak{k}$ is the Lie algebra of K.

2.2 Representations of $SU(2)$

The set of equivalence classes of the finite dimensional continuous representations of $SU(2)$ is exhaused by the symmetric tensor products τ_l $(l = 0, 1, \ldots,)$ of the standard representation. These are realized as follows.

Let V_l be the subspace consisting of homogeneous polynomials of two variables x, y in the polynomial ring $\mathbf{C}[x, y]$. For $g \in SU(2)$ with $g^{-1} = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, and $f(x, y) \in V_l$ we set

$$\tau_l(g)f(x, y) := f(ax + by, -\bar{b}x + \bar{a}y).$$

Passing to the Lie algebra $Lie(SU(2)) = \mathfrak{su}(2)$, the derivation of τ_l, denoted by the same symbol, is described as follows by using the standard basis $\{v_k = x^k y^{l-k}\ (0 \leq k \leq l)\}$ and the standard generators

$$u_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Namely we have

$$\tau_l(u_1)v_k = \sqrt{-1}(l - 2k)v_k, \quad \tau_l(X_+)v_k = (l - k)v_{k+1}, \quad \tau_l(X_-)v_k = -k \cdot v_{k-1}.$$

Here we put $X_+ = \frac{1}{2}(u_2 + \sqrt{-1}u_3)$, $X_- = \frac{1}{2}(u_2 - \sqrt{-1}u_3)$.

The condition that τ_l defines a representation of $SO(3)$ by passing to the quotient with respect $s : SU(2) \to SO(3)$ is that $\tau_l(-1_2) = (-1)^l = +1$, i.e., l is even. Therefore the dimension of V_l, $l + 1$ is odd in this case.

The representation τ_2 of $SU(2)$ is equivalent to the spinor homomorphism. Hence passing to the quotient, τ_2 is equivalent to the tautological representation $SO(3) \to GL(3, \mathbf{C})$.

2.3 Irreducible components of $\tau_2 \otimes \tau_4$ and $\tau_2 \otimes Ad_{\mathfrak{p}}$

For our later use, we want to specify the standard basis of the unique irreducible constituent τ_2 in the tensor product $\tau_2 \otimes \tau_4$.

Lemma (2.1) *Let* $\{v_i\ (i=0,1,2)\}$ *and* $\{w_j\ (0 \le j \le 4)\}$ *be the standard basis of* (τ_2, V_2) *and* (τ_4, V_4)*, respectively. Then the elements*

$$\begin{aligned} v_0' &= v_0 \otimes w_2 - 2v_1 \otimes w_1 + v_2 \otimes w_0, \\ v_1' &= v_0 \otimes w_3 - 2v_1 \otimes w_2 + v_2 \otimes w_1, \\ v_2' &= v_0 \otimes w_4 - 2v_1 \otimes w_3 + v_2 \otimes w_2 \end{aligned}$$

define a set of standard basis in $\tau_2 \subset \tau_2 \otimes \tau_4$*, which is unique up to a common scalar multiple.*

2.4 The K-module isomorphism betwenn $\mathfrak{p}_{\mathbf{C}}$ and V_4

We denote by $\mathfrak{p}_{\mathbf{C}}$ the complexification of the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ with respect to the Killing form, on which the group K acts via the adjont action $Ad_{\mathfrak{p}}$. We denote by E_{ij} the matrix unit with 1 at (i,j)-th entry and 0 at other entries. Then E_{ii} and $E_{ij} + E_{ji}$ are considered as elements in $\mathfrak{p}$. We set $H_{ij} = E_{ii} - E_{jj}$ for $i \neq j$.

Lemma (2.2) *Via the unique isomorphism* V_4 *and* $\mathfrak{p}_{\mathbf{C}}$ *as* K*-modules we have the identification*

$$\begin{aligned} w_0 &= -2\{H_{23} - \sqrt{-1}(E_{23} + E_{32})\}, \\ w_1 &= \sqrt{-1}\{(E_{12} + E_{21}) - \sqrt{-1}(E_{13} + E_{31})\}, \\ w_2 &= \tfrac{2}{3}(H_{12} + H_{13}), \\ w_3 &= \sqrt{-1}\{(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31})\}, \\ w_4 &= -2\{H_{23} + \sqrt{-1}(E_{23} + E_{32})\}. \end{aligned}$$

3 Principal series $(\mathfrak{g}, K)$-modules

3.1 The case of the class one principal series

3.1.1 The Capelli elements

A set of generators for the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{sl}_3$ is obtained as Capelli elements, because $\mathfrak{sl}_3$ is of type A_2.

Let

$$E'_{ii} = E_{ii} - \frac{1}{3}\left(\sum_{a=1}^{3} E_{aa}\right),\ E'_{ij} = E_{ij}\ (i \neq j).$$

Then $E'_{ij} \in \mathfrak{g}$. Define a matrix C of size 3 with entries in $\mathfrak{g}$ by

$$C = \begin{pmatrix} E'_{11} & E'_{12} & E'_{13} \\ E'_{21} & E'_{22} & E'_{23} \\ E'_{31} & E'_{32} & E'_{33} \end{pmatrix} - \mathrm{diag}(-1, 0, 1).$$

Then for

$$\mathcal{A} = (A_{ij})_{1 \le i,j \le 3} = x \cdot 1_3 - C \in M_3(\mathfrak{g}[x]) \subset M_3(U(\mathfrak{g})[x]),$$

we define its *vertical* determinant by

$$\det \downarrow (\mathcal{A}) = \sum_{\sigma \in \mathfrak{S}_3} \mathrm{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} A_{3\sigma(3)}.$$

Then it is written in the form $x^3 + Cp_2 x - Cp_3 \in U(\mathfrak{g})[x]$ with some elements Cp_2 and Cp_3 in $Z(\mathfrak{g})$.

Proposition (3.1) *The set $\{Cp_2, Cp_3\}$ is a system of independent generators of $Z(\mathfrak{g})$. Here are explicit formulae of Cp_2 and Cp_3:*

$$\begin{aligned} Cp_2 &= (E'_{11}-1)E'_{22} + E'_{22}(E'_{33}+1) + (E'_{11}-1)(E'_{33}+1) \\ &\quad -E_{23}E_{32} - E_{13}E_{31} - E_{12}E_{21}, \end{aligned}$$

$$\begin{aligned} Cp_3 = &\quad (E'_{11}-1)E'_{22}(E'_{33}+1) + E_{12}E_{23}E_{31} + E_{13}E_{21}E_{32} \\ & -(E'_{11}-1)E_{23}E_{32} - E_{13}E'_{22}E_{31} - E_{12}E_{21}(E'_{33}+1). \end{aligned}$$

Eigenvalues of Cp_2, Cp_3

We compute the value $Cp_2 f_0(e)$ and $Cp_3 f_0(e)$. Let $S_2(a,b,c) = ab + bc + ca$ and $S_3(a,b,c) = abc$ be the elementary symmetric functions of three variables of degree 2 and 3, respectively. Then we have the following.

Proposition (3.2) *The infinitesimal character of $\pi_{\sigma_0,\nu}$ is given by*

$$Cp_2 f_0 = S_2(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)) f_0$$

and

$$Cp_3 f_0 = S_3(\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), \frac{1}{3}(\nu_1 + \nu_2)) f_0.$$

3.2 $(\mathfrak{g}, K)$-module structure of non-spherical principal series at the minimal K-type

3.2.1 Construction of K-equivariant differential operators

Lemma (3.3) *Let $\{f_i\ (i = 0,1,2)\}$ be the set of the standard basis of the minimal K-type $\tau \subset \pi_{\sigma,\nu}$ of a non-spherical principal series representation $\pi_{\sigma,\nu} = \pi$. Define another three C^∞-elements $\{\varphi_i\ (i = 0,1,2)\}$ by the formulae:*

$$\begin{aligned} \varphi_0 &= \tfrac{2}{3}\pi(2E_{11} - E_{22} - E_{33}) f_0 \\ &\quad -2\sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31})) f_1 \\ &\quad -2\pi(E_{12} + E_{21} - \sqrt{-1}(E_{23} + E_{32})) f_2, \\ \varphi_1 &= \sqrt{-1}\pi(E_{12} + E_{21} + \sqrt{-1}(E_{13} + E_{31})) f_0 \\ &\quad -\tfrac{4}{3}\pi(2E_{11} - E_{22} - E_{33}) f_1 \\ &\quad +\sqrt{-1}\pi(E_{12} + E_{21} - \sqrt{-1}(E_{13} + E_{31})) f_2, \\ \varphi_2 &= -2\pi(E_{22} - E_{33} + \sqrt{-1}(E_{23} + E_{32})) f_0 \\ &\quad -2\sqrt{-1}\pi(E_{12} + E_{21}) + \sqrt{-1}(E_{13} + E_{31})) f_1 \\ &\qquad +\tfrac{2}{3}\pi(2E_{11} - E_{22} - E_{33}) f_2. \end{aligned}$$

Then $(\varphi_0, \varphi_1, \varphi_2)$ is a constant multiple of (f_0, f_1, f_2).

3.2.2 Computation of eigenvalues

The previous lemma tells that there exist a scalar $\lambda(\sigma,\nu)$ depending on σ and ν such that $\varphi_i = \lambda(\sigma,\nu)f_i$ $(i=0,1,2)$. We determine this eigenvalue $\lambda(\sigma,\nu)$ by using explicit models of the principal series $\pi_{\sigma,\nu}$.

To do this, we have to find functions in

$$L^2\text{-}\mathrm{Ind}_M^K(\sigma_i) = L^2_{M,\sigma_i}(K) = \{f \in L^2(K) | f(mk) = \sigma(m)f(k) \text{ for all } m \in M, k \in K\}$$

corresponding to the standard basis in the minimal K-type for each i.

In the larger space $L^2(K)$, the τ_2-isotypic component is generated by the 9 matrix elements $s_{ij}(k)$ $(1 \le i,j \le 3)$ of the tautological representation

$$k \in K \mapsto S(k) = (s_{ab}(k))_{1\le a,b\le 3} \in SO(3).$$

It is directly confirmed that $s_{ib}(k)$ $(b=0,1,2)$ belong to the subspace $L^2_{M,\sigma_i}(K)$ for each i.

Diagonalizing the action of u_1, we find that s_{i1} corresponds to v_1 for each i. And finally we find that the standard basis is given by

$$v_0 = \sqrt{-1}(s_{i2} - \sqrt{-1}s_{i3}),\ v_1 = s_{i1}, \text{ and } v_2 = \sqrt{-1}(s_{i2} + \sqrt{-1}s_{i3}).$$

We need the values of these standard functions $f_a(k) = v_a$ $(a=0,1,2)$ at the identity $e \in K$.

Lemma (3.4) *The values of the standard functions at $e \in K$ is given as follows.*

1. *If $\sigma = \sigma_1$, $(f_0(e), f_1(e), f_2(e)) = (0,1,0)$.*
2. *If $\sigma = \sigma_2$, $(f_0(e), f_1(e), f_2(e)) = (\sqrt{-1}, 0, \sqrt{-1})$.*
3. *If $\sigma = \sigma_3$, $(f_0(e), f_1(e), f_2(e)) = (1,0,-1)$.*

Now we can proceed to the compuation of the value $\lambda(\sigma_i,\nu)$.

Lemma (3.5)

$$\lambda(\sigma_1,\nu) = -\frac{4}{3}(2\nu_1 - \nu_2),\ \lambda(\sigma_2,\nu) = \frac{4}{3}(\nu_1 - 2\nu_2),\ \lambda(\sigma_3,\nu) = \frac{4}{3}(\nu_1 + \nu_2).$$

Summing up the lemmata in this section, we have the following.

Proposition (3.6) *Let $\{f_i\ (i=0,1,2)\}$ be the set of the standard basis of the minimal K-type $\tau \subset \pi_{\sigma,\nu}$ of a non-spherical principal series representation $\pi_{\sigma,\nu} = \pi$. Define another three C^∞-elements $\{\varphi_i\ (i=0,1,2)\}$ by the formulae:*

$$\begin{aligned}
\varphi_0 &= \tfrac{2}{3}\pi(H_{12}+H_{13})f_0 \\
&\quad -2\sqrt{-1}\pi(E_{12}+E_{21}-\sqrt{-1}(E_{13}+E_{31}))f_1 \\
&\quad -2\pi(H_{23}-\sqrt{-1}(2E_{23}+\tfrac{1}{2}u_1)f_2, \\
\varphi_1 &= \sqrt{-1}\pi(E_{12}+E_{21}+\sqrt{-1}(E_{13}+E_{31}))f_0 \\
&\quad -\tfrac{4}{3}\pi(H_{12}+H_{13})f_1 \\
&\quad +\sqrt{-1}\pi(E_{12}+E_{21}-\sqrt{-1}(E_{13}+E_{31}))f_2, \\
\varphi_2 &= -2\pi(H_{23}+\sqrt{-1}(2E_{23}+\tfrac{1}{2}u_1)f_0 \\
&\quad -2\sqrt{-1}\pi(E_{12}+E_{21})+\sqrt{-1}(E_{13}+E_{31}))f_1 \\
&\quad +\tfrac{2}{3}\pi(H_{12}+H_{13})f_2.
\end{aligned}$$

Then we have

$$(\varphi_0, \varphi_1, \varphi_2) = \lambda(\sigma_i, \nu)(f_0, f_1, f_2)$$

with eigenvalue $\lambda(\sigma_i, \nu)$ *given by*

$$\lambda(\sigma_1, \nu) = -\frac{4}{3}(2\nu_1 - \nu_2),\ \lambda(\sigma_2, \nu) = \frac{4}{3}(\nu_1 - 2\nu_2),\ \lambda(\sigma_3, \nu) = \frac{4}{3}(\nu_1 + \nu_2).$$

In the next section, we consider the Whittaker realization of the equation of the above proposition. Then we need the following Iwasawa decomposition of standard elements of $\mathfrak{g}$.

Lemma (3.7) *We have the following decompostion of standard generators of* $\mathfrak{g}$ *with respect to the Iwasawa decomposition* $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$. *For* $H_{ij} \in \mathfrak{a}$ *we have*

$$H_{ij} = 0 + H_{ij} + 0.$$

Since $E_{ij} + E_{ji} = 2E_{ij} - (E_{ij} - E_{ji})$, *we have*

$$E_{12}+E_{21} = 2E_{12}+0+K_3, \quad E_{13}+E_{31} = 2E_{13}+0+(-K_2), \quad E_{23}+E_{23} = 2E_{23}+0+K_1.$$

4 The holonomic system for the A-radial part of the principal series Whittaker functions

4.1 The case of the class one principal series

Let I be a non-zero Whittaker functional from the class one principal series $\pi_{\sigma_0,\nu}$ to $C^\infty\text{-Ind}_N^G(\psi)$. Let F be the restriction of the image $I(f_0)$ of the K-fixed vector f_0 to A. We write here the holonomic system for F with respect to the variables $y_1 = \eta_{12}(a) = a_1/a_2$, $y_2 = \eta_{23}(a) = a_2/a_3 = a_1/a_2^2$.

Proposition (4.1) *Put* $F(y_1, y_2) = y_1 y_2 G(y_1, y_2)$ *(note* $a^\rho = y_1 y_2$*). Then* $G(y_1, y_2)$ *satisfies the partial differential eqautions:*

$$\Delta_2 G = \frac{1}{3}(\nu_1^2 + \nu_2^2 - \nu_1\nu_2)G$$

and

$$\{\partial_1(\partial_1 - \partial_2)\partial_2 + 4\pi^2 c_2^2 y_2^2 \partial_1 - 4\pi^2 c_1^2 y_1^2 \partial_2\}G = -\frac{1}{27}(2\nu_1 - \nu_2)(2\nu_2 - \nu_1)(\nu_1 + \nu_2)G.$$

Here ∂_i *is the Euler operator* $y_i \frac{\partial}{\partial y_i}$ *for* $i = 1, 2$. *and we write*

$$\Delta_2 = (\partial_1^2 + \partial_2^2 - \partial_1\partial_2) - 4\pi^2(c_1^2 y_1^2 + c_2^2 y_2^2).$$

Remark From these equations for the monodromy exponents α_1, α_2 at the origin $y_1 = 0$, $y_2 = 0$, we have an equality of sets of complex numbers:

$$\{\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2\} = \{\frac{1}{3}(2\nu_1 - \nu_2), \frac{1}{3}(2\nu_2 - \nu_1), -\frac{1}{3}(\nu_1 + \nu_2)\}.$$

4.2 The holonomic system for the A-radial part of non-spherica Whittaker functions

Let I be a non-zero Whittaker functional from the principal series $\pi_{\sigma_i,\nu}$. For the set $\{f_i|(i=0,1,2)\}$ of standard functions, we put $F_i = I(f_i)$.

Theorem (4.4) *Let $F(a) = {}^t(F_0(a), F_1(a), F_2(a)) = (y_1y_2)^t(G_0(y), G_1(y), G_2(y))$ be the vector of the A-radial part of the standard Whittaker functions with minimal K-type of the principal series representation $\pi_{\sigma,\nu}$ with non-trivial $\sigma = \sigma_i$. Then it satisfies the following partial differential equations:*

(i):

$$\begin{pmatrix} \partial_1 & 4\pi c_1 y_1 & \partial_1 - 2\partial_2 - 4\pi c_2 y_2 \\ -2\pi c_1 y_1 & -2\partial_1 & -2\pi c_1 y_1 \\ \partial_1 - 2\partial_2 + 4\pi c_2 y_2 & 4\pi c_1 y_1 & \partial_1 \end{pmatrix} \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix} = \frac{1}{2}\lambda_i \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix},$$

(ii):

$$\Delta_2 \cdot 1_3 \cdot \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix} - 2\pi c_2 y_2 \begin{pmatrix} G_0(y) \\ 0 \\ -G_2(y) \end{pmatrix} + 2\pi c_1 y_1 \begin{pmatrix} G_1(y) \\ \frac{1}{2}(G_0(y) + G_2(y)) \\ G_1(y) \end{pmatrix} = \frac{1}{3}\mu \begin{pmatrix} G_0(y) \\ G_1(y) \\ G_2(y) \end{pmatrix}.$$

Moreover the eigenvalues λ_i and μ depending on the representation $\pi_{\sigma,\nu}$ are given by

$$\begin{cases} \lambda_1 = -\frac{4}{3}(2\nu_1 - \nu_2) & (\sigma = \sigma_1) \\ \lambda_2 = \frac{4}{3}(\nu_1 - 2\nu_2) & (\sigma = \sigma_2) \\ \lambda_3 = \frac{4}{3}(\nu_1 + \nu_2) & (\sigma = \sigma_3) \end{cases} \quad \text{and } \mu = \nu_1^2 + \nu_2^2 - \nu_1\nu_2.$$

Remark We can write the differential equations (i) and (ii) of the above Theorem as

$$\text{(i)}: \ \mathcal{D}_1 G = \lambda_i G \qquad \text{(ii)}: \ \mathcal{D}_2 G = \mu G,$$

with $\mathcal{D}_i$ $(i = 1, 2)$ 3 by 3 matrix-valued differential operators. Then we have

$$\mathcal{D}_1 \cdot \mathcal{D}_2 - \mathcal{D}_2 \cdot \mathcal{D}_1 = 0.$$

4.3 The equations via the tautological basis

Let $k \in K \mapsto S(k) = (s_{ij}(k))_{1 \le i,j \le 3}$ be the tautological representation of $K = SO(3)$. Let $I \in \mathrm{Hom}_{\mathfrak{g},K}(\pi_{\sigma_i,\nu}, \mathrm{Ind}_N^G(\psi))$ be a Whittaker functional and define function T_{ij} on A by

$$I(s_{ij})_{|A} = y_1 y_2 T_{ij}(y) \quad (1 \le i, j, \le 3).$$

Then

$$\begin{pmatrix} G_0 \\ G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1} & 1 \\ 1 & 0 & 0 \\ 0 & \sqrt{-1} & -1 \end{pmatrix} \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix}.$$

Then for each i, the equation (i) of the above theorem is transformed to

$$\begin{pmatrix} -\partial_1 & -2\pi\sqrt{-1}c_1y_1 & 0 \\ -2\pi\sqrt{-1}c_1y_1 & \partial_1-\partial_2 & -2\pi\sqrt{-1}c_2y_2 \\ 0 & -2\pi\sqrt{-1}c_2y_2 & \partial_2 \end{pmatrix} \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix} = \frac{1}{2}\lambda_i \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix},$$

and the equation (ii) to

$$\left[\Delta_2 \cdot 1_3 + \begin{pmatrix} 0 & 2\pi\sqrt{-1}c_1y_1 & 0 \\ -2\pi\sqrt{-1}c_1y_1 & 0 & 2\pi\sqrt{-1}c_2y_2 \\ 0 & -2\pi\sqrt{-1}c_2y_2 & 0 \end{pmatrix}\right] \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix} = \frac{1}{3}\mu \begin{pmatrix} T_{i1} \\ T_{i2} \\ T_{i3} \end{pmatrix}.$$

5 Power series solutions at the origin

We determine 6 linearly independent formal power series at the origin $(y_1, y_2) = (0,0)$ for generic parameter ν in this section. These formal solutions converges because the singularity at the origin is a regular singularity. These solutions do not have exponential decay at infinity, different from the unique 'good' solution given by Jacquet integral. We refer to these solutions as *secondary* Whittaker functions sometimes.

5.1 The case of the class one principal series

This case is more or less discussed in the paper of Bump [2], up to some difference of notations. We omit its explicit formula.

An integral expression of this power series solution was found by Stade ([9, Lemma 3.10], [11, Theorem 2]) as an analogue of an integral formula for Jacquet integral by Vinogradov and Takhadzhyan [12]. The same as non-spherical case discussed later, we let $\{e_1, e_2, e_3\}$ be a permutation of the three complex numbers $\{-\frac{1}{3}(2\nu_1-\nu_2), -\frac{1}{3}(2\nu_2-\nu_1), \frac{1}{3}(\nu_1+\nu_2)\} = \{\frac{1}{4}\lambda_1, \frac{1}{4}\lambda_2, \frac{1}{4}\lambda_3\}$

Theorem (5.2) *For* $\mathrm{Re}(e_2-e_1) > 2$,

$$\Phi(y_1,y_2) = \Gamma(\tfrac{e_2-e_1}{2}+1)\Gamma(\tfrac{e_3-e_1}{2}+1)\Gamma(\tfrac{e_2-e_3}{2}+1)(\pi c_1y_1)^{\frac{e_3}{2}}(\pi c_2y_2)^{-\frac{e_3}{2}}(\pi c_1)^{e_1}(\pi c_2)^{-e_2}$$
$$\cdot \frac{1}{2\pi\sqrt{-1}}\int_{|u|=1} I_{\frac{e_2-e_1}{2}}(2\pi c_1y_1\sqrt{1+1/u})I_{\frac{e_2-e_1}{2}}(2\pi c_2y_2\sqrt{1+u})\,u^{-\frac{3}{4}e_3}\frac{du}{u}.$$

5.2 The case of the non-spherical principal series

In this case also, the holonomic system obtained in Theorem (4.4) has regular singularities at the origin $(y_1, y_2) = (0,0)$ with rank 6, i.e., the order of the Weyl group of $SL(3,\mathbf{R})$, for generic values of parameter ν. We determine the characteristic indices and the convergent formal power series solutions at $y = 0$. Here to abridge the notation, we write the set of variables (y_1, y_2) as y collectively.

By inspection we find that it is convenient to introduce scalar functions $\Phi_i(y_1, y_2)$ ($i = 0, 1, 2$) by

$$F(y) = y_1y_2G(y) = y_1y_2\{\Phi_0(y)\begin{pmatrix}0\\1\\0\end{pmatrix} + \Phi_1(y)\begin{pmatrix}1\\0\\1\end{pmatrix} + \Phi_2(y)\begin{pmatrix}1\\0\\-1\end{pmatrix}\}.$$

5.3 The holonomic system for $\Phi_i(y)$

Now we can rewrite the holonomic system for G_i to that for Φ_i.

Proposition (5.3) *The holonomic system in Theorem (4.4) is equivalent to the following system for $\Phi_i = \Phi_i(y_1, y_2)$ $(i = 0, 1, 2)$.*

$$
\begin{aligned}
(1)\quad & \text{(i)}\ [\partial_1 + \tfrac{1}{4}\lambda_i]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0,\\
& \text{(ii)}\ [\partial_1 - \partial_2 - \tfrac{1}{4}\lambda_i]\Phi_1 + (2\pi c_1 y_1)\Phi_0 + (2\pi c_2 y_2)\Phi_2 = 0,\\
& \text{(iii)}\ [\partial_2 - \tfrac{1}{4}\lambda_i]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0,
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad & \text{(i)}\ [\Delta_2 - \tfrac{1}{3}\mu]\Phi_0 + (2\pi c_1 y_1)\Phi_1 = 0,\\
& \text{(ii)}\ [\Delta_2 - \tfrac{1}{3}\mu]\Phi_1 + (2\pi c_1 y_1)\Phi_0 - (2\pi c_2 y_2)\Phi_2 = 0,\\
& \text{(iii)}\ [\Delta_2 - \tfrac{1}{3}\mu]\Phi_2 - (2\pi c_2 y_2)\Phi_1 = 0.
\end{aligned}
$$

5.4 The characteristic indices at the origin $(y_1, y_2) = (0, 0)$ and the recurrence formulae.

Let

$$\Phi_k(y) = y_1^{-e_1} y_2^{e_2} \sum_{n_1, n_2 \geq 0} c_{k;n_1,n_2} (\pi c_1 y_1)^{n_1} (\pi c_2 y_2)^{n_2}, \quad (k = 0, 1, 2)$$

be a system of formal power series solutions at the origin $y = 0$.

Now we can determine the 6 pairs $(-e_1, e_2)$ of characteristic indices at the origin, and the corresponding initial values conditions for F or Φ_i. the system at the origin and to determine the first coefficients Moreover we have the recurrence relations between the coefficients.

Lemma (5.4) *When $\sigma = \sigma_i$ for $i = 1,\ 2$ or 3, we have the following:*

(1) *The characteristic indices take the six values:*

$$(-e_1, e_2) = (-\tfrac{1}{4}\lambda_k, \tfrac{1}{4}\lambda_l) \quad (1 \leq k \neq l \leq 3).$$

(2) *For each case, the set of first coefficients, or the initial values at the origin are given as follows:*

(i) *If $(-e_1, e_2) = (-\frac{1}{4}\lambda_i, \frac{1}{4}\lambda_k)$ $(k \neq i)$,*

$(y_1^{e_1} y_2^{-e_2} G)(0,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, *i.e.,* $(y_1^{e_1} y_2^{-e_2}\Phi_0)(0,0) = 1$, *and* $(y_1^{e_1} y_2^{-e_2}\Phi_j)(0,0) = 0$ *for other* j.

(ii) *If $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_l)$ $(k \neq i, l \neq i, k \neq l)$,*

$(y_1^{e_1} y_2^{-e_2} G)(0,0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, *i.e.,* $(y_1^{e_1} y_2^{-e_2}\Phi_1)(0,0) = 1$, *and* $(y_1^{e_1} y_2^{-e_2}\Phi_j)(0,0) = 0$ *for other* j.

(iii) *If* $(-e_1, e_2) = (-\frac{1}{4}\lambda_k, \frac{1}{4}\lambda_i)$ $(k \neq i)$,

$(y_1^{e_1} y_2^{-e_2} G)(0,0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, *i.e.*, $(y_1^{e_1} y_2^{-e_2}\Phi_2)(0,0) = 1$, *and* $(y_1^{e_1} y_2^{-e_2}\Phi_j)(0,0) =$ 0 *for other* j.

(3) *We have the following recurrence relations for the coefficients:*

(i) $(n_1 - e_1 + \frac{1}{4}\lambda_i)c_{0;n_1,n_2} + 2c_{1;n_1-1,n_2} = 0$;

(ii) $(n_1 - n_2 - e_1 - e_2 - \frac{1}{4}\lambda_i)c_{1;n_1,n_2} + 2c_{0;n_1-1,n_2} + 2c_{2;n_1,n_2-1} = 0$;

(iii) $(n_2 + e_2 - \frac{1}{4}\lambda_i)c_{2;n_1,n_2} - 2c_{1;n_1,n_2-1} = 0$.

5.5 Power series solutions at the origin

Now we can show the following formulae for the power series solutions.

Theorem (5.5) *Assume that* $\frac{1}{4}(\lambda_k - \lambda_l) \notin \mathbf{Z}$. *Then we have the following.*

(I) *When* $\sigma = \sigma_1$ *we have the following six independent solutions.*

$$ {}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) = y_1^{-\frac{\lambda_1}{4}} y_2^{\frac{\lambda_2}{4}} $$

$$ \cdot \begin{pmatrix} \displaystyle\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+m_2}}{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_2-\lambda_3}{8}+1)_{m_2}} \\ \displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_3}{8}+1)_{m_2}} \\ \displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2+1}}{m_1!m_2!(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_2-\lambda_3}{8}+1)_{m_2}} \end{pmatrix}, $$

$$ {}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) = y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_3}{4}} $$

$$ \cdot \begin{pmatrix} \displaystyle\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1+m_2}}{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_2}} \\ \displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1+m_2}}{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_2}} \\ \displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1+m_2}}{(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_3-\lambda_2}{8}+1)_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2+1}}{m_1!m_2!(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})_{m_2+1}} \end{pmatrix}, $$

$$ {}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) = y_1^{-\frac{\lambda_2}{4}} y_2^{\frac{\lambda_1}{4}} $$

$$ \cdot \begin{pmatrix} \displaystyle -\sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1+1}(\pi c_2 y_2)^{2m_2+1}}{m_1!m_2!(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{m_2+1}} \\ \displaystyle \sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+m_2+1}}{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_2+1}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2+1}}{m_1!m_2!(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{m_2+1}} \\ \displaystyle \sum_{m_1,m_2\geq 0} \frac{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1+m_2}}{(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_1}(\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2})_{m_2}} \cdot \frac{(\pi c_1 y_1)^{2m_1}(\pi c_2 y_2)^{2m_2}}{m_1!m_2!(\frac{\lambda_3-\lambda_2}{8}+1)_{m_1}(\frac{\lambda_1-\lambda_3}{8}+\frac{1}{2})_{m_2}} \end{pmatrix} $$

and other three solutions $\Phi_i^{1,II}, \Phi_i^{1,IV}$ and $\Phi_i^{1,VI}$ are given by exchanging the role of λ_2 and λ_3 in the expression for $\Phi_i^{1,I}, \Phi_i^{1,III}$ and $\Phi_i^{1,V}$, respectively.

(II) *When $\sigma = \sigma_2$, exchange λ_1 and λ_2 in the part* (I).
(III) *When $\sigma = \sigma_3$, exchange λ_1 and λ_3 in the part* (I).

5.6 Integral representations of the secondary Whittaker functions

In this subsection, we rewrite the power series solutions of the previous subsection by integral expressions.

Theorem (5.6) (I) *When $\sigma = \sigma_1$ we have*

$$
\begin{aligned}
{}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) &= (\pi c_1 y_1)^{\frac{\lambda_3}{8}+\frac{1}{2}}(\pi c_2 y_2)^{-\frac{\lambda_3}{8}+\frac{1}{2}} \\
&\cdot(2\pi\sqrt{-1})^{-1}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+1)(\pi c_1)^{\frac{\lambda_1}{4}}(\pi c_2)^{-\frac{\lambda_2}{4}}
\end{aligned}
$$

$$
\cdot\begin{pmatrix}
\displaystyle\int_{|u|=1} I_{\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u})I_{\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}}(2\pi c_2 y_2\sqrt{1+u})\,u^{-\frac{3}{16}\lambda_3+\frac{1}{4}}\frac{du}{u} \\
\displaystyle(-1)\int_{|u|=1} I_{\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u})I_{\frac{\lambda_2-\lambda_1}{8}-\frac{1}{2}}(2\pi c_2 y_2\sqrt{1+u})\,u^{-\frac{3}{16}\lambda_3-\frac{1}{4}}(1+u)^{\frac{1}{2}}\frac{du}{u} \\
\displaystyle(-1)\int_{|u|=1} I_{\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u})I_{\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2}}(2\pi c_2 y_2\sqrt{1+u})\,u^{-\frac{3}{16}\lambda_3-\frac{1}{4}}\frac{du}{u}
\end{pmatrix}
$$

for $\mathrm{Re}(\frac{\lambda_2-\lambda_1}{8}) > \frac{3}{2}$,

$$
\begin{aligned}
{}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) &= (\pi c_1 y_1)^{\frac{\lambda_1}{8}}(\pi c_2 y_2)^{-\frac{\lambda_1}{8}} \\
&\cdot(2\pi\sqrt{-1})^{-1}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+1)(\pi c_1)^{\frac{\lambda_2}{4}}(\pi c_2)^{-\frac{\lambda_3}{4}}
\end{aligned}
$$

$$
\cdot\begin{pmatrix}
\displaystyle(\pi c_1 y_1)\int_{|u|=1} I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_1 y_1\sqrt{1+1/u})I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_2 y_2\sqrt{1+u})\,u^{-\frac{3}{16}\lambda_3-\frac{1}{2}}\frac{du}{u} \\
\displaystyle(-1)\int_{|u|=1}\Big[\pi c_1 y_1\sqrt{1+1/u}\,I_{\frac{\lambda_3-\lambda_2}{8}-1}(2\pi c_1 y_1\sqrt{1+1/u})+(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2}) \\
\displaystyle\cdot I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_1 y_1\sqrt{1+1/u})\Big]I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_2 y_2\sqrt{1+u})\,u^{-\frac{3}{16}\lambda_3-\frac{1}{2}}\frac{du}{u} \\
\displaystyle(-1)(\pi c_2 y_2)\int_{|u|=1} I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_1 y_1\sqrt{1+1/u})I_{\frac{\lambda_3-\lambda_2}{8}}(2\pi c_2 y_2\sqrt{1+u})\,u^{-\frac{3}{16}\lambda_3+\frac{1}{2}}\frac{du}{u}
\end{pmatrix}
$$

for $\mathrm{Re}(\frac{\lambda_3-\lambda_2}{8}) > 1$,

$$
\begin{aligned}
{}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) &= (\pi c_1 y_1)^{\frac{\lambda_3}{8}+\frac{1}{2}}(\pi c_2 y_2)^{-\frac{\lambda_3}{8}+\frac{1}{2}} \\
&\cdot(2\pi\sqrt{-1})^{-1}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+1)(\pi c_1)^{\frac{\lambda_2}{4}}(\pi c_2)^{-\frac{\lambda_1}{4}}
\end{aligned}
$$

$$
\cdot\begin{pmatrix}
\displaystyle(-1)\int_{|u|=1} I_{\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u})I_{\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}}(2\pi c_2 y_2\sqrt{1+u})\,u^{-\frac{3}{16}\lambda_3+\frac{1}{4}}\frac{du}{u} \\
\displaystyle\int_{|u|=1} I_{\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u})I_{\frac{\lambda_1-\lambda_2}{8}+\frac{1}{2}}(2\pi c_2 y_2\sqrt{1+u})\,u^{-\frac{3}{16}\lambda_3-\frac{1}{4}}(1+u)^{\frac{1}{2}}\frac{du}{u} \\
\displaystyle\int_{|u|=1} I_{\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2}}(2\pi c_1 y_1\sqrt{1+1/u})I_{\frac{\lambda_1-\lambda_2}{8}-\frac{1}{2}}(2\pi c_2 y_2\sqrt{1+u})\,u^{-\frac{3}{16}\lambda_3-\frac{1}{4}}\frac{du}{u}
\end{pmatrix}
$$

for $\mathrm{Re}(\frac{\lambda_1-\lambda_2}{8}) > \frac{3}{2}$.

To have the integral expression for $\Phi_i^{1,II}, \Phi_i^{1,IV}$ *and* $\Phi_i^{1,VI}$, *we have to exchange the role of* λ_2 *and* λ_3 *in the expression for* $\Phi_i^{1,I}, \Phi_i^{1,III}$ *and* $\Phi_i^{1,V}$, *respectively.*

(II) *When* $\sigma = \sigma_2$, *exchange* λ_1 *and* λ_2 *in* (I).
(III) *When* $\sigma = \sigma_3$, *exchange* λ_1 *and* λ_3 *in* (I).

6 Evaluation of Jacquet integrals

We give explicit descriptions of Jacquet integrals for non-spherical principal series Whittaker functions here. These are similar to the class one case ([12]).

6.1 Jacquet integrals

Let us denote by $g = n(g)a(g)k(g)$ the Iwasawa decomposition of $g \in G$. We define *Jacquet integral* J_{ij} for $\sigma_i \in \widehat{M}$ $(1 \leq i, j \leq 3)$ as

$$J_{ij}(g) = \int_N \psi(n)^{-1} a(s_0^{-1}ng) s_{ij}(k(s_0^{-1}ng))dn$$

for $1 \leq j \leq 3$. Here

$$s_0 = \begin{pmatrix} & & -1 \\ & -1 & \\ -1 & & \end{pmatrix}$$

the longest element in the Weyl group of $SL(3,\mathbf{R})$ and $s_{ij}(k)$ is the element of the tautological representation of K (cf. [4, (7.1)]).

Since

$$v_0 = \sqrt{-1}(s_{i2} - \sqrt{-1}s_{i3}), v_1 = s_{i1}, \; v_2 = \sqrt{-1}(s_{i2} + \sqrt{-1}s_{i3})$$

(§3.2.2) and

$$\Phi_0 = G_1, \; 2\Phi_1 = G_0 + G_2, \; 2\Phi_2 = G_0 - G_2,$$

(§5.2) the vector of integrals ${}^t(J_{i1}, \sqrt{-1}J_{i2}, J_{i3})$ has the same K-type as ${}^t(\Phi_0, \Phi_1, \Phi_2)$.

For an element $a \in A$, we use the coordinates $(y_1, y_2) = (a_1/a_2, a_1a_2^2)$. In the Iwasawa decomposition of the element $s_0^{-1}na$ its A-part $a(s_0^{-1}na)$ is given by

$$a(s_0^{-1}na) = \left(\frac{y_1^{\frac{1}{3}} y_2^{\frac{2}{3}}}{\sqrt{\Delta_1}}, \left(\frac{y_2}{y_1}\right)^{\frac{1}{3}} \sqrt{\frac{\Delta_1}{\Delta_2}} \right)$$

with

$$\Delta_1 = y_1^2y_2^2 + y_1^2n_2^2 + (n_1n_2 - n_3)^2, \quad \Delta_2 = y_1^2y_2^2 + y_2^2n_1^2 + n_3^2.$$

Under the symbol above

$$J_{ij}(y) = y_1^{(2\nu_1-\nu_2)/3+1} y_2^{(\nu_1+\nu_2)/3+1} \cdot \int_{\mathbf{R}^3} \Delta_1^{(\nu_2-\nu_1-1)/2} \Delta_2^{(-\nu_2-1)/2} k_{ij} \exp(-2\pi\sqrt{-1}(c_1n_1 + c_2n_2)) \, dn_1dn_2dn_3.$$

Here $(k_{ij})_{1\leq i,j\leq 3} = k(s_0^{-1}na)$.

6.2 Integral representations of Jacquet integrals

To write down our results, we use the following notation.

Notation.

$$K(\alpha,\beta,\gamma,\delta\,;y) := 4\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}(y_1y_2)(\pi|c_1|y_1)^{\frac{\lambda_2}{8}}(\pi|c_2|y_2)^{-\frac{\lambda_2}{8}}$$
$$\cdot\int_0^\infty K_{\frac{\lambda_3-\lambda_1}{8}+\alpha}(2\pi|c_1|y_1\sqrt{1+1/v})K_{\frac{\lambda_3-\lambda_1}{8}+\beta}(2\pi|c_2|y_2\sqrt{1+v})\,v^{-\frac{3}{16}\lambda_2+\gamma}(1+v)^{\delta}\,\frac{dv}{v}$$

with $K_\nu(z)$ the K-Bessel function.

6.2.1 The case of the class one principal series

In the case of class one, the Jacquet integral $J_0(y)$ is

Theorem (6.2) ([12]) *For* $\mathrm{Re}(\lambda_2-\lambda_1)>0$, $\mathrm{Re}(\lambda_3-\lambda_2)>0$,

$$J_0(y)=\frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})}K(0,0,0,0\,;y).$$

6.2.2 The case of the non-spherical principal series

Theorem (6.3) *For* $\mathrm{Re}(\lambda_2-\lambda_1)>0$, $\mathrm{Re}(\lambda_3-\lambda_2)>0$, *the Jacquet integrals* J_{ij} *can be written as follows.*

$$\begin{pmatrix} J_{11}(y)\\ J_{12}(y)\\ J_{13}(y)\end{pmatrix}=\frac{(\pi|c_1|)^{\frac{1}{2}}(\pi|c_2|)^{\frac{1}{2}}(\pi|c_1|y_1)^{\frac{1}{2}}(\pi|c_2|y_2)^{\frac{1}{2}}}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)}\cdot\begin{pmatrix}\varepsilon_1\varepsilon_2K(-\frac{1}{2},-\frac{1}{2},\frac{1}{4},0\,;y)\\ -\sqrt{-1}\,\varepsilon_2K(\frac{1}{2},-\frac{1}{2},-\frac{1}{4},\frac{1}{2}\,;y)\\ -K(\frac{1}{2},\frac{1}{2},-\frac{1}{4},0\,;y)\end{pmatrix}$$

$$\begin{pmatrix} J_{21}(y)\\ J_{22}(y)\\ J_{23}(y)\end{pmatrix}=\frac{1}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})}\cdot\begin{pmatrix}-\sqrt{-1}\,\varepsilon_1K(0,0,-\frac{1}{2},0\,;y)\\ -K(0,0,\frac{1}{2},-1\,;y)\\ \sqrt{-1}\,\varepsilon_2K(0,0,\frac{1}{2},0\,;y)\end{pmatrix},$$

$$\begin{pmatrix} J_{31}(y)\\ J_{32}(y)\\ J_{33}(y)\end{pmatrix}=\frac{(\pi|c_1|)^{\frac{1}{2}}(\pi|c_2|)^{\frac{1}{2}}(\pi|c_1|y_1)^{\frac{1}{2}}(\pi|c_2|y_2)^{\frac{1}{2}}}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)}\cdot\begin{pmatrix}-K(\frac{1}{2},\frac{1}{2},\frac{1}{4},0\,;y)\\ \sqrt{-1}\,\varepsilon_1K(-\frac{1}{2},\frac{1}{2},-\frac{1}{4},\frac{1}{2}\,;y)\\ \varepsilon_1\varepsilon_2K(-\frac{1}{2},-\frac{1}{2},-\frac{1}{4},0\,;y)\end{pmatrix}.$$

Here ε_i $(i=1,2)$ *means* 1 *if* $c_i>0$ *and* -1 *if* $c_i<0$.

7 Integral expression of Mellin-Barnes type

As in [9], we consider the Mellin-Barnes integral expression for $J_{ij}(y)$ to find linear relations between Jacquet integrals J_{ij} and power series solutions $\Phi_k^{i,*}$. We discuss only the non-spherical case.

Lemma (7.1) *For* $p,q\in\mathbf{C}$,

$$(\pi|c_1|y_1)^p(\pi|c_2|y_2)^q\int_0^\infty K_\alpha(2\pi|c_1|y_1\sqrt{1+1/v})K_\beta(2\pi|c_2|y_2\sqrt{1+v})\,v^\gamma(1+v)^\delta\,\frac{dv}{v}$$
$$=\frac{1}{2^4(2\pi\sqrt{-1})^2}\int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty}\int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty}V_0(s_1,s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}ds_1ds_2,$$

$$V_0(s_1,s_2) = \frac{\Gamma(\frac{s_1+p+\alpha}{2})\Gamma(\frac{s_1+p-\alpha}{2})\Gamma(\frac{s_1+p+2\gamma}{2})\Gamma(\frac{s_2+q+\beta}{2})\Gamma(\frac{s_2+q-\beta}{2})\Gamma(\frac{s_2+q-2\gamma-2\delta}{2})}{\Gamma(\frac{s_1+s_2+p+q}{2}-\delta)}.$$

Here the lines of integration are taken as to the right of all poles of the integrand.

Proposition (7.2) *Let*

$$\begin{aligned} &M(a_1,a_2,a_3\,;b_1,b_2,b_3\,;c\,;y) \\ &= \frac{1}{(2\pi\sqrt{-1})^2}\int_{\rho_1-\sqrt{-1}\infty}^{\rho_1+\sqrt{-1}\infty}\int_{\rho_2-\sqrt{-1}\infty}^{\rho_2+\sqrt{-1}\infty} V(s_1,s_2)(\pi|c_1|y_1)^{-s_1}(\pi|c_2|y_2)^{-s_2}ds_1ds_2, \end{aligned}$$

with

$$V(s_1,s_2) = \frac{\Gamma(\frac{s_1+a_1-\lambda_1}{2})\Gamma(\frac{s_1+a_2-\lambda_2}{2})\Gamma(\frac{s_1+a_3-\lambda_3}{2})\Gamma(\frac{s_1+b_1+\lambda_1}{2})\Gamma(\frac{s_1+b_2+\lambda_2}{2})\Gamma(\frac{s_1+b_3+\lambda_3}{2})}{\Gamma(\frac{s_1+s_2+c}{2})}.$$

Here the lines of integration are taken as to the right of all poles of the integrand. Then

$$\begin{pmatrix} J_{11}(y) \\ J_{12}(y) \\ J_{13}(y) \end{pmatrix} = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \cdot \begin{pmatrix} \varepsilon_1\varepsilon_2 M(0,1,1\,;1,0,0\,;1\,;y) \\ -\sqrt{-1}\,\varepsilon_2 M(1,0,0\,;1,0,0\,;0\,;y) \\ -M(1,0,0\,;0,1,1\,;1\,;y) \end{pmatrix}$$

$$\begin{pmatrix} J_{21}(y) \\ J_{22}(y) \\ J_{23}(y) \end{pmatrix} = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})} \cdot \begin{pmatrix} -\sqrt{-1}\,\varepsilon_1 M(1,0,1\,;0,1,0\,;1\,;y) \\ -M(0,1,0\,;0,1,0\,;0\,;y) \\ \sqrt{-1}\,\varepsilon_2 M(0,1,0\,;1,0,1\,;1\,;y) \end{pmatrix}$$

$$\begin{pmatrix} J_{31}(y) \\ J_{32}(y) \\ J_{33}(y) \end{pmatrix} = \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \cdot \begin{pmatrix} -M(1,1,0\,;0,0,1\,;1\,;y) \\ \sqrt{-1}\,\varepsilon_1 M(0,0,1\,;0,0,1\,;0\,;y) \\ \varepsilon_1\varepsilon_2 M(0,0,1\,;1,1,0\,;1\,;y) \end{pmatrix}.$$

Proof. It is obvious from Lemma (7.1). □

Remark. In view of this proposition, we can see the following symmetry for J_{ij} with respect to the parameter $(\lambda_1,\lambda_2,\lambda_3)$. This is natural but is not immediately seen from the formulae for J_{ij} (Theorem 6.3). We denote

$$\tilde{J}_i(\lambda_1,\lambda_2,\lambda_3) = \left(\frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{\Gamma(\frac{\lambda_2-\lambda_1}{8}+p_i)\Gamma(\frac{\lambda_3-\lambda_2}{8}+q_i)\Gamma(\frac{\lambda_3-\lambda_1}{8}+r_i)}\right)^{-1} {}^t(J_{i1}(y),J_{i2}(y),J_{i3}(y))$$

with $(p_i,q_i,r_i) = (1,\frac{1}{2},1)$ $(i=1)$, $(1,1,\frac{1}{2})$ $(i=2)$, $(\frac{1}{2},1,1)$ $(i=3)$. Then

$$\tilde{J}_2(\lambda_1,\lambda_2,\lambda_3) = (-\sqrt{-1})\,\varepsilon_2\tilde{J}_1(\lambda_2,\lambda_1,\lambda_3), \quad \tilde{J}_3(\lambda_1,\lambda_2,\lambda_3) = -\varepsilon_1\varepsilon_2\tilde{J}_1(\lambda_3,\lambda_2,\lambda_1).$$

8 Relation between Jacquet integrals and power series solutions.

We omit the case of the class one principal series here, which is discussed by other people. In the same way of [9] for class one case, we move the lines of Mellin-Barnes ɜgral expression in Proposition (7.2) to the left and sum up the residues at the poles. ɜn we obtain the following.

eorem (8.2)

$$
\begin{aligned}
{}^t(J_{11}(y), J_{12}(y), J_{13}(y)) &= \frac{\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \\
&\cdot\Big[\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\,{}^t(\Phi_0^{1,I}, \Phi_1^{1,I}, \Phi_2^{1,I}) \\
&+\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8})\,{}^t(\Phi_0^{1,II}, \Phi_1^{1,II}, \Phi_2^{1,II}) \\
&-\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8})\,{}^t(\Phi_0^{1,III}, \Phi_1^{1,III}, \Phi_2^{1,III}) \\
&-\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\,{}^t(\Phi_0^{1,IV}, \Phi_1^{1,IV}, \Phi_2^{1,IV}) \\
&-(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8})\,{}^t(\Phi_0^{1,V}, \Phi_1^{1,V}, \Phi_2^{1,V}) \\
&-(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8})\,{}^t(\Phi_0^{1,VI}, \Phi_1^{1,VI}, \Phi_2^{1,VI})\Big],
\end{aligned}
$$

$$
\begin{aligned}
{}^t(J_{21}(y), J_{22}(y), J_{23}(y)) &= \frac{-\sqrt{-1}\,\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+1)\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+\frac{1}{2})} \\
&\cdot\Big[\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8})\,{}^t(\Phi_0^{2,I}, \Phi_1^{2,I}, \Phi_2^{2,I}) \\
&+\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\,{}^t(\Phi_0^{2,II}, \Phi_1^{2,II}, \Phi_2^{2,II}) \\
&-(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\,{}^t(\Phi_0^{2,III}, \Phi_1^{2,III}, \Phi_2^{2,III}) \\
&-(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8})\,{}^t(\Phi_0^{2,IV}, \Phi_1^{2,IV}, \Phi_2^{2,IV}) \\
&-\varepsilon_2(\pi|c_1|)^{\frac{-\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8})\,{}^t(\Phi_0^{2,V}, \Phi_1^{2,V}, \Phi_2^{2,V}) \\
&-\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8})\,{}^t(\Phi_0^{2,VI}, \Phi_1^{2,VI}, \Phi_2^{2,VI})\Big],
\end{aligned}
$$

$$
\begin{aligned}
{}^t(J_{31}(y), J_{32}(y), J_{33}(y)) &= \frac{-\pi^{\frac{3}{2}}(\pi|c_1|)^{\frac{\lambda_3}{4}}(\pi|c_2|)^{-\frac{\lambda_1}{4}}y_1y_2}{4\,\Gamma(\frac{\lambda_2-\lambda_1}{8}+\frac{1}{2})\Gamma(\frac{\lambda_3-\lambda_2}{8}+1)\Gamma(\frac{\lambda_3-\lambda_1}{8}+1)} \\
&\cdot\Big[(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8})\,{}^t(\Phi_0^{3,I}, \Phi_1^{3,I}, \Phi_2^{3,I}) \\
&+(\pi|c_1|)^{-\frac{\lambda_3}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\,{}^t(\Phi_0^{3,II}, \Phi_1^{3,II}, \Phi_2^{3,II}) \\
&-\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_1}{4}}\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_2}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\,{}^t(\Phi_0^{3,III}, \Phi_1^{3,III}, \Phi_2^{3,III}) \\
&-\varepsilon_1(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_2}{4}}\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_3-\lambda_1}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8})\,{}^t(\Phi_0^{3,IV}, \Phi_1^{3,IV}, \Phi_2^{3,IV}) \\
&-\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_2}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_2}{8})\,{}^t(\Phi_0^{3,V}, \Phi_1^{3,V}, \Phi_2^{3,V}) \\
&-\varepsilon_1\varepsilon_2(\pi|c_1|)^{-\frac{\lambda_1}{4}}(\pi|c_2|)^{\frac{\lambda_3}{4}}\Gamma(\tfrac{\lambda_2-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_1-\lambda_3}{8}+\tfrac{1}{2})\Gamma(\tfrac{\lambda_2-\lambda_1}{8})\,{}^t(\Phi_0^{3,VI}, \Phi_1^{3,VI}, \Phi_2^{3,VI})\Big].
\end{aligned}
$$

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