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Modelling for Convective Heat Transport Based on Mixing Length Theory
混合距離理論を用いた対流のモデル化

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Introduction

Convection is the most important mechanism for the Earth's internal dynamics, and plays a substantial role in its evolution. When investigating the thermal history of the Earth, convective heat transport should be taken into account. However, it is difficult to treat precisely full convective flow throughout the Earth's entire history. As a result, parameterized convection was developed and has been widely used [*Schubert et al.*, 1979; *Sharpe and Peltier*, 1979].

Convection occurring in the Earth's interior has some complicated aspects, including a large variation in viscosity, internal heating, and phase boundaries. In particular, the viscosity contrast has a significant effect on the efficiency of convective heat transport. Parameterized convection treats viscosity variation artificially, and therefore has many limitations. We developed an alternative method based on the concept of "mixing length theory". The basic concept of this theory is that heat is transported by vertical motion of a fluid parcel, and after migrating for mixing length, the parcel loses it's individuality. We can relate the local thermal gradient to the local convective velocity of the fluid parcel and define the effective thermal diffusivity as the effect of convective heat transport. Then, we can calculate a horizontally averaged temperature profile and heat flux in a convective fluid by solving a mere thermal conduction problem. When estimating the parcel's velocity, we can include effects such as that caused by variable viscosity.

In this study, through comparison with experimental results, we confirm that the temperature profile can be calculated correctly by this method. We further determine the effect of the viscosity contrast on the temperature structure of the convective fluid, and calculate the relationship between the Nusselt number and a representative Rayleigh number for the layer.

Formulation

As described above, here we simply treat the convective heat flow using mixing length theory, of which the basic premise is that the velocity of the fluid parcel is related to the local thermal gradient.

Mixing length theory was firstly developed in the field of astrophysical studies in order to estimate heat flux for convective fluid with low Prandtl number and high Rayleigh number [*Vitense*, 1953]. This formulation was derived by neglecting viscous drag, and the vertical velocity of the convective fluid parcel was estimated from free fall velocity by considering that all gravitational energy was changed into kinematic energy. The viscosity, then, does not appear in the formula. *Sasaki and Nakazawa* [1986] and *Abe* [1993] extended the theory and formulated for highly viscous fluids. This formulation was based on the estimation of the vertical velocity of a parcel from Stokes velocity, namely on the concept that the buoyancy force is balanced with viscous drag. These formulations were derived from the perturbation equations of energy and momentum.

In this study, we re-formulate this theory more simply and intuitively, especially for highly viscous fluids. Because the idea of this theory is that the fluid parcel migrates for a mixing length and loses it's individuality, the mixing length can be regarded as a type of mean free path. Therefore, the effective thermal diffusivity, κ_{conv}, can be defined as

$$\kappa_{conv} = v \times l \tag{1}$$

where v is the velocity of the fluid parcel, and l is the mixing length. The temperature difference between the parcel and the surrounding temperature of the fluid, generated because the parcel moves for l vertically, is estimated as

$$\Delta T = \left[\left(\frac{dT}{dz}\right)_{ad} - \left(\frac{dT}{dz}\right)\right] l \tag{2}$$

where $\left(\frac{dT}{dz}\right)_{ad}$ is the adiabatic temperature gradient. In this study, the size of the parcel is assumed to be identified with the mixing length. The fluid parcel moves against Stoke's resistance. The velocity of the parcel, then, is defined as

$$v = \frac{4\alpha g l^2}{15\nu} \Delta T \tag{3}$$

$$= \frac{4\alpha g}{15\nu}\left[\left(\frac{dT}{dz}\right)_{ad} - \left(\frac{dT}{dz}\right)\right] l^3 \tag{4}$$

where α is the thermal expansivity, g is the gravitational accerelation, and ν is the kinematic viscosity. The effective thermal diffusivity and convective heat flux are calculated as follows;

$$\kappa_{conv} = v \times l = \frac{4\alpha g}{15\nu}\left[\left(\frac{dT}{dz}\right)_{ad} - \left(\frac{dT}{dz}\right)\right] l^4 \tag{5}$$

$$J_{conv} = \rho C_P \kappa_{conv} \frac{\Delta T}{l} = \rho C_P \kappa_{conv}\left[\left(\frac{dT}{dz}\right)_{ad} - \left(\frac{dT}{dz}\right)\right] \tag{6}$$

where ρ is the density and C_P is the heat capacity. Therefore, the temporal change of the horizontally averaged temperature profile in the convective fluid can be estimated by solving the conduction equation,

$$\rho C_P \frac{\partial T}{\partial t} = \mathrm{div}\left(k\frac{\partial T}{\partial z} - J_{conv}\right) + H \tag{7}$$

where H is the heat generation. These formulae are the same as those derived by *Sasaki and Nakazawa* [1986], except for the coefficient, although the process of formulation is different.

Convective heat flux

In this method, the mixing length 'l' is the most important parameter. We assume that the mixing length is equal to the distance from the boundary, as adopted by *Sasaki and Nakazawa* [1986] and *Abe* [1993]. This means that the fluid parcel has a size that is the same as the distance from the boundary to its generating point, and it moves for the distance of its size. This concept is illustrated in Figure 1. To compare with experimental results, we set the adiabatic temperature gradient to zero and calculated the heat flux based on equation (6). We assumed that the viscosity in the fluid layer was constant. Figure 2 shows the Nusselt number derived by the calculated heat flux as a function of Rayleigh number. The Nusselt number increases in proportion to the Rayleigh number to the power of $\frac{1}{3}$:

$$Nu \propto Ra^{\beta} \tag{8}$$

$$\beta = \frac{1}{3} \tag{9}$$

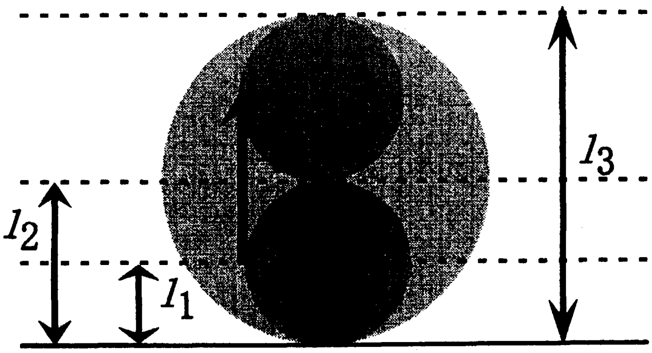

図 1: The concept of the mixing length and the parcel size, when the parcel size is regarded as same as the mixing length and the mixing length is assumed to be the distance from the boundary of the convective fluid.

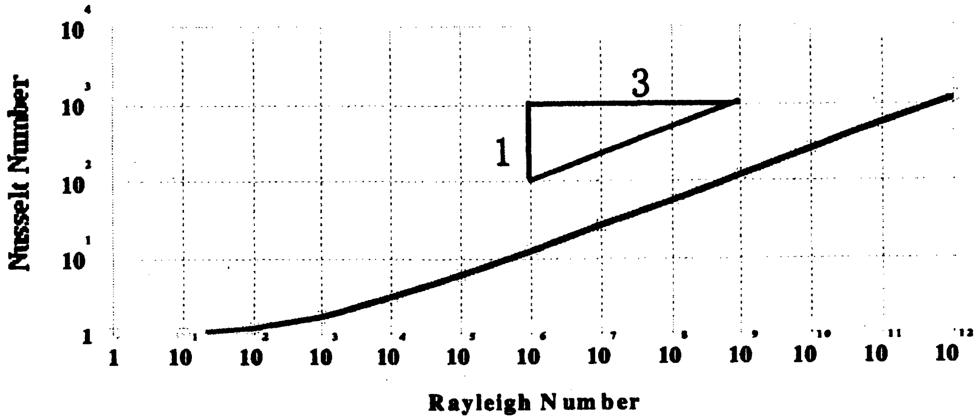

図 2: Nusselt number - Rayleigh number relation obtained by the mixing length theory.

and agrees well with the experimentally measured value. Therefore, the method for treating convective heat flow developed here can calculate the temperature structure in the convective layer accurately and easily. However, the Nusselt number is slightly overestimated by this calculation at low Rayleigh numbers. If the Rayleigh number is below the critical Rayleigh number, which is the value for the onset of convection, the Nusselt number should be a unity. Here, the calculated Nusselt number is larger than 1 under the critical Rayleigh number. This is because in this method, the critical Rayleigh number is 1, which is much smaller than the experimental or linear stable analytic critical Rayleigh number ($\sim 10^3$). But the surplus heat flux is relatively small and cannot affect the system significantly.

Temperature dependence of viscosity

In considering mantle convection, viscosity is strongly variable due to its temperature dependence. In previous parameterized convection models, it is implicitly assumed that the Nusselt-Rayleigh number relationship is not affected by spatial variation in viscosity. When using viscosity calculated from the mean temperature between the top and the bottom boundary, it is experimentally found that although the viscosity depends on temperature, the Nusselt-Rayleigh number relationship is the same as for constant viscosity convection [*Booker*, 1976; *Richter et al.*, 1983]. Some experimental and computational studies indicate that when the viscosity has an extremely high

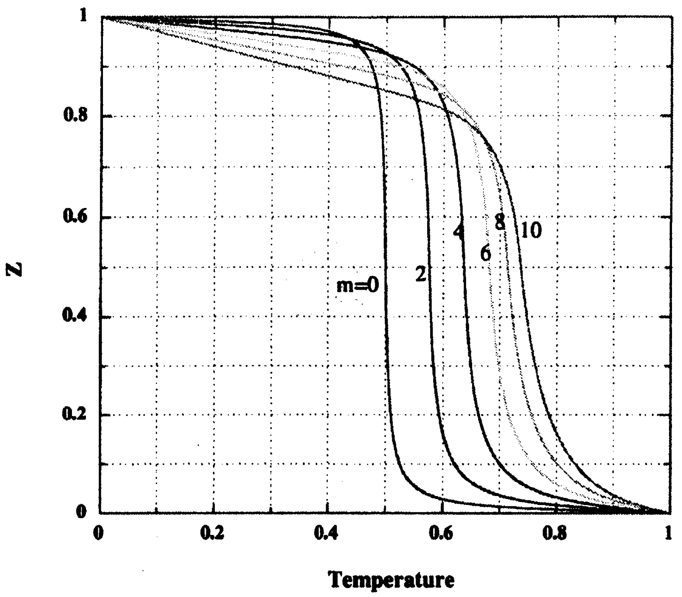

図 3: Horizontal mean temperature profile in convective layer with strongly temperature dependent viscosity. m is indicator of temperature dependency of viscosity

contrast between the top and the bottom boundaries, the dependence of Nusselt number on the Rayleigh number decreases [*Christensen*, 1984]. Namely, in equation (8), β drops below $\frac{1}{3}$.

In the method we propose here, only the local value of viscosity is needed. It is unnecessary to calculate the Rayleigh number of the convective layer before getting the Nusselt number. Therefore, the variation of the viscosity due to its temperature dependence can be taken into account directly, without an artificial treatment like parameterized convection. In addition, as we only solve a simple conduction equation, less computational effort is needed, even though the temperature dependence of viscosity is strong and convection is very active. When the viscosity strongly depends on temperature, the 2 or 3 dimensional calculations require extremely large computational efforts, and therefore are difficult to carry out. Experiments are also more difficult under such situations. Consequently, the value of β has not been clarified for convection with highly variable viscosity, and it has not been determined which viscosity is adequate for defining the system's Rayleigh number.

Figure 3 shows the temperature profiles in a convective layer with strongly temperature dependent viscosity. The viscosity is given by

$$\nu = \nu_0 \exp\left(-A(T - T_0)\right) \tag{10}$$

where T_0 is the criterion temperature, which is usually assumed to be the temperature at the cold or the hot boundary, and ν_0 is the viscosity at T_0. The indicator of the temperature dependence of viscosity, m, is defined by

$$\frac{\nu(T_t)}{\nu(T_b)} = 10^m \tag{11}$$

where T_t and T_b are the temperatures at the top and the bottom boundaries, respectively. Figure 3 indicates that a greater m corresponds to a thicker surface conductive layer and a higher core temperature, as shown in 2-D computational calculations [*Moresi and Solomatov*, 1995].

Next we used the method developed here to estimate β in equation (8), when the viscosity depends strongly on temperature. We can confirm that when the temperature dependence of the viscosity becomes strong, the dependence of the Nusselt number on the Rayleigh number decreases, namely β becomes smaller, as previous

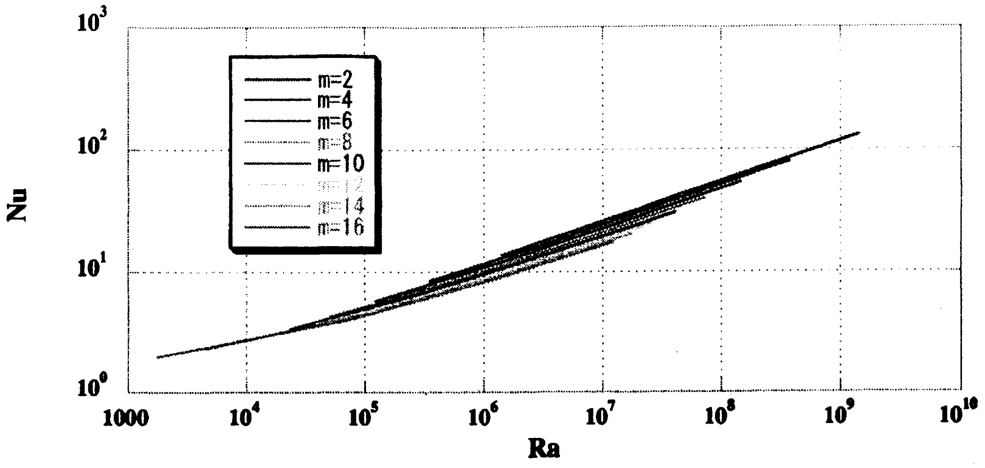

図 4: Nusselt number - Rayleigh number relation with strongly temperature dependent viscosity. Here Rayleigh number is calculated by the core temperature in the convective layer

studies indicated. When using the viscosity at the bottom temperature to calculate the Rayleigh number, the value of β decreases slightly from 0.33 to around 0.3 as increasing m. By contrast, in the case of using the viscosity at the top boundary, the value of β gets much smaller below 0.1. If the viscosity is estimated at the core temperature, β changes from 0.33 to 0.24. Figure 4 shows that relationship between the Nusselt number and the Rayleigh number, for a range in the temperature dependence of the viscosity, when the Rayleigh number is estimated by the viscosity at the core temperature.

It is found that the value of β differs significantly between various definitions of the Rayleigh number. If we want to investigate the thermal evolution of planetary bodies, the surface is the only place where the viscosity can be put to be constant throughout history. Therefore it might be better to choose the viscosity at the surface of the bodies to calculate the Rayleigh number, In this case, the value of β to use would be much smaller than the value adopted by many previous studies. If the value of the β is assumed to be $\frac{1}{3}$ and the Rayleigh number is estimated from the viscosity at the mean temperature between the top and the bottom boundaries, the Earth would have cooled very rapidly. In the method developed in this study, we do not need a Rayleigh number to calculate the heat flux, and therefore we are free from the definition of a representative Rayleigh number and the value of β.

Application for the thermal history of the Earth

In this study, we developed a simple method for treating the convective heat flux based on the concept of mixing length theory, and showed that this method can calculate the temperature structure in the convective layer correctly, and can take into account strongly temperature dependent viscosity very easily. Of course, internal heating can also be taken into account very easily by only adding a heat generation term to the conduction equation. As described above, the mixing length is the most important parameter in this method, and by assuming the mixing length adequately, it is possible to extend this method to layered convection. In addition, it can be applied to porous media by an alteration to the velocity of the fluid parcel. The mushy region between solidus and liquidus can be regarded as a type of porous media, therefore, this extended method can treat the phase change regime in the planets. The phase change also can be considered easily, through the simple conduction problem we solve in this method. Temperature dependence of the viscosity, porous media, phase change, and layered convection: all of the obstacles to calculating the thermal histories of the planetary bodies can be avoided by using this method. Therefore, we argue that this method is an adequate and powerful tool for investigating the one dimensional thermal structural evolution of the Earth.

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