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# ON THE FEKETE-SZEGÖ PROBLEM FOR STRONGLY $\alpha$-LOGARITHMIC QUASICONVEX FUNCTIONS

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ABSTRACT. The purpose of the present paper is to introduce the classes $\mathcal{M}^{\alpha}(\beta)$ and $\mathcal{Q}^{\alpha}(\beta)$, respectively, of normalized strongly $\alpha$-logarithmic convex and quasiconvex functions of order $\beta$ in the open unit disk and to obtain sharp Fekete-Szegö inequalities for functions belonging to the classes be the class $\mathcal{M}^{\alpha}(\beta)$ and $\mathcal{Q}^{\alpha}(\beta)$.

## 1. Introduction

Let $\mathcal{S}$ denote the class of analytic functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are univalent in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A classical theorem of Fekete and Szegö [8] states that for $f \in \mathcal{S}$ given by (1.1),

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq 0, \\ 1 + 2e^{-2\mu/(1-\mu)} & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 3 & \text{if } \mu \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each $\mu$ there exists a function in $\mathcal{S}$ such that equality holds. Recently, Pfluger [17,18] has considered the problem when $\mu$ is complex. In the case of $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{K}$, the subclasses of convex, starlike and close-to-convex functions, respectively, the above inequality can be improved [10,11]. Also, Darus and Thomas [5] studied the class $\mathcal{M}^{\alpha}$ of $\alpha$-logarithmic convex functions and they also have solved the Fekete-Szegö problem for the class $\mathcal{M}^{\alpha}$. Furthermore, London [14] have extended the results of Abdel-Gawad and Thomas [1], Keogh and

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Merkes [10] and Koepf [11,12] to the class $\mathcal{K}(\beta)$ of strongly close-to-convex functions of order $\beta$. Now we introduce new classes which incorporate well-known classes of univalent functions.

**Definition 1.1.** A function $f \in \mathcal{S}$ given by (1.1) is said to be strongly logarithmic $\alpha$-convex of order $\beta$ if

$$\left|\arg\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right)^{\alpha}\right\}\right| \leq \frac{\pi}{2}\beta \quad (\alpha \geq 0;\ 0 < \beta \leq 1;\ z \in \mathcal{U}). \tag{1.2}$$

Denote by $\mathcal{M}^{\alpha}(\beta)$ the class of strongly $\alpha$-logarithmic convex functions of order $\beta$. The class $\mathcal{M}^{\alpha}(\beta)$ was introduced by Chiang [4]. In particular, the classes $\mathcal{M}^{\alpha}(1) = \mathcal{M}^{\alpha}$ and $\mathcal{M}^{0}(\beta)$ have been extensively studied by Lewandowski, Miller and Zlotkiewiez [13] and Bramnan and Kirwan [2](also, see [7,20]), respectively.

**Definition 1.2.** A function $f \in \mathcal{S}$ given by (1.1) is said to be $\alpha$-logarithmic quasiconvex of order $\beta$ if there exists a function $g \in \mathcal{C}$ such that

$$\left|\arg\left\{\left(\frac{f'(z)}{g'(z)}\right)^{1-\alpha}\left(\frac{(zf'(z))'}{g'(z)}\right)^{\alpha}\right\}\right| \leq \frac{\pi}{2}\beta \quad (\alpha, \beta \geq 0;\ z \in \mathcal{U}). \tag{1.3}$$

We denote by $\mathcal{Q}^{\alpha}(\beta)$ the class of strongly $\alpha$-logarithmic quasiconvex functions of order $\beta$. Clearly, $\mathcal{Q}^{0}(1)$ and $\mathcal{Q}^{1}(1)$ are the classes of close-to-convex functions and quasiconvex functions introduced by Kaplan [9] and Noor [15](also, see [16]), respectively. Also we note that $\mathcal{Q}^{0}(\beta) = \mathcal{K}(\beta)$.

In the present paper, we derive sharp Fekete-Szegö inequalities for functions belonging to the classes $\mathcal{M}^{\alpha}(\beta)$ and $\mathcal{Q}^{\alpha}(\beta)$, which imply the results obtained by Abdel-Gawad and Thomas [1], Darus and Thomas [5], Keogh and Merkes [10], Koepf [11,12], and London [14].

## 2. Results

To prove our main results, we need the following

**Lemma 2.1.** *Let $p$ be analytic in $\mathcal{U}$ and satisfy* Re $\{p(z)\} > 0$ *for $z \in \mathcal{U}$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then*

$$|p_n| \leq 2 \quad (n \geq 1) \tag{2.1}$$

*and*

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_1|^2}{2}. \tag{2.2}$$

The inequality (2.1) was first proved by Carathéodory [3](also, see Duren [6, p. 41]) and the inequality (2.2) can be found in [19, p.166].

With the help of Lemma 2.1, we now derive

**Theorem 2.1.** *Let $f \in \mathcal{M}^{\alpha}(\beta)$ and be given by (1.1). Then for complex number $\mu$,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{\beta}{1+2\alpha}\max\left\{1, \frac{|3(1+3\alpha) - 4\mu(1+2\alpha)|\beta}{(1+\alpha)^2}\right\}.$$

*For each $\mu$, there is a function in $\mathcal{M}^{\alpha}(\beta)$ such that equality holds.*

*Proof.* From (1.2), we can write

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right)^{\alpha} = p^{\beta}(z),$$

where $p$ is given by Lemma 2.1. Equating coefficents, we obtain

$$a_2 = \frac{\beta}{1+\alpha}p_1 \tag{2.3}$$

and

$$a_3 = \frac{1}{4(1+2\alpha)}\left(\beta(\beta-1)p_1^2 + 2\beta p_2 - (\alpha^2 - 7\alpha - 2)\left(\frac{\beta p_1}{1+\alpha}\right)^2\right).$$

Then we have

$$a_3 - \mu a_2^2 = \frac{\beta}{2(1+2\alpha)}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{(3+9\alpha-4\mu(1+2\alpha))\beta^2 p_1^2}{4(1+2\alpha)(1+\alpha)^2}. \tag{2.4}$$

Hence (2.4) and Lemma 2.1 give

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &\leq \frac{\beta}{2(1+2\alpha)}\left(2 - \frac{|p_1|^2}{2}\right) + \frac{|3+9\alpha-4\mu(1+2\alpha)|\beta^2|p_1|^2}{4(1+2\alpha)(1+\alpha)^2} \\ &\leq \frac{\beta}{1+2\alpha} + \frac{\left\{|3+9\alpha-4\mu(1+2\alpha)|\beta^2 - (1+\alpha)^2\beta\right\}|p_1|^2}{4(1+2\alpha)(1+\alpha)^2}.\end{aligned}$$

Therefore, by using $|p_1| \leq 2$, we have

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \frac{\beta}{1+2\alpha}, & \text{if } k(\alpha) \leq \frac{(1+\alpha)^2}{\beta}, \\ \frac{|3+9\alpha-4\mu(1+2\alpha)|\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } k(\alpha) \geq \frac{(1+\alpha)^2}{\beta}, \end{cases}$$

where

$$k(\alpha) = |3(1+3\alpha) - 4\mu(1+2\alpha)|.$$

Equality is attained for functions in $\mathcal{M}^\alpha(\beta)$, respectively, given by

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right)^{\alpha} = \left(\frac{1+z^2}{1-z^2}\right)^{\beta} \tag{2.5}$$

and

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha}\left(\frac{(zf'(z))'}{f'(z)}\right)^{\alpha} = \left(\frac{1+z}{1-z}\right)^{\beta}. \tag{2.6}$$

**Remark 2.1.** It follows at once from (2.3) that $|a_2| \le 2\beta/(1+\alpha)$ and Theorem 2.1 gives

$$|a_3| \le \begin{cases} \frac{\beta}{1+2\alpha}, & \text{if } (1+\alpha)^2 \ge 3(1+3\alpha)\beta, \\ \frac{3(1+3\alpha)\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } (1+\alpha)^2 \le 3(1+3\alpha)\beta, \end{cases}$$

The inequality for $|a_2|$ is sharp when $f$ is defined by (2.6) and the inequalities for $|a_3|$ are sharp when $f$ is defined by (2.5) and (2.6), respectively.

Next, we consider the real number $\mu$ as follows.

**Theorem 2.2.** *Let $f \in \mathcal{M}^\alpha(\beta)$ and be given by (1.1). Then for real number $\mu$,*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{(3(1+3\alpha)-4(1+2\alpha)\mu)\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \le \frac{3(1+3\alpha)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{\beta}{1+2\alpha}, & \text{if } \frac{3(1+3\alpha)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta} \le \mu \le \frac{3(1+3\alpha)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{(4(1+2\alpha)\mu-3(1+3\alpha))\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \ge \frac{3(1+3\alpha)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}. \end{cases}$$

*For each $\mu$, there is a function in $\mathcal{C}^\alpha(\beta)$ such that equality holds in all cases.*

*Proof.* We consider two cases. At first, we suppose that $\mu \le 3(1+3\alpha)/(4(1+2\alpha))$. Then (2.3) and Lemma 2.1 give

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{\beta}{2(1+2\alpha)}\left(2 - \frac{|p_1|^2}{2}\right) + \frac{(3+9\alpha-4\mu(1+2\alpha))\beta^2|p_1|^2}{4(1+2\alpha)(1+\alpha)^2} \\ &\le \frac{\beta}{1+2\alpha} + \frac{((3+9\alpha-4\mu(1+2\alpha))\beta^2-(1+\alpha)^2\beta)|p_1|^2}{4(1+2\alpha)(1+\alpha)^2}. \end{aligned}$$

So, by using the fact that $|p_1| \le 2$, we obtain

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{(3(1+3\alpha)-4(1+2\alpha)\mu)\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \le \frac{3(1+3\alpha)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{\beta}{1+2\alpha}, & \text{if } \frac{3(1+3\alpha)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta} \le \mu \le \frac{3(1+3\alpha)}{4(1+2\alpha)}, \end{cases}$$

Equality is attained by choosing $p_1 = p_2 = 2$ and $p_1 = 0$, $p_2 = 2$, respectively, in (2.3).

Next, we suppose that $\mu \ge 3(1+3\alpha)/(4(1+2\alpha))$. In this case, it follows again from (2.3) and Lemma 2.1 that

$$\begin{aligned} |a_3-\mu a_2^2| &\le \frac{\beta}{2(1+2\alpha)}\left(2-\frac{|p_1|^2}{2}\right)+\frac{(4\mu(1+2\alpha)-(3+9\alpha))\beta^2|p_1|^2}{4(1+2\alpha)(1+\alpha)^2} \\ &\le \frac{\beta}{1+2\alpha}+\frac{((4\mu(1+2\alpha)-(3+9\alpha))\beta^2-\beta(1+\alpha)^2)|p_1|^2}{4(1+2\alpha)(1+\alpha)^2}, \end{aligned}$$

and so, as in the first case, we have

$$|a_3-\mu a_2^2| \le \begin{cases} \frac{\beta}{1+2\alpha}, & \text{if } \frac{3(1+3\alpha)}{4(1+2\alpha)} \le \mu \le \frac{3(1+3\alpha)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{(4(1+2\alpha)\mu-3(1+3\alpha))\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \ge \frac{3(1+3\alpha)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}. \end{cases}$$

The results are sharp by choosing $p_1 = 0$, $p_2 = 2$ and $p_1 = 2i$, $p_2 = -2$, respectively, in (2.3).

**Remark 2.2.** If we take $\beta = 1$ in Theorem 2.1 and Theorem 2.2, then we obtain the results by Darus and Thomas [5].

Finally, we prove

**Theorem 2.3.** *Let $f \in \mathcal{Q}^\alpha(\beta)$ and be given by (1.1). Then for $\alpha \ge 0$ and $\beta \ge 0$, we have*

$$3(2\alpha+1)\,|a_3-\mu a_2^2| \le \begin{cases} 1+\frac{(1+\beta)^2(2(3\alpha+1)-3(2\alpha+1)\mu)}{(\alpha+1)^2}, & \text{if } \mu \le \frac{2\alpha(1-\alpha)+2\beta(3\alpha+1)}{3(2\alpha+1)(1+\beta)}, \\ 1+2\beta+\frac{2(2(3\alpha+1)-3(2\alpha+1)\mu)}{2(\alpha+1)^2-\beta(2(3\alpha+1)-3(2\alpha+1)\mu)}, & \text{if } \frac{2\alpha(1-\alpha)+2\beta(3\alpha+1)}{3(2\alpha+1)(1+\beta)} \le \mu \le \frac{2(3\alpha+1)}{3(2\alpha+1)}, \\ 1+2\beta, & \text{if } \frac{2(3\alpha+1)}{3(2\alpha+1)} \le \mu \le \frac{2(\alpha+1)^2+2(3\alpha+1)(1+\beta)}{3(2\alpha+1)(1+\beta)}, \\ -1+\frac{(1+\beta)^2(3(2\alpha+1)\mu-2(3\alpha+1))}{(\alpha+1)^2}, & \text{if } \mu \ge \frac{2(\alpha+1)^2+2(3\alpha+1)(1+\beta)}{3(2\alpha+1)(1+\beta)}. \end{cases}$$

*For each $\mu$, there is a function in $\mathcal{Q}^\alpha(\beta)$ such that equality holds in all cases.*

*Proof.* Let $f \in Q^{\alpha}(\beta)$. Then it follows from (1.2) that we may write

$$\left(\frac{f'(z)}{g'(z)}\right)^{1-\alpha}\left(\frac{(zf'(z))'}{g'(z)}\right)^{\alpha} = p^{\beta}(z), \tag{2.7}$$

where $g$ is convex and $p$ has positive real part. Let $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ and let $p$ be given in the Lemma above. Then by comparing the coefficients of both sides of (2.7), we obtain

$$2(\alpha+1)a_2 = \beta p_1 + 2b_2$$

and

$$\begin{aligned}3(2\alpha+1)a_3 = 3b_3 &+ \frac{2\alpha(1-\alpha)}{(\alpha+1)^2}b_2^2 + \beta\left(p_2 - \frac{1}{2}p_1^2\right)\\ &+ \frac{\beta^2(3\alpha+1)}{2(\alpha+1)^2}p_1^2 + \frac{2\beta(3\alpha+1)}{(\alpha+1)^2}p_1b_2.\end{aligned}$$

So, with

$$x = \frac{2(3\alpha+1) - 3(2\alpha+1)\mu}{(\alpha+1)^2},$$

we have

$$\begin{aligned}3(2\alpha+1)(a_3 - \mu a_2^2) &= 3\left(b_3 + \frac{1}{3}(x-2)b_2^2\right)\\ &\quad + \beta\left(p_2 + \frac{1}{4}(\beta x - 2)p_1^2\right) + \beta x p_1 b_2.\end{aligned} \tag{2.8}$$

Since rotations of $f$ also belong to $Q^{\alpha}(\beta)$, without loss of generality, we may assume that $a_3 - \mu a_2^2$ is positive. Thus we now estimate Re $(a_3 - \mu a_2^2)$.

Since $g \in \mathcal{C}$, there exists $h(z) = 1 + k_1z + k_2z^2 + \cdots$ $(|z| < 1)$ with positive real part, such that $g'(z) + zg''(z) = g'(z)h(z)$. Hence, by equating coefficients, we get that $b_2 = k_1/2$ and $b_3 = (k_2 + k_1^2)/6$. So, by Lemma 2.1,

$$\begin{aligned}3\mathrm{Re}\left(b_3 + \frac{1}{3}(x-2)b_2^2\right) &= \frac{1}{2}\mathrm{Re}\left(k_2 - \frac{1}{2}k_1^2\right) + \frac{1}{4}(x+1)\mathrm{Re}k_1^2\\ &\le 1 - \rho^2 + (x+1)\rho^2\cos 2\phi,\end{aligned} \tag{2.9}$$

where $b_2 = k_1/2 = \rho e^{i\phi}$ for some $\rho$ $(0 \le \rho \le 1)$. We also have

$$\begin{aligned}\beta\mathrm{Re}\left(p_2+\frac{1}{4}(\beta x-2)p_1^2\right)&=\beta\mathrm{Re}\left(p_2-\frac{1}{2}p_1^2\right)+\frac{1}{4}\beta^2x\mathrm{Re}p_1^2\\&\leq 2\beta(1-r^2)+\beta^2xr^2\cos 2\theta,\end{aligned}\tag{2.10}$$

where $p_1=2re^{i\theta}$ for some $r$ $(0\leq r\leq 1)$. From (2.8-10), we obtain

$$\begin{aligned}3\mathrm{Re}(2\alpha+1)(a_3-\mu a_2^2)\leq 1-\rho^2+(x+1)\rho^2\cos 2\phi\\+2\beta(1-r^2)+\beta^2xr^2\cos 2\theta+2\beta xr\rho\cos(\theta+\phi),\end{aligned}\tag{2.11}$$

and we now proceed to maximize the right-hand of (2.11). This function will be denoted by $\psi(x)$ whenever all the parameters except $x$ are held constant.

We consider first the case

$$\frac{2\alpha(1-\alpha)+2\beta(3\alpha+1)}{3(2\alpha+1)(1+\beta)}\leq\mu\leq\frac{2(3\alpha+1)}{3(2\alpha+1)},$$

so that $0\leq x\leq 2/(1+\beta)$. Since the expression $-2t^2+t^2\beta x\cos 2\theta+2xt$ is the largest when $t=x/(2-\beta x\cos 2\theta)$, we have

$$-2t^2+t^2\beta x\cos 2\theta+2xt\leq\frac{x^2}{2-\beta x\cos 2\theta}\leq\frac{x^2}{2-\beta x}.$$

Thus

$$\begin{aligned}\psi(x)&\leq x+1+\beta\Big(2+\frac{x^2}{2-\beta x}\Big)\\&=1+2\beta+\frac{2\{2(3\alpha+1)-3(2\alpha+1)\mu\}}{2(\alpha+1)^2-\beta\{(2(3\alpha+1)-3(2\alpha+1)\mu\}}\end{aligned}$$

and with (2.11) this establishes the second inequality in the theorem. Equality occurs only if

$$p_1=\frac{2x}{2-\beta x}=\frac{2\{2(3\alpha+1)-3(2\alpha+1)\mu\}}{2(\alpha+1)^2-\beta\{2(3\alpha+1)-3(2\alpha+1)\mu\}},\quad p_2=2,\quad b_2=b_3=1,$$

and the corresponding function $f$ is defined by

$$(f'(z))^{1-\alpha}((zf'(z))')^{\alpha}=\frac{1}{(1-z)^2}\Big(\lambda\frac{1+z}{1-z}+(1-\lambda)\frac{1-z}{1+z}\Big)^{\beta},$$

$$\lambda = \frac{2(\alpha+1)^2 + (1-\beta)(2(3\alpha+1) - 3(2\alpha+1)\mu)}{4(\alpha+1)^2 - 2\beta(2(3\alpha+1) - 3(2\alpha+1)\mu)}.$$

We now prove the first inequality. Let

$$\mu \leq \frac{2\alpha(1-\alpha) + 2\beta(3\alpha+1)}{3(2\alpha+1)(1+\beta)},$$

so that $x \geq 2/(1+\beta)$. With $x_0 = 2/(1+\beta)$, we have

$$\begin{aligned}\psi(x) &= \psi(x_0) + (x - x_0)(\rho^2 \cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\beta\rho r \cos(\theta + \phi)) \\ &\leq \psi(x_0) + (x - x_0)(1+\beta)^2 \\ &\leq 1 + \frac{(1+\beta)^2\{2(3\alpha+1) - 3(2\alpha+1)\mu\}}{(\alpha+1)^2}\end{aligned}$$

as required. Equality occurs only if $p_1 = p_2 = 2,\ \ b_2 = b_3 = 1$, and the corresponding function $f$ is defined by

$$(f'(z))^{1-\alpha}((zf'(z))')^{\alpha} = \frac{1}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\beta}.$$

Let $x_1 = -2/(1+\beta)$. We shall find that $\psi(x_1) = 1 + 2\beta$, and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain

$$\begin{aligned}\psi(x) &\leq \psi(x_1) + |x - x_1|(1+\beta)^2 \\ &\leq -1 + \frac{(1+\beta)^2\{3(2\alpha+1)\mu - 2(3\alpha+1)\}}{(\alpha+1)^2}.\end{aligned}$$

if $x \leq x_1$, that is,

$$\mu \geq \frac{2(\alpha+1)^2 + 2(3\alpha+1)(1+\beta)}{3(2\alpha+1)(1+\beta)}.$$

Equality occurs only if $p_1 = 2i,\ \ b_2 = i,\ \ p_2 = -2,\ \ b_3 = -1$, and the corresponding function $f$ is defined by

$$(f'(z))^{1-\alpha}((zf'(z))')^{\alpha} = \frac{1}{(1-iz)^2}\left(\frac{1+iz}{1-iz}\right)^{\beta}.$$

Also, for $0 \leq \lambda \leq 1$, we note that

$$\begin{aligned}\psi(\lambda x_1) &= \lambda\psi(x_1) + (1-\lambda)\psi(0)\\ &\le \lambda(1+2\beta) + (1-\lambda)(1+2\beta) = 1+2\beta,\end{aligned}$$

so $\psi(x) \le 1+2\beta$ for $x_1 \le x \le 0$, that is,

$$\frac{2(3\alpha+1)}{3(2\alpha+1)} \le \mu \le \frac{2(\alpha+1)^2 + 2(3\alpha+1)(1+\beta)}{3(2\alpha+1)(1+\beta)}.$$

Equality occurs only if $p_1 = b_2 = 0$, $p_2 = 2$, $b_3 = 1/3$, and the corresponding function $f$ is defined by

$$(f'(z))^{1-\alpha}((zf'(z))')^{\alpha} = \frac{1}{1-z^2}\left(\frac{1+z^2}{1-z^2}\right)^{\beta} = \frac{(1+z^2)^{\beta}}{(1-z^2)^{1+\beta}}.$$

We now show that $\psi(x_1) \le 1+2\beta$. Since

$$\begin{aligned}&(-2+\beta x_1\cos 2\theta)t^2 + 2x_1 t\rho\cos(\theta+\phi)\\ &= (-2+\beta x_1\cos 2\theta)\left\{t + \frac{x_1\rho\cos(\theta+\phi)}{-2+\beta x_1\cos 2\theta}\right\}^2 + \frac{x_1^2\rho^2\cos^2(\theta+\phi)}{2-\beta x_1\cos 2\theta}\end{aligned}$$

for all real $t$ and

$$2-\beta x_1\cos 2\theta = 2 + \frac{2\beta}{1+\beta}\cos 2\theta \ge 2 - \frac{2\beta}{1+\beta} \ge 0,$$

we have

$$\psi(x_1) - (1+2\beta) \le \rho^2\left(-1 + (x_1+1)\cos 2\phi + \frac{\beta x_1^2(1+\cos 2(\theta+\phi))}{2(2-\beta x_1\cos 2\theta)}\right).$$

Thus we consider the inequality

$$\beta x_1^2(1+\cos 2(\theta+\phi)) + 2(2-\beta x_1\cos 2\theta)(-1+(x_1+1)\cos 2\phi \le 0.$$

After some simplifications, this becomes

$$4(\beta^2(\cos 2\phi+1)(\cos 2\phi-1) - \beta(1+\cos 2\theta+\sin 2\theta\sin 2\phi) - 1 - \cos 2\phi) \le 0,$$

which is true if

$$2\beta^2\cos^2\theta\sin^2\phi + 2\beta\cos\theta\sin\theta\cos\phi\sin\phi + \cos^2\phi \ge 0. \tag{2.12}$$

Now, for all real $t$,

$$2t^2 + 2t\sin\theta\cos\phi + \cos^2\phi \geq 0,$$

so, by taking $t = \beta\cos\theta\sin\phi$, we obtain (2.12). This completes the proof of the theorem.

**Remark.** Letting $\alpha = 0$ in Theorem 2.3, we have the corresponding result obtained by London [14], which extend the earlier results by several authors [1,5,10-12].

For $\alpha = 1$ in Theorem, we have the following

**Corollary 2.1.** *Let $f \in Q^1(\beta)$ and be given by (1.1). Then for $\beta \geq 0$, we have*

$$9\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 1 + \frac{(1+\beta)^2(8-9\mu)}{4} & \textit{if } \mu \leq \frac{8\beta}{9(1+\beta)}, \\ 1 + 2\beta + \frac{2(8-9\mu)}{8-\beta(8-9\mu)} & \textit{if } \frac{8\beta}{9(1+\beta)} \leq \mu \leq \frac{8}{9}, \\ 1 + 2\beta & \textit{if } \frac{8}{9} \leq \mu \leq \frac{8(2+\beta)}{9(1+\beta)}, \\ -1 + \frac{(1+\beta)^2(9\mu-8)}{4} & \textit{if } \mu \geq \frac{8(2+\beta)}{9(1+\beta)}. \end{cases}$$

*For each $\mu$, there is a function in $Q^1(\beta)$ such that equality holds in all cases.*

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